Mergeable Summaries

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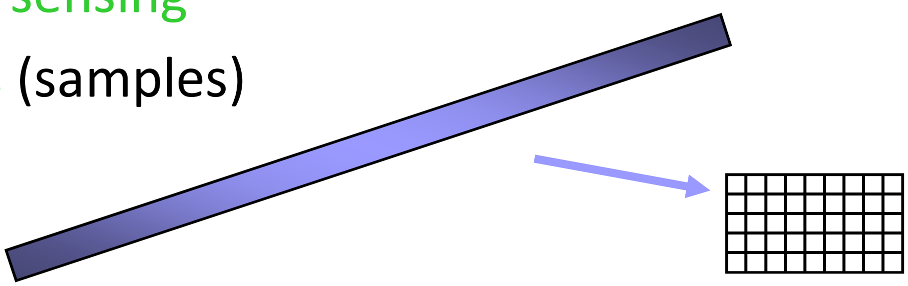
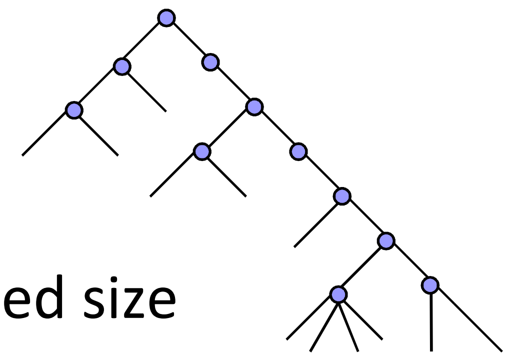
Summaries

- Summaries allow approximate computations:
  - Euclidean distance (Johnson-Lindenstrauss lemma)
  - Vector Inner-product, Matrix product (sketches)
  - Distinct items, Distinct Sampling (Flajolet-Martin onwards)
  - Frequent Items (Misra-Gries onwards)
  - Compressed sensing
  - Subset-sums (samples)



Mergeability

- Ideally, summaries are algebraic: associative, commutative
  - Allows arbitrary computation trees
    (see also synopsis diffusion [Nath+04], MUD model)
  - Distribution "just works", whatever the architecture



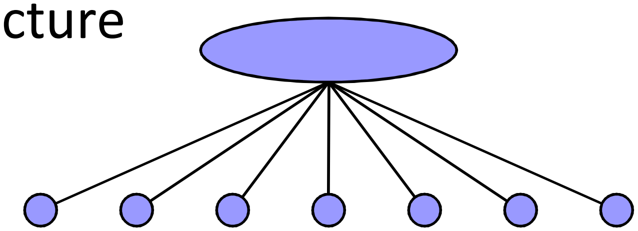
- Summaries should have bounded size
  - Ideally, independent of base data size
  - Or sublinear in base data (logarithmic, square root)
  - Should **not** depend linearly on number of merges
  - Rule out "trivial" solution of keeping union of input

Approximation Motivation

- Why use approximate when data storage is cheap?
  - Parallelize computation: partition and summarize data
    - Consider holistic aggregates, e.g. median finding
  - Faster computation (only work with summaries, not full data)
    - Less marshalling, load balancing needed
  - Implicit in some tools
    - E.g. Google Sawzall for data analysis requires mergability
  - Allows computation on data sets too big for memory/disk
    - When your data is "too big to file"

Models of Summary Construction

- Offline computation: e.g. sort data, take percentiles
- Streaming: summary merged with one new item each step
- One-way merge: each summary merges into at most one
  - Single level hierarchy merge structure
  - Caterpillar graph of merges

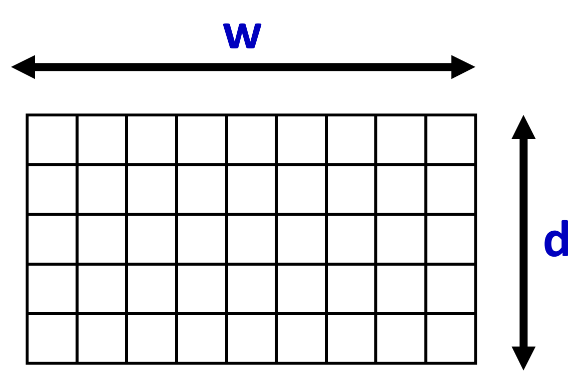


- Equal-size merges: can only merge summaries of same arity
- **Full mergeability** (algebraic): allow arbitrary merging schemes
  - Our main interest

