IN DISCRETE MATHEMATICS

Using Discrete Mathematics in the Classroom

Issue #2 *October 1992*

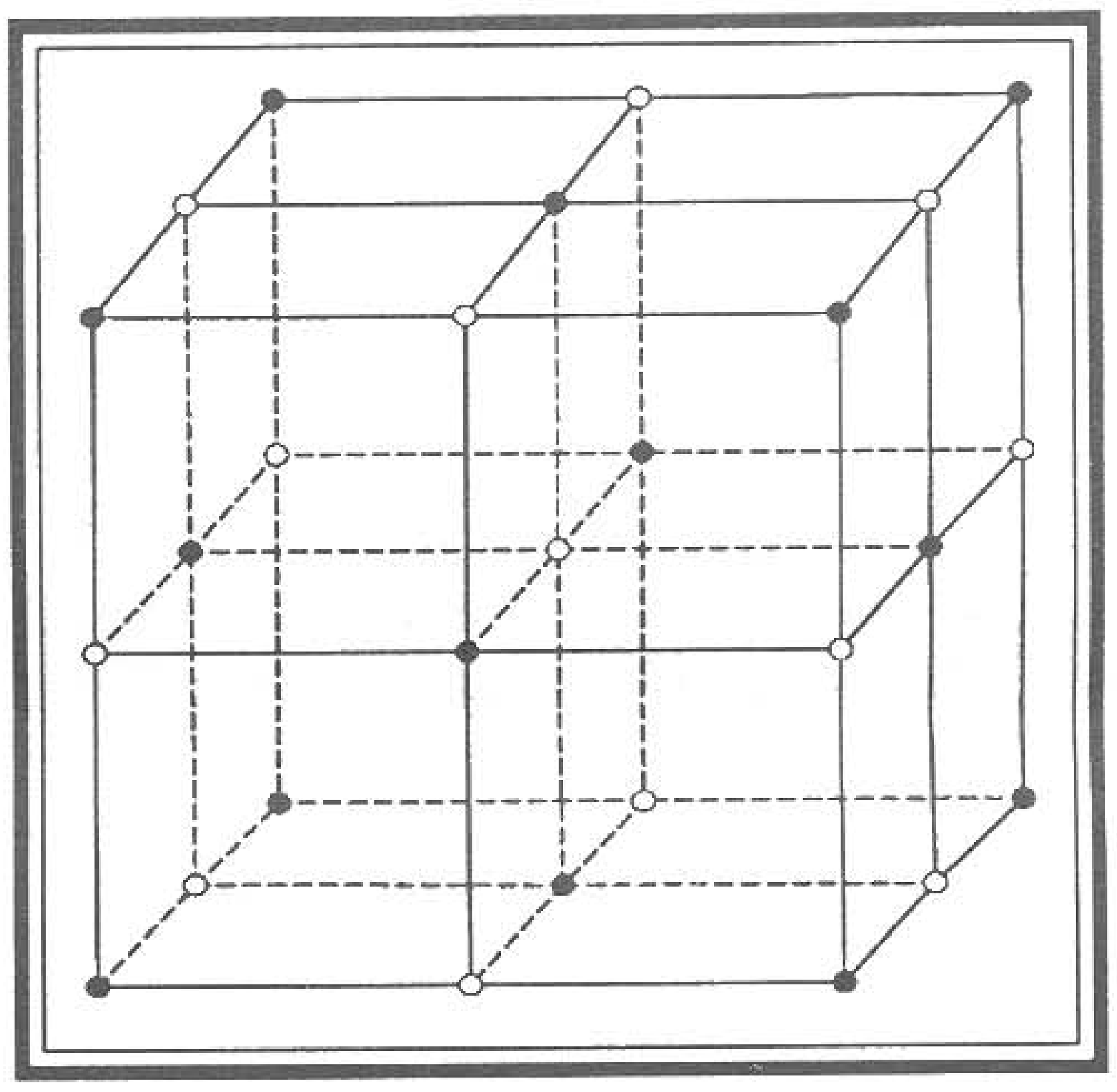
Speaking discretely...

by Joseph G. Rosenstein

This is the second issue of the Newsletter -- it took longer to produce than we intended, but here it is, and we hope that it meets your expectations!

The success of this newsletter depends on your participation. We have targeted it to teachers who already are using discrete mathematics in their classrooms, but can use the assistance and support of their colleagues. So send us assistance and support and we will pass it on.

Please communicate with us -- share with us your ideas, your classroom activities and experiences, your successes and failures, and your questions about implementing discrete mathematics in the schools. We are not asking for much -- a one page summary of an interesting lesson, a few paragraphs about a chapter in a book that you found valuable, a paragraph about your students' response to a new topic, a reaction to an article in the Newsletter, ... Your colleagues will find it valuable.



In this issue, we include articles on combinations, fractals, graphs, elections, Pythagorean triples, and hungry mice (see graph at right and the article on the bottom of page 3); discrete mathematics embraces a variety of topics.

A major focus of the issue is apportionment, represented by the "Have-you-seen ..." article at the right and the "Mini-bibliography" on page 9, both by Joseph Malkevitch. We also feature reviews of the ground-breaking *For All Practical Purposes* and the more recent treatment of similar topics in *Excursions in Modern Mathematics*. Finally, you will find here descriptions of two major opportunities to learn more about discrete mathematics during the summer. Enjoy!

In This Issue

Have-you-seen...

by Joseph Malkevitch

... the many articles this past year dealing with the census, apportionment and the Supreme Court?

The United States Constitution has been a wonderful source for mathematics problems involving the census and apportionment. First, Article I, Section 2 calls for a regular population census. The information from the census is very important since it is used, for example, in deciding the financial share of various national programs to which each state is entitled. If rural populations are counted properly but urban populations are not, this will result in "shortchanging" cities and states that contain large cities.

The last census was unusually controversial because the response rate to mail questionnaires was lower than expected and because the growing phenomenon of homelessness raised issues concerning how to count homeless people accurately. One fundamental insight from statistics is that for large populations (whether of cars, people, or anything else) it is often more accurate to get the information one desires from carefully constructed samples drawn from the population, rather than from trying to examine every element of the population, as the census currently tries to do. Considerable controversy developed after the 1990 census about the desirability of using statistical methods to "adjust" the census data to make it more accurate.

The census is also used in deciding how many seats each state is entitled to in the House of Representatives (HR below). Article I, Section 2 of the Constitution states: "Representatives...shall be apportioned among the several States...according to their

(Continued on page 8)

Teaching briefs... Garage-Door Openers

by Lillis Weber

Recently on several occasions my husband and I have come home to discover our large overhead garage door open. Since we always shut it using an electronic closer, we wondered if someone else on the street had the same frequency. When our clicker failed to open any of the surrounding doors, it seemed a mystery until I heard our door go up about noon one day. I ran outside to see if I could spot a car. There were none in sight but an airplane was directly overhead coming in for a landing at Jacksonville Naval Air Station. Sure enough, when I called the Air Station, they said this could be the problem and that I should reset the dip switches on the opener and the remote control units. This prompted me to explore the number of settings on a garage door opener.

(1) How many combinations of settings are available on a garage door opener with 8 dip switches? Each switch is either on or off. They can all be on or all be off or any combination of on's and off's. *Answer:* Since each switch is either on or off and you have 8 switches, there are altogether $2^8 = 256$ possibilities!

(2) Suppose you have 9 dip switches, how many more frequencies do you get? (The newer garage door openers have 9 switches) *Answer:* $2^9 = 512$, double the number for 8 switches, which provides 256 extra combinations.

(3) What is the probability that we have now reset ours to match a neighbor's if there are 20 houses nearby with garage door openers similar to ours? *Answer:* $20/255 = 4/51$. ■

Spreading the Word...

by Kathy Blackwood

In the fall of 1986 I attended a four-session workshop called "New Directions in Math" led by the +PLUS+ Project of Los Angeles Educational Partnership (LAEP). It was the first valuable in-service that I had ever attended, so I had taken some trig tests with me to correct, thinking that I'd have plenty of time to mark them. The presenter was Bill Lucas, a professor of mathematics at Claremont Graduate School in Southern California. I was completely captivated by the content of discrete mathematics, i.e., fair division, game theory, graph theory, voting, and apportionment; and those trig tests remained uncorrected until I returned home. The following year, I coordinated the sessions for the program, and have presented the workshop for the last two years to other math teachers. For the last three summers I have also been on the faculty at the NSF-funded Institute for Mathematics and Computer Science Education (IFMACSE) in Kent, Ohio.

