Coverable functions

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Coverable functions

- Let us recall that given a Boolean function $f$, we denote by:
  - $cnf(f)$ - minimum number of clauses needed to represent $f$ by a CNF.
  - $ess(f)$ - maximum number of pairwise disjoint essential sets of implicates of $f$.
- A function $f$ is **coverable**, if $cnf(f) = ess(f)$.
Talk outline

~ We already know from the previous talk, that not every function is coverable.

~ We shall show, that quadratic, acyclic, quasi-acyclic, and CQ Horn functions are coverable.

~ Before that we shall show, that in case of Horn functions we can restrict our attention to only pure Horn functions.
Negative implicates

$\sim$ Let $f$ be a Horn function.

$\sim$ Let $\mathcal{X}$ be an exclusive set of implicates of $f$, such that no two clauses in $\mathcal{E} = \mathcal{I}(f) \setminus \mathcal{R}(\mathcal{X})$ are resolvable.

$\sim$ Then there exists an integer $k$, and pairwise disjoint essential sets $\mathcal{Q}_1, \ldots, \mathcal{Q}_k \subseteq \mathcal{E}$, such that for every CNF $\mathcal{C}$ representing $f$:

$\sim$ $|\mathcal{C} \cap \mathcal{Q}_j| = 1, j = 1, \ldots, k$

$\sim$ $\mathcal{C}$ does not contain other elements of $\mathcal{E}$. 
Negative implicates

- We can use this proposition to negative implicates, if we put:

- $\mathcal{X} = $ pure Horn implicates of $f$, and

- $\mathcal{E} = $ negative implicates of $f$.

- Now we can observe that:

$$ess(f) = ess(\mathcal{X}) + k$$

- Therefore we can restrict our attention to only pure Horn functions.
For a Horn CNF $\varphi$ let $G_\varphi = (N, A_\varphi)$ be the digraph defined as:

- $N$ is the set of variables of $\varphi$.
- $(x, y)$ belongs to $A_\varphi$, if there is a clause $C$ in $\varphi$, which contains $\overline{x}$ and $y$.
- $G_f$, where $f$ is the function represented by $\varphi$, is transitive closure of $G_\varphi$. 

CNF Graph
Quadratic functions

- A quadratic function is a function, which can be represented by a CNF $\varphi$, in which every clause consists of at most two literals.

- Minimization algorithm for pure Horn quadratic functions:
  - Make $\varphi$ prime and irredundant.
  - Construct CNF graph $G_\varphi$.
  - Find strong components of $G_\varphi$.
  - Replace strong components by cycles.
Example

Let us consider the following CNF:

\[(\overline{a} \lor b) \land (\overline{b} \lor c) \land (\overline{c} \lor d) \land (\overline{d} \lor c) \land (\overline{c} \lor e) \land (\overline{e} \lor c)\]

CNF graph follows:

```
        d
       / \
  ----/    \----
     /      \   /
    /        \ d
   /          /
 a---b------c
    \
     \
    /   \
   /     \
  e------c
```
Example

- A shortest CNF:

$$(\overline{a} \lor b) \land (\overline{b} \lor c) \land (\overline{c} \lor d) \land (\overline{d} \lor e) \land (\overline{e} \lor c)$$

- and its CNF graph:
Disjoint essential sets for quadratic functions

Let us have a clause \((\overline{x} \lor y)\) and let us define essential set \(\mathcal{E}\) for this clause.

If \(x\) and \(y\) belong to different strong components of \(G_f\), we put \((\overline{u} \lor v)\) into \(\mathcal{E}\), if \(u\) belongs to the same strong component as \(x\) and \(v\) belongs to the same strong component as \(y\).
Disjoint essential sets ...

- If $x$ and $y$ belong to the same component of $G_f$, we put $(\overline{u} \lor y)$ into $\mathcal{E}$ for every $u$ in this component.

- It is easily possible to find vector based definition of these sets as well.

- If the input CNF is minimum, the sets are disjoint.
Example

For our shortest CNF

$$(\overline{a} \lor b) \land (\overline{b} \lor c) \land (\overline{c} \lor d) \land (\overline{d} \lor e) \land (\overline{e} \lor c)$$

we would have:

$$(\overline{a} \lor b) \rightarrow \{(\overline{a} \lor b)\}$$

$$(\overline{b} \lor c) \rightarrow \{(\overline{b} \lor c)\}$$

$$(\overline{c} \lor d) \rightarrow \{(\overline{c} \lor d), (\overline{e} \lor d)\}$$

$$(\overline{d} \lor e) \rightarrow \{(\overline{d} \lor e), (\overline{c} \lor e)\}$$

$$(\overline{e} \lor c) \rightarrow \{(\overline{e} \lor c), (\overline{d} \lor c)\}$$
Essentiality of defined sets I

Let us assume, that $x$ and $y$ belong to different strong components of $G_f$.

We have $u$ in the same SC as $x$, $v$ in the same SC as $y$, and $(\overline{u} \lor v) = R(\overline{u} \lor z, \overline{z} \lor v)$ for some $z$.

If $z$ does not belong to the same SC as $x$ or $y$, then $(\overline{x} \lor y)$ is redundant.