Merging: sketches

- Example: most sketches (random projections) fully mergeable
- Count-Min sketch of vector $x[1..U]$:
  - Creates a small summary as an array of $w \times d$ in size
  - Use d hash functions h to map vector entries to $[1..w]$
  - Estimate $x[i] = \min_j CM[h_j(i), j]$
  - Error $2|x|_1/w$ with probability $1 - ½^d$
- Trivially mergeable: $CM(x + y) = CM(x) + CM(y)$

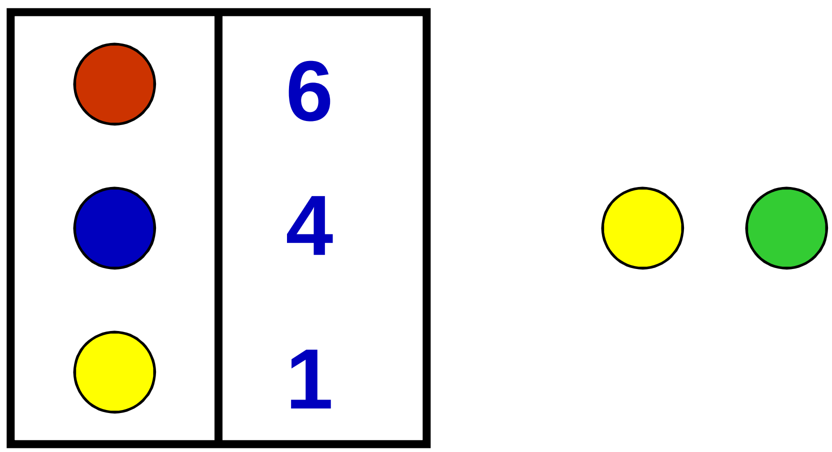
Array: CM[i,j]



Merging: sketches

- Consequence of sketch mergability:
  - Full mergability of quantiles, heavy hitters, F0, F2, dot product…
  - Easy, widely implemented, used in practice
- Limitations of sketch mergeability:
  - Probabilistic guarantees
  - May require discrete domain (ints, not reals or strings)
  - Some bounds are logarithmic in domain size

Deterministic Summaries for Heavy Hitters



- Misra-Gries (MG) algorithm finds up to k items that occur more than 1/k fraction of the time in a stream [MG82]
- Keep k different candidates in hand. For each item in stream:
  - If item is monitored, increase its counter
  - Else, if < k items monitored, add new item with count 1
  - Else, decrease all counts by 1

Streaming MG analysis

- N = total weight of input
- M = sum of counters in data structure
- Error in any estimated count at most (N-M)/(k+1)
  - Estimated count a lower bound on true count
  - Each decrement spread over (k+1) items: 1 new one and k in MG
  - Equivalent to deleting (k+1) distinct items from stream
  - At most (N-M)/(k+1) decrement operations
  - Hence, can have “deleted” (N-M)/(k+1) copies of any item
  - So estimated counts have at most this much error

Merging two MG Summaries

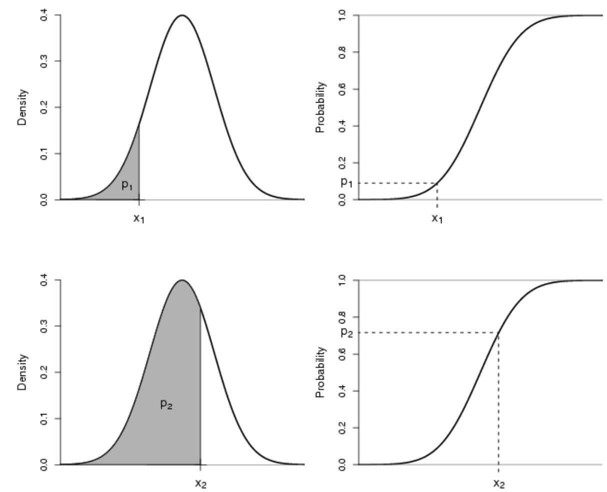
- Merging alg:
  - Merge two sets of k counters in the obvious way
  - Take the $(k+1)$th largest counter $= C_{k+1}$, and subtract from all
  - Delete non-positive counters
  - Sum of remaining (at most k) counters is M_{12}
- This alg gives full mergeability:
  - Merge subtracts at least $(k+1)C_{k+1}$ from counter sums
  - So $(k+1)C_{k+1} \leq (M_1 + M_2 - M_{12})$
  - By induction, error is
    $((N_1-M_1) + (N_2-M_2) + (M_1+M_2-M_{12}))/(k+1)=((N_1+N_2) -M_{12})/(k+1)$

    (prior error) (from merge) (as claimed)