For the last five years we have offered a course in Discrete Mathematics at Venice High School in Los Angeles for seniors who don't want to take the Advanced Placement Calculus course for one reason or another. Some have been "turned off" to traditional math courses and most are humanities students who don't see a need for calculus in their future. Student comments range from "This is the first time I've seen a practical use for math since I started algebra," to "This is almost fun". It is very exciting to see the light go on again for many students.

As a result of the +PLUS+ workshops and IFMACSE, several other schools that I know about have started separate courses in discrete math and

(Continued on page 11)

Inservice workshops...

Are you looking for a one-day workshop in your own school or district dealing with discrete mathematics and how it can be incorporated into your math classes and curricula?

Participants in the *Leadership Program in Discrete Mathematics* at Rutgers University (see pages 6-7) have developed and are available to conduct such workshops. Priced moderately, the workshops are appropriate for teachers and administrators at all levels. Materials are provided for classroom activities. For further information, use the address or phone number on page 6.

In each issue...

... you will find a variety of articles under the following headings:

Teaching briefs...
suggestions for classroom activities
Spreading the word...
communicating with teachers and administrators
Have-you-seen...
recent articles about discrete mathematics in the news
Mini-bibliography...
helping you find your way into a topic
Topics...
articles to introduce you to various topics
Reports...
on happenings and events
Announcements...
of opportunities and events
Ask a discrete question...
of the editors or other readers
Reader responses...
letters to the editor

... and you are invited to submit your own comments, letters, and articles under any of these (or other) headings. Please use the Newsletter address on page 6.

Teaching briefs... Shortest Connection Problems: Applied Computation

by Susan H. Picker

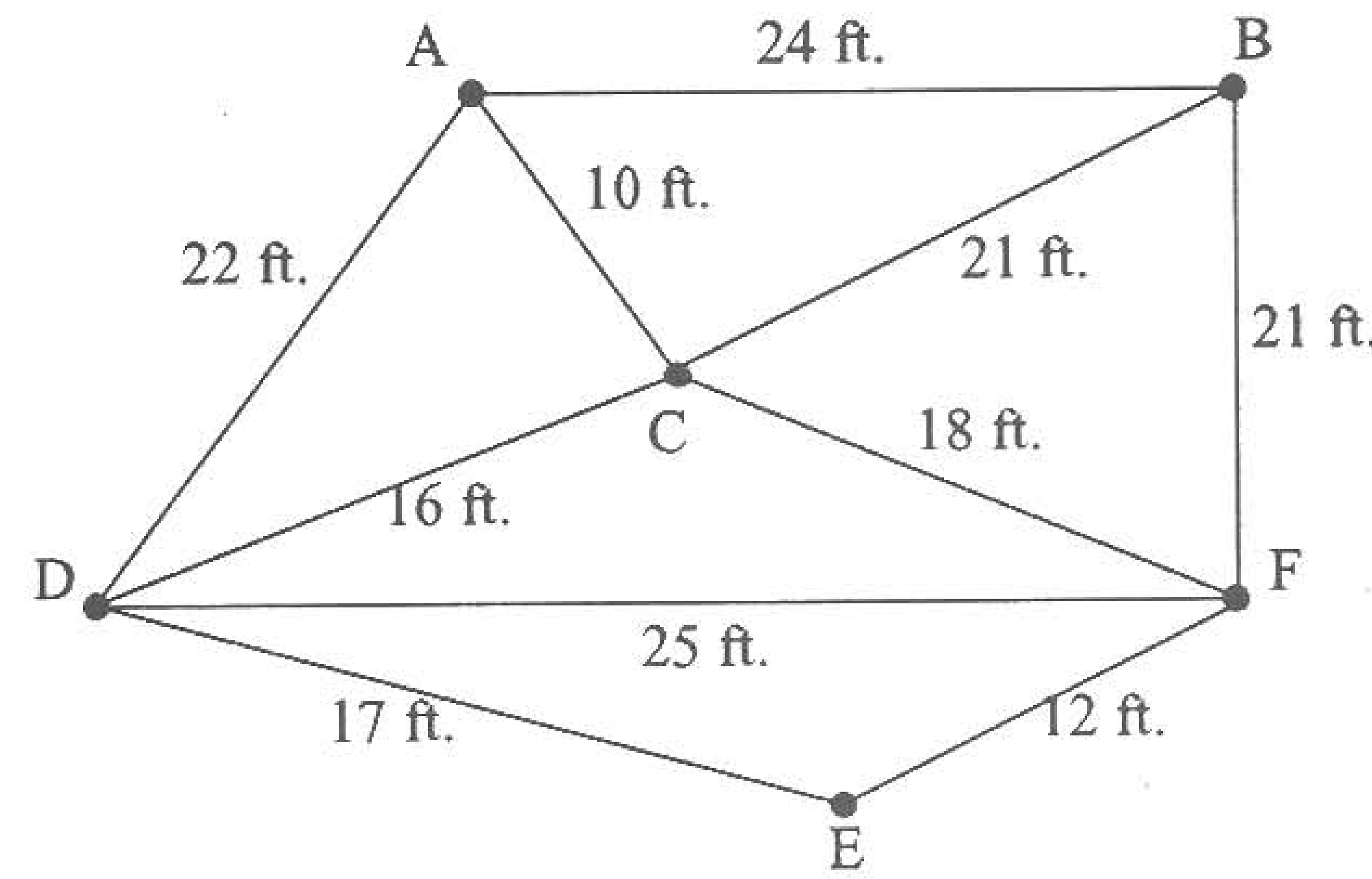
For students who are weak in computational skills, shortest connection problems which use weighted graphs are an excellent way to reinforce arithmetic concepts and introduce applied problem solving. Too often, students with weak skills are made to do pages of a particular problem or are placed in front of computers which generate one example after another and praise correct answers. It is no wonder that many students come to think of mathematics as a stifling subject.

Shortest connection problems provide interesting real-life problems. To solve them, computation is necessary, but only as the means for arriving at a solution. Among the other advantages of these problems: there often is more than one way to arrive at the solution; the problems require students to work with and understand an abstract model - a graph; they are excellent problems to use with cooperative learning and writing in the classroom; and they give rise to wonderful discussions about efficiency, as students seek a minimum length of wire or pipeline or cable.

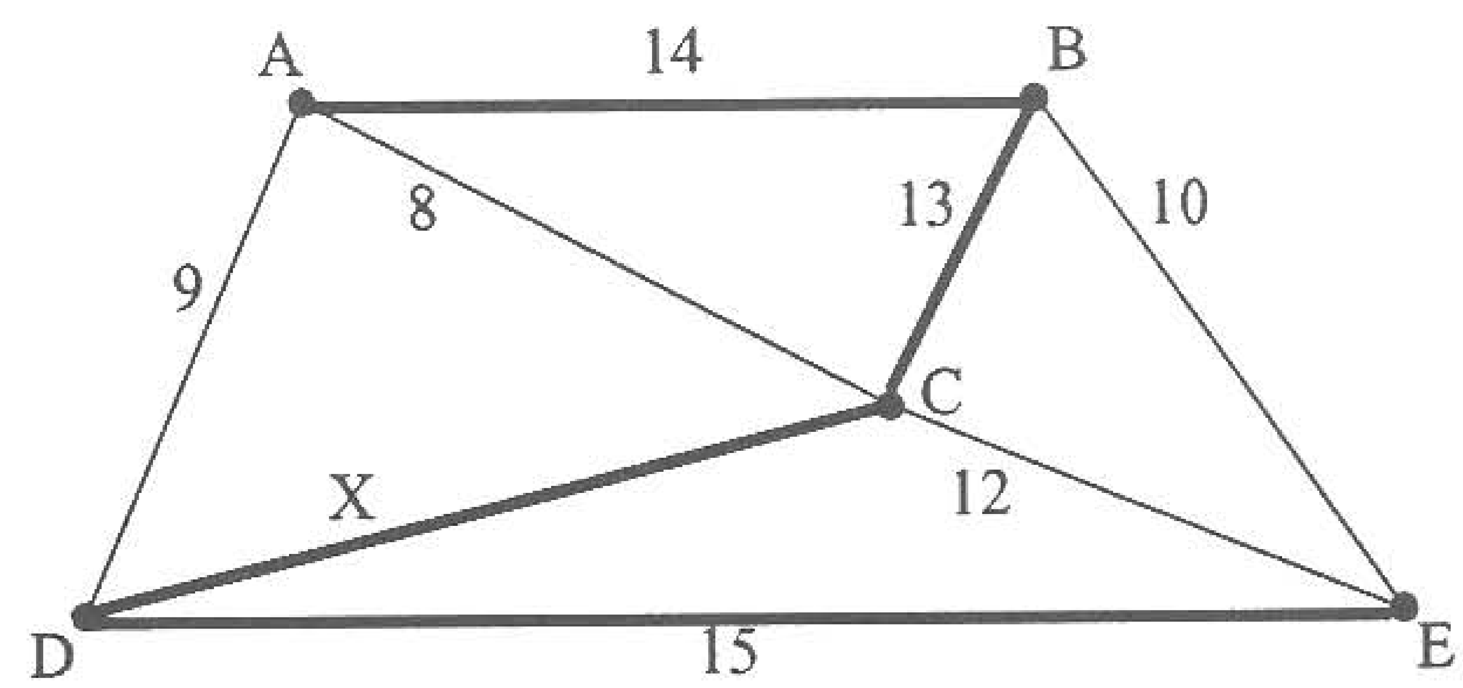
I give students a few of the problems and encourage them to try to find a rule for finding the shortest connection in the graph. Working together, students often come to see that the most efficient technique is always to link the vertex which is nearest to any vertex already connected in the solution.

In the two problems below, which are from The Decision Maths Pack, (The Spode Group, Australian Edition by P. Galbraith & A. Carr, Edward Arnold, Australia, 1988) extra information must be generated by the student. In #1, the student has to convert from yards to feet to reach a conclusion as to whether the available cable is sufficient. In #2, students have to use algebra to determine the amount of wire at x, then see if there is a shorter connection possible. Of course if students are not familiar with algebra, the unknown length can be given explicitly. (See "*Solutions...*" on page 6.) ■

1. Jack and Jill plan to hang electric lanterns in their backyard for a party. They have 25 *yards* of cable. Will this be enough to stretch between the six points shown?



2. Andrew, Bob, Crystal, Dwayne and Ezra are camping out near Bear Mountain in the pouring rain. They have all set up their tents and want to have an intercom system between them so that they can talk without getting wet. They have five transmitters and need to decide how to wire them. A possible wiring system is shown by the **dark** lines using 49 meters of wire. Can you find a way of connecting all five friends that uses *less* wire?



Topics... Two Problems Involving Graphs (reprise)

by Joseph G. Rosenstein

Last Newsletter's article with the above title posed two problems involving graphs. The first problem was solved but the second problem was left to the reader -- "A mouse eats her way through a 3x3x3 cube of cheese by tunnelling through all of the 27 1x1x1 minicubes. If she starts at one corner of the cube and always moves to an adjacent uneaten minicube, can she finish at the center of the cube?" The two problems were provided a common context -- that of bipartite graphs -- and readers were invited to see how the second problem fit into this context. The vertices of a bipartite graph can be colored using two colors -- say, red and black -- so that adjacent vertices have different colors. The 27 minicubes of cheese can be viewed as the 27 vertices of a graph, where two vertices are adjacent (in the graph theory sense) if the corresponding minicubes are adjacent (in the physical sense). This graph is bipartite -- see the diagram on page 1. Now the mouse wants to take a walk through this graph starting from a corner cube, ending at the center cube, and passing through each vertex exactly once. As she walks, she must always move from a red vertex to a black one, and vice versa. Thus if the first vertex on the walk is red, then the 27th vertex on the walk will also be red. This means that if the corner cube is red and the mouse can complete its task then the center cube must also be red. But in fact, it is black. Sorry, mouse, it can't be done! But can 25 children whose seats form a 5x5 square simultaneously move to adjacent seats? ■

Teaching briefs... An Update On Fractals

by Elyse Magram

In the summer of 1991 I attended a one week program on "Chaos, Fractals and Iteration" at Rutgers University. The program was part of the ongoing Leadership Program in Discrete Mathematics, to which the previous year's "graduates" were invited back to learn about fractal geometry. I was particularly enthusiastic about participating because I had heard about fractals the prior summer, had done extensive research on my own, and had introduced fractals to four of my classes with great enthusiasm. I wrote about this in the previous issue of this Newsletter.

Dr. Terry Perciante, professor of mathematics at Wheaton College in Illinois, was the dynamic force behind the week-long program. He introduced us to the material from the new NCTM book "Fractals for the Classroom" of which he is a coauthor.

The course exposed me to such an enormous amount of mathematics, and to so much correlation of math with fractals, that I couldn't wait to teach the topic.

This year, I used worksheets from "Fractals for the Classroom" on the Pascal triangle and cellular automata to introduce the relationship between modular arithmetic, the Pascal triangle, and the Sierpinski triangle. We then did the "chaos game" described on the right, and viewed programmed computer pictures of the Sierpinski triangle and the Sierpinski carpet. We discussed "why" this worked, using probability and area intersections. The students found all of the geometric, algebraic, probability, and arithmetic correlations amazing and thought-provoking.

Each student was responsible for doing one "colored" fractal, and urged to color as creatively as possible. We now have more than one hundred fifty of these. We have "fractal monsters" with faces and secret messages in the Hilbert curve, a Koch curve with "West," the name of our school, and a 3-D Menger Sponge of foam core. One student oil-painted her Koch snowflake (see issue #1, page 4).

We discussed the perimeter, and then the area, of the nth level of the Sierpinski triangle and carpet. I was amazed at the participation and the logic displayed by the students in finding these "limiting terms."

I spent two and one-half weeks prior to the holidays on this topic in Precalculus and two weeks in my slow, non-regents applications class. In the slower class, we didn't discuss limits.

I felt that the material provided was fantastic. The gridded worksheets in the NCTM book in the program saved a lot of time that I had previously used to measure segments and angles. Students were extremely interested in the topic, saw many applications and really "got into it." I was amused when one of my students told me how thrilled he had been to see his fractal image shaving that morning. He had rigged up an extra mirror so that his image was reflected to infinity. ■