Other heavy hitter summaries

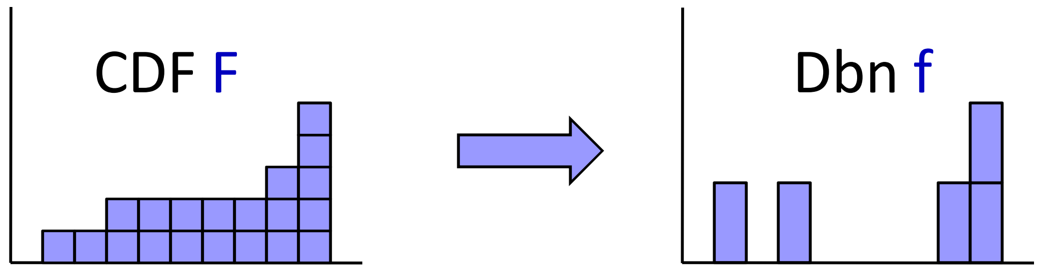
- The "SpaceSaving" (SS) summary also keeps k counters [MAA05]
  - If stream item not in summary, overwrite item with least count
  - SS seems to perform better in practice than MG
- Surprising observation: SS is actually isomorphic to MG!
  - An SS summary with k+1 counters has same info as MG with k
  - SS outputs an upper bound on count, which tends to be tighter than the MG lower bound
- Isomorphism is proved inductively
  - Show every update maintains the isomorphism
- Immediate corollary: SS is fully mergeable
  - Just merge as if it were an MG structure

Quantiles (order statistics)



- Quantiles generalize median:
  - Exact answer: $CDF^{-1}(\phi)$ for $0 < \phi < 1$
  - Approximate version: tolerate answer in $CDF^{-1}(\phi - \varepsilon) \ldots CDF^{-1}(\phi + \varepsilon)$
  - Quantile summaries solve dual problem: estimate $CDF(x) \pm \varepsilon$
- Hoeffding bound: sample of size $O(1/\varepsilon^2 \log 1/\delta)$ suffices
- Fully mergeable samples of size s via "Min-wise sampling":
  - Pick a random "tag" for samples in [0…1]
  - Merge two samples: keep the s items with smallest tags
  - Tags of $O(\log N)$ bits suffice whp
    - Can draw tie-breaking bits when needed

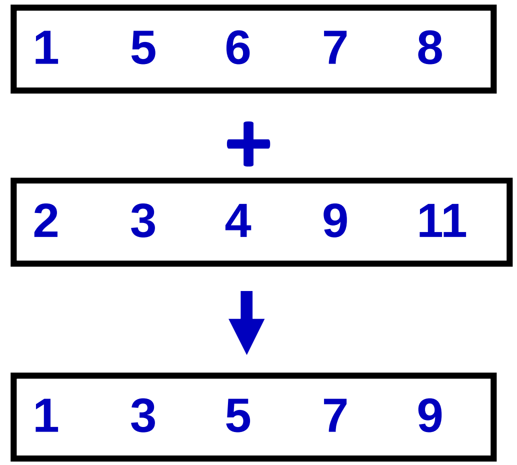
One-way mergeable quantiles



- Easy result: one-way mergeability in O(1/ε log (εn))
  - Assume a streaming summary (e.g. [Greenwald Khanna 01])
  - Extract an approximate CDF F from the summary
  - Generate corresponding distribution f over n items
  - Feed f to summary, error is bounded
  - Limitation: repeatedly extracting/inserting causes error to grow

Equal-weight merging quantiles

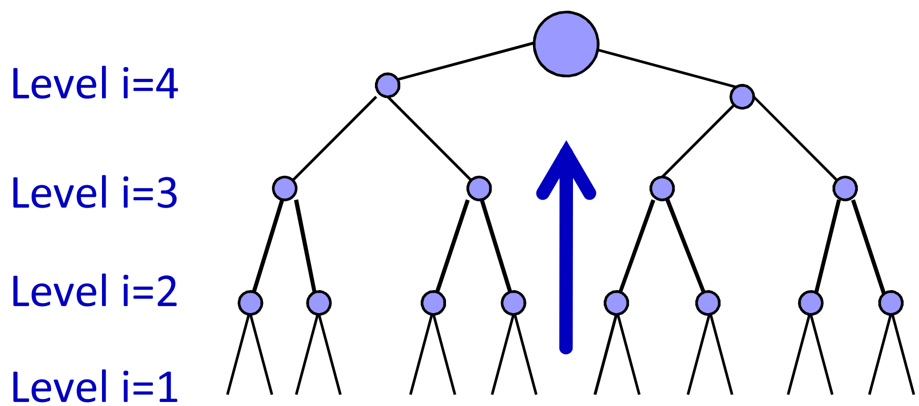
- A classic result (Munro-Paterson '78):
  - Input: two summaries of equal size k
  - Base case: fill summary with k input items
  - Merge, sort summaries to get size 2k
  - Take every other element