The Chaos Game

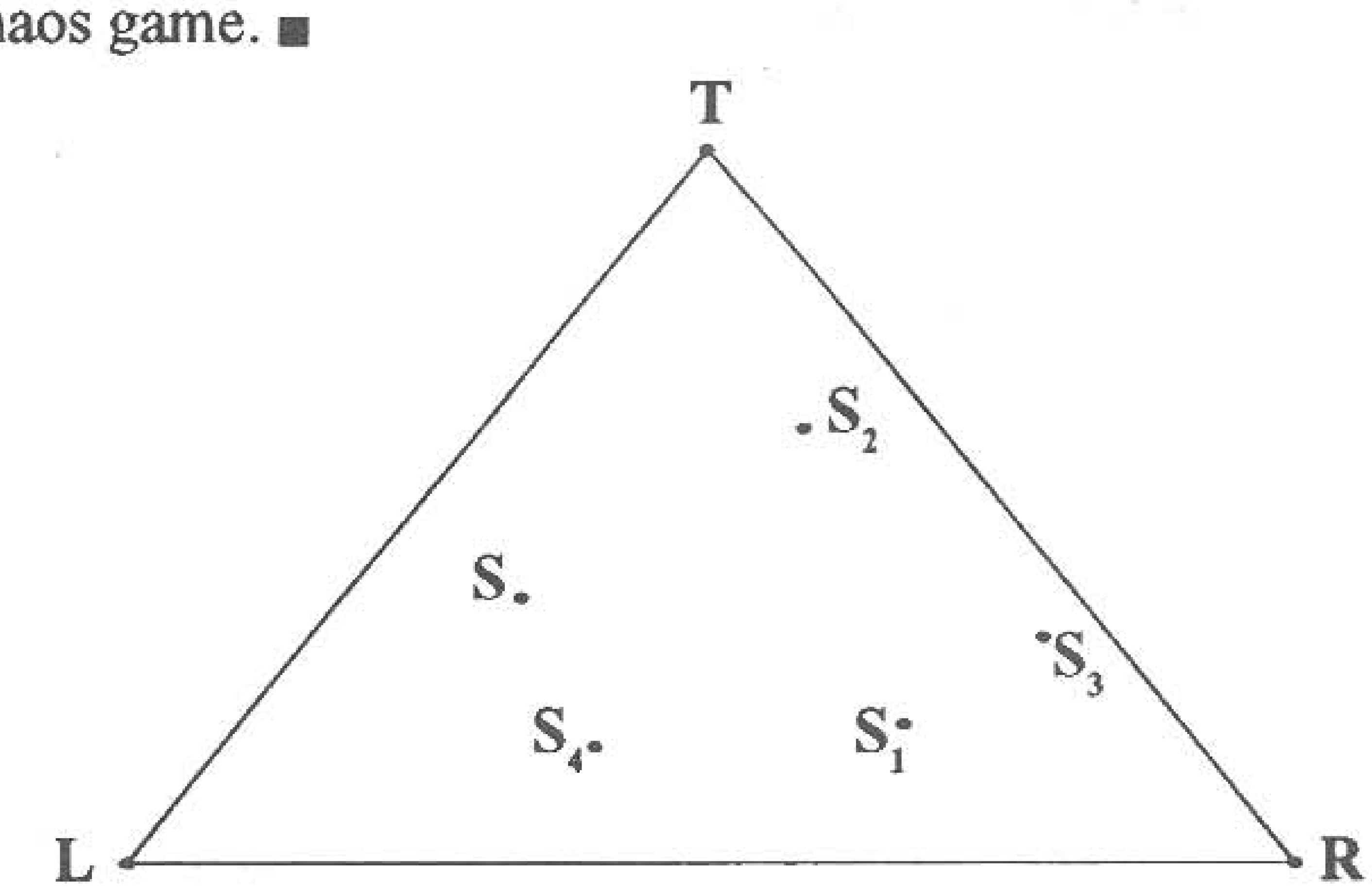
The chaos game magically collects randomly generated points into the familiar, highly structural Sierpinski triangle.

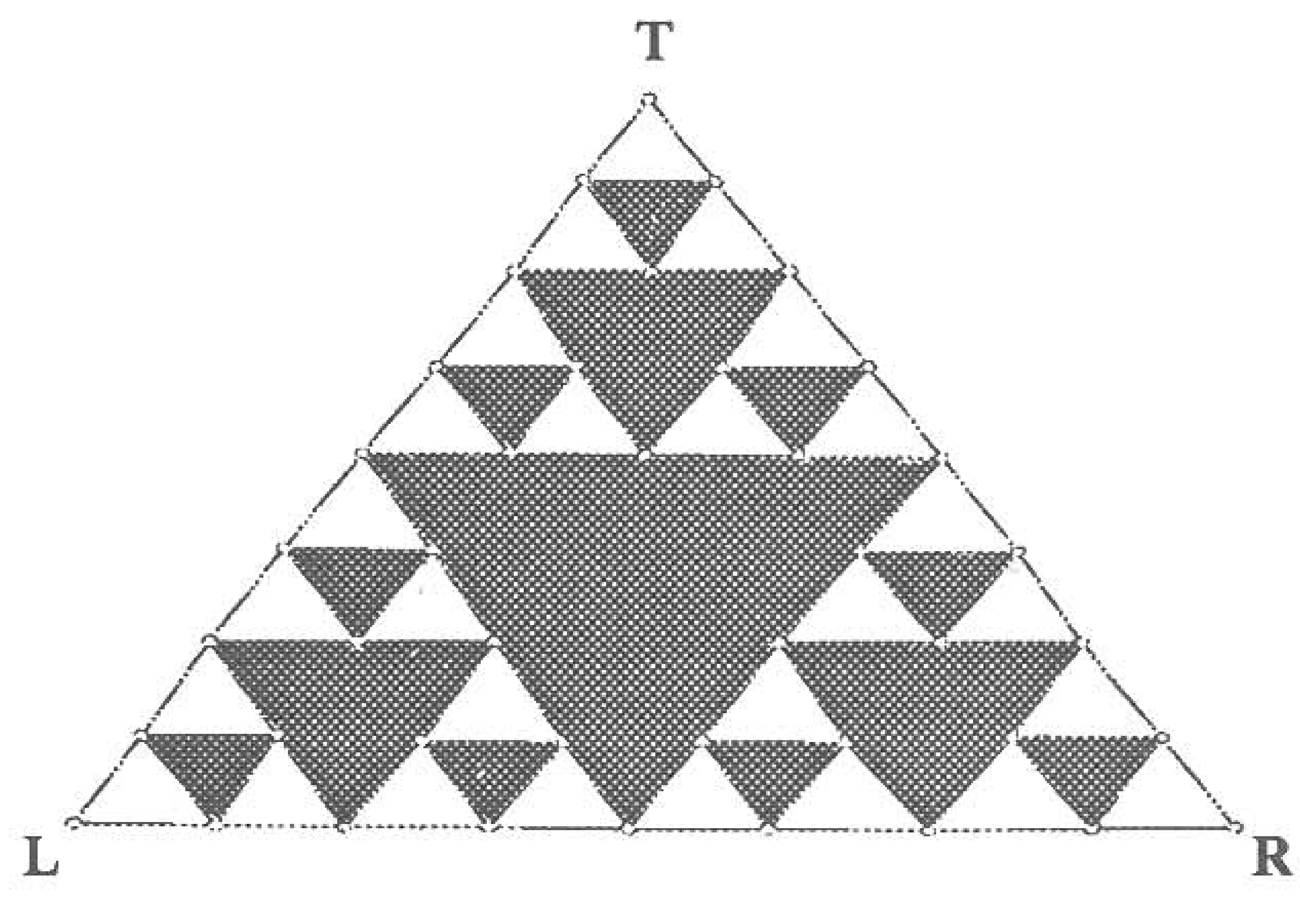
Start with three fixed points L(left), R(right) and T(top), vertices of an equilateral triangle, and a random point S within the triangle. Choose L, R, or T randomly and move halfway from S toward that vertex to get the point S_1. (For example, roll a die and assume a roll of 1 or 2 corresponds to vertex T, a roll of 3 or 4 corresponds to vertex L, and a roll of 5 or 6 corresponds to vertex R.) Choose L, R, or T randomly and move halfway from S_1 to the vertex to get the point S_2. Repeat this process, so that S_{n+1} is the midpoint of the line segment from S_n to a randomly chosen vertex L, R, or T.

In the first diagram, S_1 is halfway from S to R, S_2 is halfway from S_1 to T, S_3 is halfway from S_2 to R, and S_4 is halfway from S_3 to L. The midpoints S_1, S_2, S_3,... fill up the shaded portions of the second diagram, forming what is called (after infinite iteration) the Sierpinski triangle.

To enhance the image, ask different students to play the chaos game on transparencies marked only with identically placed triangles, and stack the resulting transparencies on the overhead projector.

You can involve geometry, algebra, probability, limits, and other topics in your classroom discussion of the chaos game. ■





Teaching briefs... Investigating Pythagorean Triples

by Ken Sullins

For the past two years I've begun my class *Finite/Discrete Math: An Introduction* with an investigation of the four sets of Pythagorean Triples on the right.

The students are given the four sets and are allowed to investigate the sets for a couple of days on their own and in small groups, with minimal directions and hints from me. I want them to have the freedom to ask their questions, make their observations, make their conjectures, and make their connections as they investigate the individual or groups of sets.

We then discuss Set 1 in class and I write the summary of our conjectures, observations, discoveries, and extensions for the next class discussion.

The next day we discuss Set 2 and write a report together in class. This writing has taken one or two days. It has been a very interesting experience for the class; trying to organize so many thoughts into a collaborative report is difficult, but provides a great experience. Remember, this is probably their first experience in writing a report in mathematics--their experiences in writing science lab reports are helpful, but the writing of mathematical concepts and explanations is likely new.

Once we have completed the class report in rough form, they select either Set 3 or Set 4 for their report. My basic rule for working together is that students can discuss anything from beginning to end (this freedom is another new experience for most of them), but when something needs to be written (in a report or on a take home quiz) it should be done after the discussion is completed and be their own writing. Time and patience by both student and teacher are important here.

The one question that always comes up in one way or another is "How do I know when I'm done?" My answer is either "When you've got no more questions to ask" or "When you're satisfied there's nothing else to find." The students come to realize that you really only take a break from the problem--you keep coming back over a period of time to look for something else!

I believe these are the types of investigations the *Standards* support. Give these sets to your classes (or those on page 7), turn them loose and see what happens! Let their questions and answers, investigations and discoveries, and conjectures, take them to a different level of problem solving.

Here are excerpts from the class report for Set 1.

We first considered the number in column **a**. These are the odd numbers and we wanted a general expression for them. To represent any even number we can use the expression $2n$, where n is any natural number; so to get the odds we can use the expression $2n - 1$.

We then looked for something relating the rows of numbers, which are Pythagorean triples. It was noted that the numbers in each row were related by $a^2 + b^2 = c^2$ and that the numbers in columns **b** and **c** were consecutive integers. Dave also noticed that if you add the numbers from the columns **a** and **b** in one row to the column **a** number in the next row you get the number in column **b** of that second row!

We then looked at a new problem solving approach, graphing, to see what type of relation might exist between the term number (we created a new column) and the numbers in the columns. The column **a** numbers were graphed (n, a), where n was the term number. We could clearly see that this did give us a line; that line was what we had found earlier---$a = 2n - 1$. For the first time students were exposed to the connection between discrete (dots) and continuous (line) graphs.

The first thing we noticed for column **b** was that the numbers are multiples of 4; Mr. Sullins told us that the second factors were called triangular numbers and explained

(Continued on page 7)

SET ONE

1	0	1
3	4	5
5	12	13
7	--	25
9	40	--
11	--	--
--	--	85

SET TWO

4	3	5
8	15	17
12	35	37
16	--	---
20	--	101
--	--	---

SET THREE

12	5	13
20	21	29
28	--	53
36	77	--
44	--	--
--	--	--

SET FOUR

9	0	9
15	8	17
21	20	29
27	36	--
--	--	65
39	--	--
--	--	--

Complete the above tables.
Discover the patterns.
Find relationships among each triple.
Seek anything else within each set.

Encouraging words...

This is *your* Newsletter -- that means that its success will be dependent on the willingness of you the readers to share your discrete thoughts and classroom experiences -- your use of written materials and software -- your information about resources -- your questions and responses -- your cartoons and problems -- your articles and announcements.

So while you are going about your way in discrete mathematics, keep the Newsletter in mind, and if you notice something that might be of interest, write a few paragraphs to submit to the Newsletter.

You will be thanked profusely by the other readers of *IN DISCRETE MATHEMATICS... Using Discrete Mathematics in the Classroom.*

IN DISCRETE MATHEMATICS...
Using Discrete Mathematics in the Classroom

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Subscriptions...

Please send us the name, address, phone number, and school of any teacher who should receive a copy of this Newsletter, and we will include him/her on our mailing list.

Credits...

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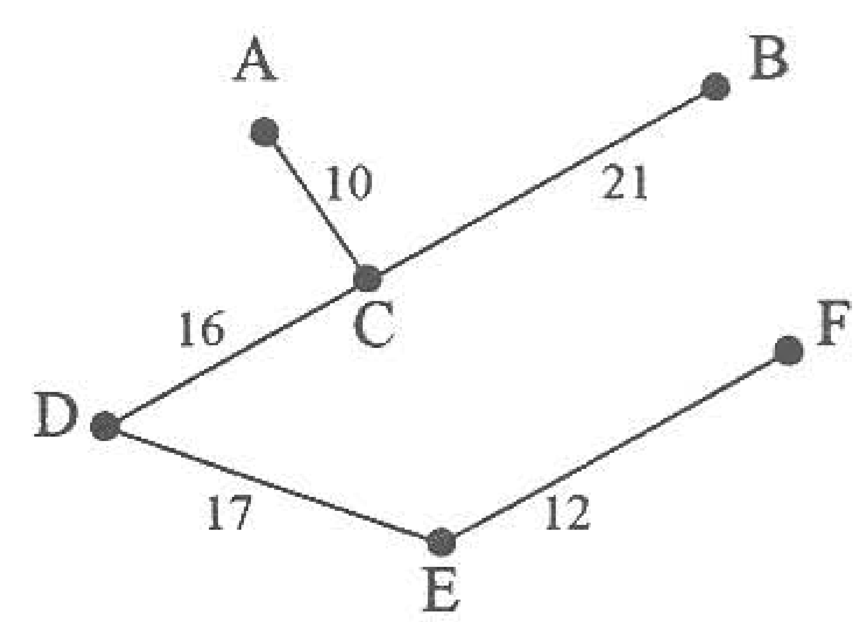
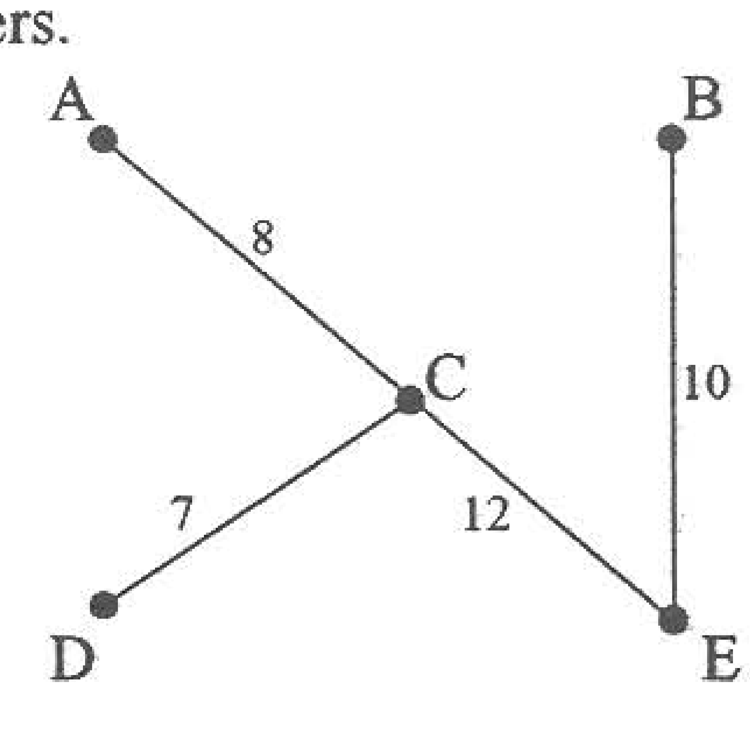
DIMACS is a national Science and Technology Center (STC) founded by the National Science Foundation (NSF); it was formed in 1989 as a consortium of four institutions -- Rutgers University, Princeton University, AT&T Bell Laboratories, and Bell Communications Research.

Solutions... *(from page 3)*

1. The length of the shortest network is 76 feet, so 75 feet of cable is insufficient.



2. Total length of wire needed: 37 meters.



Announcements ... Summer Programs

The *Leadership Program in Discrete Mathematics* will conduct its fifth annual summer program at Rutgers University this summer during the three weeks from June 28 - July 16, 1993. Two residential institutes will be offered -- one primarily for high school teachers and supervisors, and the other primarily for middle school teachers and elementary mathematics specialists and teachers. Participants will be expected to attend follow-up sessions during the school year and a two-week follow-up program during the summer of 1994. The 1993 institutes will focus on applications of graphs, graphs and algorithms, and combinatorics; other discrete mathematics topics, such as fractals and mathematical notions of fairness, will be the focus of the 1994 follow-up sessions, but will be introduced during the 1993 program. Graduate credit will be available. Funding by the National Science Foundation provides for all costs of the institute and a stipend for $300 per week. Participants will be expected to assume leadership roles in bringing discrete mathematics to their classrooms and schools, and in introducing their colleagues to these topics, both within their districts and beyond. The staff of the institutes include college faculty members who have specialized in these areas and high school and middle school teachers who have participated in past programs and have used the materials extensively in their own classrooms. For further information and application forms, please call Stephanie Micale at 908/932-4065, or write to Leadership Program, P. O. Box 10867, New Brunswick, New Jersey 08906. ■

The *Implementing the NCTM Standard in Discrete Mathematics Project* will be conducting workshops for grades 7-12 teachers at six sites during the three weeks from July 12-30, 1993 -- Boston College, University of North Carolina at Chapel Hill, Illinois State University at Normal, Southwest Texas State University at San Marcos, Portland State University, and California Polytechnic University at Pomona. Staff of these workshops will be leadership teams of high school teachers who worked together in the summer of 1992 to prepare this inservice model based on the recommendations of the Standards. Topics included will be social choice, graph theory, counting and finite probability, matrices and recursion. Participants will be expected to implement the materials they acquire in their own mathematics classes and to conduct some inservice activities in their own districts. They will return to campus once during the school year to share experiences with their colleagues and to plan outreach activities. For additional information contact the program director, Margaret Kenney at 617/552-3775 or write to Mathematics Institute, Boston College, Chestnut Hill, Massachusetts 02167. ■

Information about other summer programs will be included in the next issue of the Newsletter if received by January 1.