- Deterministic bound:
  - Error grows proportional to height of merge tree
  - Implies $O(1/\varepsilon \log^2 n)$ sized summaries (for n known upfront)
- Randomized twist:
  - Randomly pick whether to take odd or even elements

Equal-sized merge analysis: absolute error

- Consider any interval I over sample S from a single merge
- Estimate $2|I \cap S|$ has absolute error at most 1
  - $|I \cap D|$ is even: $2|I \cap S| = |I \cap X|$ (no error)
  - $|I \cap D|$ is odd: $2|I \cap S| - |I \cap X| = \pm 1$
  - Error is zero in expectation (unbiased)
- Analyze total error after multiple merges inductively
  - Binary tree of merges



Equal-sized merge analysis: error at each level

- Consider j'th merge at level i of $L^{(i-1)}$, $R^{(i-1)}$ to $S^{(i)}$
  - Estimate is $2^i \mid I \cap S^{(i)} \mid$
  - Error introduced by replacing L, R with S is

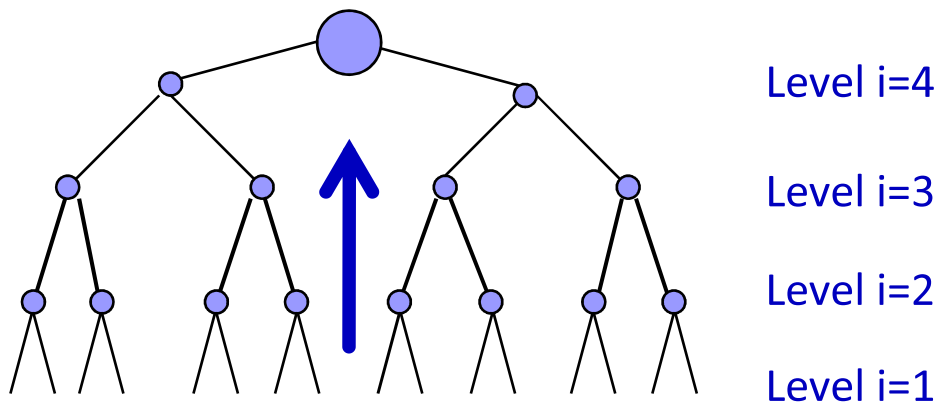
$$X_{i,j} = 2^i \mid I \cap S^i \mid \quad - \quad (2^{i-1} \mid I \cap (L^{(i-1)} \cup R^{(i-1)}) \mid)$$

(new estimate) (old estimate)

  - Absolute error $|X_{i,j}| \leq 2^{i-1}$ by previous argument
- Bound total error over all m merges by summing errors:
  - $M = \sum_{i,j} X_{i,j} = \sum_{1 \leq i \leq m} \sum_{1 \leq j \leq 2^{m-i}} X_{i,j}$
  - Analyze sum of unbiased bounded variables via Chernoff bound

Equal-sized merge analysis: Chernoff bound

- Give unbiased variables Y_j s.t. $|Y_j| \leq y_j$:
  $$\Pr[\,\mathrm{abs}(\textstyle\sum_{1 \leq j \leq t} Y_j) > \alpha\,] \leq 2\exp(-2\alpha^2 / \textstyle\sum_{1 \leq j \leq t} (2y_j)^2)$$
- Set $\alpha = h\,2^m$ for our variables:
  - $2\alpha^2/(\sum_i \sum_j (2\max(X_{i,j})^2)$
    $= 2(h2^m)^2 / (\sum_i 2^{m-i} \,.\, 2^{2i})$
    $= 2h^2\, 2^{2m} / \sum_i 2^{m+i}$
    $= 2h^2 / \sum_i 2^{i-m}$
    $= 2h^2 / \sum_i 2^{-i}$
    $\geq 2h^2$