Teaching briefs... Investigating Pythagorean Triples

(Continued from page 5)

why using triangles. This didn't help us get what we wanted, but did leave an unanswered question [discussed and answered later] -- Is there an expression representing the triangular numbers? Also, Tom noticed that the triangular numbers occur in Pascal's triangle.

We also noticed that the column **b** numbers are multiples of the term numbers. This led to the discovery that we had one factor being n and the other factor an even number, not $2n$, but $2n - 2$. This gave us the expression for the numbers in column **b** -- $n(2n - 2)$, which can be factored and written $2n(n - 1)$. Since the numbers in column **c** are one more than the numbers in column **b**, we get the expression $2n(n - 1) + 1$ for column **c**. ■

More Sets of Pythagorean Triples to Investigate

SET FIVE

25	0	25
35	12	37
45	28	53
---	48	73
---	---	97
---	100	---
85	---	---
---	---	---
---	---	233

SET SIX

---	9	41
56	33	65
72	---	97
88	---	---
---	---	185
---	209	---
---	---	---

SET SEVEN

---	11	61
---	39	89
100	75	125
120	---	---
---	---	221
160	231	281
---	299	---
---	---	---

SET EIGHT

---	7	---
36	27	45
---	---	73
---	---	109
72	---	---
---	187	205
---	---	---
108	315	333
---	---	409

SET NINE

---	---	49
63	---	65
77	36	85
91	60	---
---	60	---
119	---	---
---	156	---
147	196	245
---	---	---

Have-you-seen...

(Continued from page 1)

respective numbers.... Each State shall have at least one representative."

Suppose a state has 10 percent of the population. Since there are 435 seats in the HR, the state should be entitled to 43.5 seats. The problem is how to treat the fractions which inevitably arise! Mathematics has been used since the beginnings of our nation to analyze how the seats in the HR should be apportioned. The history of this interaction has been long and fascinating; see "An Apportionment Problem" at the right of this page for a description of the methods of Alexander Hamilton and Thomas Jefferson and see the mini-bibliography on page 9 for further references.

The method currently in use is known as the Hill-Huntington Method (or, in a misnomer, the method of Equal Proportions). However, on December 17, 1991 the New York Times carried an article about the decision of a United States District Court, involving the state of Montana, declaring the method unconstitutional! (See [1] and also [2], [3], [4].) The Supreme Court immediately decided to hear the case, since state redistricting decisions and the legislative process itself depend on knowing that the HR is legally constituted.

In a related development, a Federal Court allowed the state of Massachusetts to maintain 11 seats in the HR rather than have one of its seats given to Washington. The decision was based on the fact that giving the seat to Washington was based on census data which included Americans living abroad. It was successfully argued by Massachusetts that data for Americans living abroad was so unreliable that only domestic data should be used. Based on domestic data only, Massachusetts was entitled to maintain 11 seats. (See [5], [6])

The Supreme Court, by votes of 9-0 in each case, overturned both the lower court decisions [8,9,10,11]. In an April 1st article in the NY Times dealing with the Supreme Court decision in the Montana case [8], it is stated that in his decision Justice Stevens was of the opinion that the method of equal proportions led to the least bias with regard to the number of seats given to small and large states. In fact, the relatively recent mathematical research of Balinski and Young [1, mini-bibliography] shows that Justice Stevens is mistaken.

Understanding the mathematics of apportionment begins with arithmetic and opens up exciting ideas in the field of discrete optimization. ■

References:

1. **High Court to Weigh Redistricting Case.** NY Times, Dec. 17, 1991, pg. A18.
2. **Court to hear Montana suit on House.** Ruth Marcus. The Washington Post, Dec. 17, 1991, v115 pg. A4
3. **High court to weigh constitutionality of apportioning House seats to states.** Paul M. Barrett. The Wall Street Journal, Dec. 17,1991, pgs. A16(W) & A22(E)
4. **Remapping of the States Up to the Courts.** LA Times, September 13, 1991.
5. **US Panel allows Massachusetts to Retain its 11 Representatives.** NY Times, February 21, 1992, pg. A16.
6. **Justices to hear Massachusetts case involving census.** Paul M. Barrett. The Wall Street Journal, March 21, 1992, pgs. B3C(W) & B6B(E)
7. **Supreme Court agrees to hear census dispute; number of House seats for 2 states is at stake.** Linda Greenhouse. The New York Times, March 21, 1992, v141 pg. 6
8. **Supreme Court upholds method used in apportionment of House.** (National Pages) Linda Greenhouse. The New York Times, April 1, 1992, v141 pgs. A18(N) & B8(L)
9. **Supreme Court decision ends Montana bid to keep House seat.** David G. Savage. Los Angeles Times, April 1, 1992, v111 pg. A10
10. **Justices back method of apportionment used to distribute House seats to states.** Paul M. Barrett. The Wall Street Journal, April 1, 1992, pg. A14(W), pg, A18(E)
11. **Massachusetts loses House seat as Washington State gains.** The New York Times, June 27, 1992, v 141, pg. 11

An Apportionment Problem

Suppose there are 10 seats in the legislature for a country having 3 states (named A, B, and C) having populations 740, 170, and 90, respectively. How many seats should each state get in the HR if each state must get at least one seat?

Alexander Hamilton's method (the method of largest remainders) would work in this way. Since A, B, and C have 74, 17, and 9 percent of the population, begin by computing .74(10) = 7.4, .17(10) = 1.7, and .09(10) = .9 (i.e. the percentage population times the size of the legislature). We assign each state the integer part of this product. Thus, A gets 7 seats, B gets 1 seat and C gets 0 seats. Since 7 + 1 + 0 = 8, we must assign 2 more seats. Order the fractional remainders above in decreasing order: .9(C), .7(B), and .4(A). The two extra seats go to C and B in this order, so that the final apportionment is A = 7, B = 2, and C = 1. B and C are over-represented and A is under-represented.

Here is a different approach using the same data. It is sometimes called Jefferson's method. Consider the table on page 9. The first line consists of the original population P of each state. The second line consists of the original data divided by 2, the third line consists of the original data divided by 3, and so on.

(Continued on page 9)

Mini-Bibliography... Apportionment

by Joseph Malkevitch

The apportionment problem belongs to the part of mathematics which deals with fairness questions. As a simple example, suppose that a modest sized corporation decides to add 22 employees. The corporation has 4 divisions, which account for 18, 23, 29, and 30 percent of the corporation's business, respectively. How many of the new employees should be assigned to each division? The division with 30 percent of the business has as its "fair share" .30 of 22 or 6.6! Since a fraction of a person can not be assigned to a division, how is the problem to be resolved? This same question arises in many settings, including how to assign states to seats in the House of Representatives based on their populations (see Mathematics in the News in this issue) or the apportioning of seats to different parties in European Parliamentary democracies based on the portion of the vote that each party received. The crux of the problem deals with the fact that an integer must be written as the sum of other integers, rather than the sum of decimals!

Balinski, M. and H.P. Young, ***Fair Representation: Meeting the Ideal of One Man, One Vote,*** *Yale University Press, 1982.* This is the definitive book on the apportionment problem. It is highly readable and includes a detailed historical account of the different procedures and approaches to apportioning the House of Representatives (US). The story is wonderfully rich in history, involving such Americans as Alexander Hamilton, Thomas Jefferson, John Quincy Adams, and Daniel Webster. The mathematics appears in appendices at the end.

Bradberry, R., ***A Geometric View of Some Apportionment Paradoxes,*** *Math. Magazine, 65 (1992) 3-17.* A somewhat technical but illuminiating discussion of some of the unexpected phenomena associated with apportionment problems. It contains many diagrams to help understand small apportionment examples.

Eisner, M.J., ***Methods of Congressional Apportionment,*** *COMAP (Lexington, MA) Module, #620.* This is an example-driven, condensed account of apportionment written for college and high school students.

Lucas, W., ***The Apportionment Problem, in Political and Related Models,*** *S. Brams, W. Lucas, and P. Straffin, (eds.), Springer-Verlag, New York, 1978.* Excellent survey article in a wonderful book dealing with applications of mathematics in political science.

Rae, D., ***The Political Consequences of Electoral Law,*** *Rev. Ed., Yale University Press, 1971.* This unusual book attempts to account for differences in the stability of different countries based on the methods that are used to apportion their parliaments. (Emphasis is on European democracies.)

Robertson, J. et al., ***The Apportionment Problem: The Search for the Perfect Democracy,*** *HIMAP Module #8, COMAP, Lexington, MA 1986.* This a module that deals with different approaches to the apportionment problem, written for high school teachers and their students. Different concepts of equity are explored, the historical context mentioned, and various mathematical implementations are treated.

Saari, D., ***Apportionment Methods and the House of Representatives,*** *Amer. Math. Monthly, 85 (1978) 792-802.* A survey of mathematical ideas involving apportionment.

Steen, L., (ed.), ***For All Practical Purposes (2nd ed.),*** *W.H. Freeman, New York, 1991.* Chapter 11 has an elementary treatment of the apportionment problem.

Young, H.P. (ed.), ***Fair Allocation,*** *American Mathematical Society, Providence, 1985.* The volume is a collection of articles on fairness. It includes an article entitled the **Apportionment of Representation** by Balinski and Young. This is an expository treatment (though technical) of the work in their more technical papers, and includes an excellent bibliography. The remaining articles in this volume deal with fairness in taxes, auctions, and voting. ■

An Apportionment Problem *(Continued from page 8)*

We begin by giving each state one seat. To indicate this we have shaded the numbers in the top row. Now which state should get the next seat? If state B gets the 4th seat, there would be one representative for 85 people. This seems unfair since state A at this stage would then have only 1 seat for 740. Hence state A has the best claim to the next seat. We continue distributing seats in this manner until all 10 available seats are gone. Once A has 5 seats, it has one seat per 148 people, so the next seat, the 8th one would go to A. In the general case, the state having the next largest number in the table which is not yet shaded gets the next seat. In this situation, A gets 8 seats, B gets 1 seat, and C gets 1 seat (as witnessed by the shaded numbers in the table). A and C are over-represented and B is under-represented. Had there been an 11th seat to distribute, it would have gone to B rather than A, since in the table above 85 is bigger than 82.2. Other apportionment methods use similar priority tables but the divisors for each row are different from the 2, 3, 4, etc. used for the Jefferson method. ■

	A	B	C
P	740	170	90
P/2	370	85	45
P/3	246.6	56.7	30
P/4	185	42.5	22.5
P/5	148	34	18
P/6	123.3	28.3	15
P/7	105.7	24.3	12.9
P/8	92.5	21.2	11.3
P/9	82.2	18.9	10

Resources...

by Ethel Breuche

... a review of *Excursions in Modern Mathematics* by Peter Tannenbaum/Robert Arnold, Prentice Hall, 1992.

I used this text as a resource book for the last month when I had the opportunity to introduce topics in discrete math in my calculus class.

Based upon my only using Part 1 (The Mathematics of Social Choice), I feel that this text is wonderfully **rich**. It is rich with examples and simple and thorough explanations. It is rich in discussion and exploration. Most of all it is rich in exercises at the end of each chapter in which the problems are divided into *Walking, Jogging, and Running*. As the titles imply, the problems increase in level of difficulty and creative problem solving. Some chapters are followed by an additional appendix of information usually referred to in the chapter or one of the exercises. For example, in the voting theory section, the voting scheme for the nominations for the Academy Awards is described in detail. Every chapter offers references for further research and readings. The text is written with deliberate thoughtfulness with regard to racial and gender equity.

As the title states, the text offers a collection of "trips" into four main topics of discrete math which include: Part 1--*The Mathematics of Social Choice* (Election Theory, Power Measurement, Fair Division), Part 2--***Management Science*** (Euler Circuits, Traveling Salesman Problem, Hamiliton circuits, Minimum Network Problems, Spanning Trees, Scheduling Problems), Part 3--***Growth and Symmetry*** (Spiral Growth, Growth of Population, Symmetry of Motion, Symmetry of Scales and Fractals), and Part 4--***Statistics***.

The authors have succeeded in making the connection between mathematics and down-to-earth, concrete real-life problems. In general, the choice of topics is such that a heavy mathematical background is not needed. This material although straightforward is nevertheless not necessarily easy nor superficial, and much of the mathematics in the book has been discovered in this century.

Whether using the book as a resource or as a classroom text, I cannot praise it enough. Although written for a college-level liberal arts math course, high school college-intending juniors and seniors whose reading ability is average or above can use this text as well.

The instructor's manual is divided into *Notes and Comments, Solutions to Exercises* and a *Test Bank of Multiple Choice* questions for each chapter and final exams. The Notes and Comments offer everything from suggestions for class activities to suggestions for special projects. The test bank of multiple choice questions are relatively easy; these should not be used as the sole method of assessment in the classroom. Computer software will be available shortly. A supplement of recent New York Times articles that are appropriate for various topics in the text is also available. ■

Resources...

by Anthony Piccolino

... a review of *For All Practical Purposes* by Lynn A. Steen et. al., COMAP, W.H. Freeman Co., 1988.

If you are looking for a textbook which addresses real-life situations, emphasizes mathematical modeling, encourages students to make mathematical connections, and devotes extraordinary efforts to changing students' narrow view of mathematics, then FAPP, published by W.H. Freeman & Co., is the book of choice for you and your students.

For three years, I taught a mathematics elective course to high school seniors using this textbook. Many of the students enrolled in this course would normally have taken no mathematics course as seniors. Most of the students I had over the three-year period were fascinated by the topics in the text and the accompanying videos. One of my students remarked, "It doesn't look like a math textbook---the chapters deal with topics that actually interest me!"

Indeed they do! *For All Practical Purposes* covers a broad range of topics including graph theory, probability and statistics, voting schemes, fair division, apportionment, decision-making, game theory, fractals, and computer graphics in a style that makes for enjoyable reading for the student. Each strand in the textbook is accompanied by an overview video and each chapter is accompanied by a 28 minute video focusing on the major highlights of the chapter.

Each chapter in the text is written in a lucid and readable style complemented by eye-catching "spotlights" which focus on human-interest aspects of the topic being discussed. The exercise set is not just a collection of exercises which ask students to model what was covered in exposition, but also contains a collection of problems for which the student must go beyond the text material and utilize a variety of problem-solving strategies. Each chapter incudes a vocabulary review list and a list of suggested readings.

I found this book to be most effective when used in conjunction with the 26-program video series of the same name. The animation and the delivery style are wonderfully motivating and give students an excellent sense of the "big picture" for each chapter before having students delve into various detailed aspects of the chapter. In addition to the videos, the text supplements include an excellent instructor's guide and a telecourse guide to accompany the videos.

Although the text contains more material than can be covered in one semester, there is a wide variety of topics from which the instructor can choose. A good choice for a one-semester introduction to discrete math could include: Chapters 1&2 (graph theory), Chapter 3 (planning and scheduling), Chapter 4 (linear programming), Chapter 9 (social choice), Chapter 10 (weighted voting systems, measuring power), Chapter 11 (fair division and apportionments), Chapter 12 (game theory), and Chapter 17 (patterns).

I recommend this book with great enthusiasm! ■

Calling that mathematician... *(see cartoon on page 12)*

L. Charles Biehl and Joseph G. Rosenstein

The election is over. All 430 votes have all been tallied. But there is no winner, for the top three candidates all received the same number of votes. Enter preferential ballots.

In preferential balloting, each person votes for *all* candidates, indicating his/her order of preference of the candidates. Thus, with four candidates in the race, there are altogether $4! = 24$ possible ways a person can vote, reflecting the 24 possible orderings, or **preference schedules**, of Frumpf, Gluck, Ray, and Smiff. The results of the election can then be described by listing how many people voted for each of the 24 preference schedules. This is depicted in the cartoon on page 12, and, more fully, at the bottom of this page.