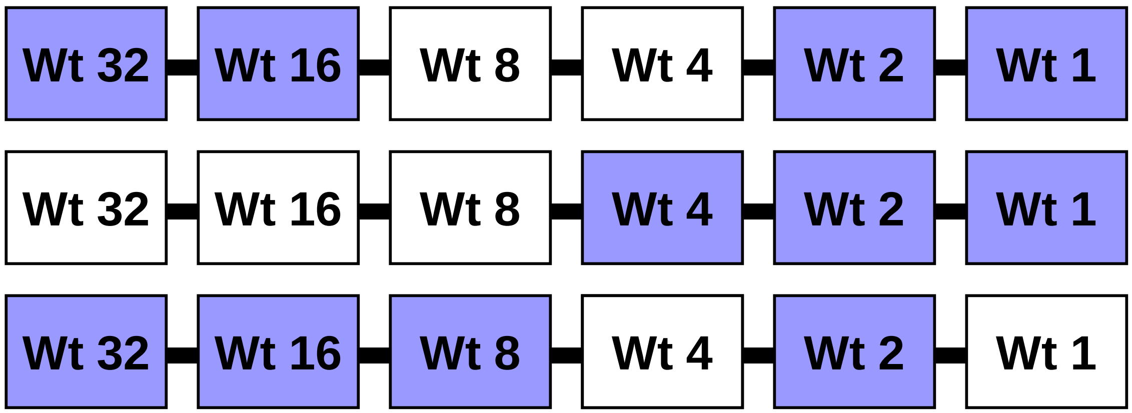
- From Chernoff bound, error probability is at most $2\exp(-2h^2)$
  - Set $h = O(\log^{1/2} \delta^{-1})$ to obtain $1-\delta$ probability of success

Equal-sized merge analysis: finishing up

- Chernoff bound ensures absolute error at most $\alpha = h2^m$
  - m is number of merges = log (n/k) for summary size k
  - So error is at most hn/k
- Set size of each summary k to be $O(h/\varepsilon) = O(1/\varepsilon \log^{1/2} 1/\delta)$
  - Guarantees give εN error with probability $1-\delta$
  - Neat: naïve sampling bound gives $O(1/\varepsilon^2 \log 1/\delta)$
  - Tightens randomized result of [Suri Toth Zhou 04]

Fully mergeable quantiles

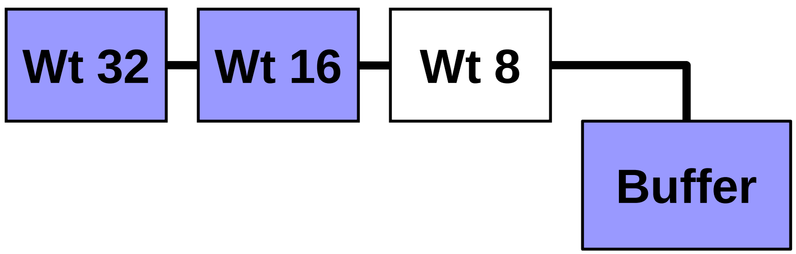
- Use equal-size merging in a standard logarithmic trick:



- Merge two summaries as binary addition
- Fully mergeable quantiles, in $O(1/\varepsilon \log(\varepsilon n) \log^{1/2} 1/\delta)$
  - n = number of items summarized, **not** known a priori
- But can we do better?

Hybrid summary

- Observation: when summary has high weight, low order blocks don't contribute much
  - Can't ignore them entirely, might merge with many small sets



- Hybrid structure:
  - Keep top O(log 1/ε) levels as before
  - Also keep a "buffer" sample of (few) items
  - Merge/keep equal-size summaries, and sample rest into buffer
  - When buffer is "full", extract points as a sample of lowest weight

Hybrid analysis (sketch)

- Keep the buffer (sample) size to $O(1/\varepsilon)$
  - Accuracy only $\sqrt{\varepsilon n}$
  - If buffer only summarizes $O(\varepsilon n)$ points, this is OK
- Analysis rather delicate:
  - Points go into/out of buffer, but always moving "up"
  - Number of "buffer promotions" is bounded
  - Similar Chernoff bound to before on probability of large error
  - Gives constant probability of accuracy in $O(1/\varepsilon \log^{1.5}(1/\varepsilon))$ space

Other Fully Mergeable Summaries

- ε-approximations generalize quantiles for range queries in multiple dimensions
  - Generalize the "odd-even" trick to low-discrepancy colorings
  - ε-approx for constant VC-dimension v queries in $\tilde{O}(\varepsilon^{-2v/(v+1)})$
- ε-kernels in d-dimensional space approximately preserve the projected extent in any direction
  - ε-kernels in $O(\varepsilon^{(1-d)/2})$ for "fat" pointsets: bounded ratio between extents in any direction
- Equal-weight merging for k-median implicit from streaming
  - Implies O(poly n) fully-mergeable summary via logarithmic trick

Open Problems

- Weight-based sampling over non-aggregated data
- Fully mergeable ε-kernels without assumptions
- More complex functions, e.g. cascaded aggregates
- Lower bounds for mergeable summaries
- Implementation studies (e.g. in Hadoop)