But how do you tell who won? Strange things can happen when you conduct preferential balloting without specifying the method of tabulation in advance -- and then you really need a mathematician!

There are a number of ways of tallying the ballots. In the **plurality method**, the winner is the candidate who received the largest number of first-place votes; preferences are in effect ignored, as is the case with standard elections where each person votes for just one candidate. In the **run-off method**, all but the two candidates A and B who received the most first-place votes are eliminated; the winner is the one of A and B who, accordingly to the preference schedules, would have received the most votes in a two-candidate race. In the **sequential run-off method**, the candidate with the least number of first place votes is eliminated, the preference schedules are retabulated and the process is repeated; when two candidates are left, the plurality method (or run-off method) is applied to determine the winner. In the **Borda count method** with four candidates, each candidate receives 3 points for each first-place vote, 2 points for each second-place vote, and 1 point for each third-place vote; the winner is the candidate who receives the greatest total count. In the **Condorcet method**, each candidate A is compared to each other candidate, and is assigned a number of points equal to the number of candidates that A would have defeated in a one-on-one election.

Your students can determine who would win the election by each of the methods above (answers are provided below) and then use preferential balloting for other situations, such as determining their favorite soft drink, rock star, or mathematician.

What is striking is that it is possible for these methods to lead to different winners. A situation where the methods give four different winners is featured in the videotapes accompanying *For All Practical Purposes*. Explanations and further information about preferential ballots can be found in both books reviewed on page 10. ■

F	G	R	S	F	G	R	S
G	F	F	G	G	F	G	F
R	R	G	R	S	S	F	G
S	S	S	F	R	R	S	R
14	22	16	31	11	9	27	17

Left Wall

F	G	R	S	F	G	R	S
R	R	F	F	R	R	G	G
G	S	S	R	S	F	F	F
S	F	G	G	G	S	S	R
6	11	37	38	36	22	7	3

Front Wall

F	G	R	S	F	G	R	S
S	S	S	R	S	S	S	R
G	F	F	G	R	R	G	F
R	R	G	F	G	F	F	G
10	25	7	19	33	21	6	2

Right Wall

Answers: Neither the **plurality method** nor the **run-off method** determines a winner, since three candidates are tied for first place. Using the **sequential run-off method**, Ray is eliminated first, and his name is deleted from each of the preference schedules -- for example, the 16 votes for preference schedule R-F-G-S now become 16 votes for F-G-S ; Smiff is eliminated next, and finally, we see that Frumpf defeats Gluck. Using the **Borda count method**, we find that Frumpf receives 694 points, Smiff 645 points, Ray 641 points, and Gluck 590 points, so that this method also gives the nod to Frumpf. Finally, using the **Condorcet method**, we find that F defeats G by 227-203, R by 224-206, and S by 243-188, so that Frumpf defeats all three other candidates in one-on-one elections and receives the maximum of 3 points. On the whole, a convincing win for Frumpf. ■

Spreading the Word... *(Continued from page 2)*

many teachers are integrating the materials into existing courses. It is a very exciting time to be a mathematics teacher with all the changes going on in structure, content, and pedagogy. Using discrete math is a way of being involved in all three.

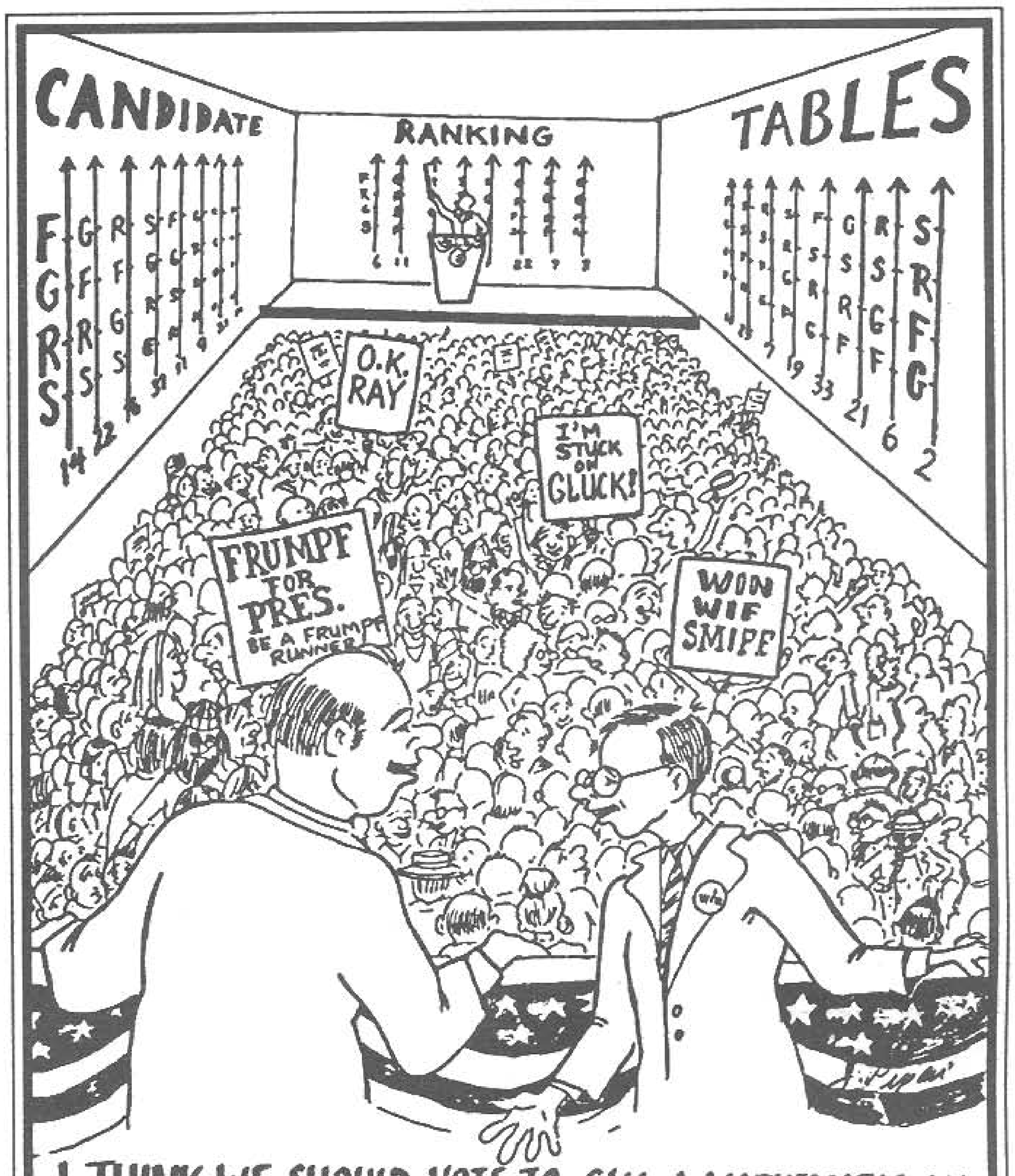
Your newsletter is wonderful! Knowing that there's a whole network of others around the country, I no longer feel like an isolated disciple of discrete math.

What Do Mathematicians Do?

If one asks the "person on the street" what plumbers, electricians, chemists, or geologists do, they are likely to give you a reasonable answer. Put in more dramatic terms, when home-owners see water cascading through the ceiling, they do not call a carpenter or a mathematician. But few people on the street know when to call a mathematician.

One thing that we can do about mathematics' image problem is to discuss how mathematics affects people's lives, even if we cannot always do proper justice to the mathematics involved. For example, we can say that mathematicians (not chemists or plumbers) study waiting lines, and show that this can be applied at banks, airports, and in computers. Or, we can say that mathematicians find shortest paths and networks, and show that this can be applied to travel arrangements and telephone connections.

The above paragraphs are adapted from "Mathematics' Image Problem" by Joseph Malkevitch (see address on page 11). The cartoon on the right was drawn by Joe Pipari one of whose colleagues at Thomas McKean High School (Wilmington, Delaware) participated in a discussion with Malkevitch at the 1990 Leadership Program in Discrete Mathematics. The situation presented in the cartoon is discussed on page 11. ■



I THINK WE SHOULD VOTE TO CALL A MATHEMATICIAN.

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