Therefore one of parent clauses belongs to $E$. 

\[
\begin{align*}
&\text{Essentiality of defined sets I} \\
&\text{At first let us assume, that } x \text{ and } y \text{ belong to different strong components of } G_f. \\
&\text{We have } u \text{ in the same SC as } x, v \text{ in the same SC as } y, \text{ and } (\overline{u} \lor v) = R(\overline{u} \lor z, \overline{z} \lor v) \text{ for some } z. \\
&\text{If } z \text{ does not belong to the same SC as } x \text{ or } y, \text{ then } (\overline{x} \lor y) \text{ is redundant.} \\
&\text{Therefore one of parent clauses belongs to } E.
\end{align*}
\]
Essentiality of defined sets I

At first let us assume, that \( x \) and \( y \) belong to different strong components of \( G_f \).

We have \( u \) in the same SC as \( x \), \( v \) in the same SC as \( y \), and \( (\overline{u} \lor v) = R(\overline{u} \lor z, \overline{z} \lor v) \) for some \( z \).

If \( z \) does not belong to the same SC as \( x \) or \( y \), then \( (\overline{x} \lor y) \) is redundant.

Therefore one of parent clauses belongs to \( \mathcal{E} \).
Essentiality of defined sets I

At first let us assume, that \(x\) and \(y\) belong to different strong components of \(G_f\).

We have \(u\) in the same SC as \(x\), \(v\) in the same SC as \(y\), and \((\overline{u} \lor v) = \mathcal{R}(\overline{u} \lor z, \overline{z} \lor v)\) for some \(z\).

If \(z\) does not belong to the same SC as \(x\) or \(y\), then \((\overline{x} \lor y)\) is redundant.

Therefore one of parent clauses belongs to \(E\).
Now let us assume, that $x$ and $y$ belong to the same strong component of $G_f$.

We have $u$ in this strong component and $z$, for which $(\overline{u} \lor y) = R(\overline{u} \lor z, \overline{z} \lor y)$.

It follows, that $z$ belong to the same strong component as well.
Acyclic functions

A function $f$ is \textit{acyclic}, if its CNF graph is acyclic.

Prime and irredundant CNF is the only minimum representation of an acyclic function.

Given the only prime and irredundant acyclic CNF $\varphi$, we define for each clause $C \in \varphi$ an essential set $\mathcal{E}_C = \{C\}$.

This set is essential due to similar reasons as in the case of quadratic functions.

Vector based definition is also possible.
Quasi-acyclic functions

- A function $f$ is **quasi-acyclic**, if every two variables $x$ and $y$, which belong to the same strong component of $G_f$, are logically equivalent.

- Definition of essential sets is a combination of cases of quadratic and acyclic function.
CQ functions

A Horn CNF $\varphi$ is CQ, if in every clause $C \in \varphi$ at most one subgoal belongs to the same strong component as its head.

A Horn function $f$ is CQ, if it can be represented by a CQ CNF.

$$(\bar{a} \lor \bar{b} \lor c) \land (\bar{c} \lor b)$$

is CQ

$$(\bar{a} \lor \bar{b} \lor c) \land (\bar{c} \lor b) \land (\bar{c} \lor a)$$

is CQ
CQ and essential sets

Any prime CNF representation of a CQ function is a CQ CNF.

In order to be able to define disjoint essential sets, we have to investigate structure of minimum CQ CNFs and minimization algorithm for CQ functions.
Decomposition lemma

Let us have:

\(~\) a function \( f \),
\(~\) a chain of exclusive subsets \( \emptyset = \mathcal{X}_0 \subseteq \mathcal{X}_1 \subseteq \cdots \subseteq \mathcal{X}_t \) in which \( \mathcal{R}(\mathcal{X}_t) = \mathcal{I}(f) \),
\(~\) minimal subsets \( \mathcal{C}_i^* \subseteq \mathcal{X}_i \setminus \mathcal{X}_{i-1}, i = 1, \ldots, t \), such that \( \mathcal{R}(\mathcal{X}_{i-1} \cup \mathcal{C}_i^*) = \mathcal{R}(\mathcal{X}_i) \).

Then:

\(~\) \( \mathcal{C}^* = \bigcup_{i=1}^{t} \mathcal{C}_i^* \) is a minimal representation of \( f \).

If we can find these sets effectively and solve corresponding subproblems effectively, we are done.
Let $\varphi$ be a pure Horn CNF representing a function $f$, we define clause graph $D_\varphi = (V_\varphi, E_\varphi)$ as follows:

- $V_\varphi = \varphi$
- $(A \lor u, B \lor v) \in E_\varphi$ if and only if:
  - $v$ can be reached from $u$ by a path in $G_\varphi$, and
  - for every $a \in A$, $(B \lor a)$ is an implicate of $f$. 
Properties of clause graphs

By $D_f = (V_f, E_f)$ we denote $D_\mathcal{I}(f)$.

By $\text{Cone}_H(u)$, where $H$ is a digraph and $u$ one of its vertices, we denote the set of vertices, from which there is a path to $u$ in $H$.

If $C = R(C_1, C_2)$, then $(C_1, C) \in E_f$ and $(C_2, C) \in E_f$.

Therefore $\text{Cone}_{D_f}(C)$ is an exclusive set.

If $K$ is a strong component of $D_f$ containing $C$, then $\text{Cone}_{D_f}(C) \setminus K$ is again an exclusive set.

Although the size of $D_f$ may be exponentially larger than $\varphi$, it is sufficient to work with $D_\varphi$, which can be constructed in polynomial time.
Back to decomposition lemma

\[ \begin{aligned} &\sim \text{ Let } K_1, \ldots, K_t \text{ be strong components of } D_f \text{ in topological order, and} \\
&\sim \text{ let us define } \mathcal{X}_i = \bigcup_{j=1}^{i} K_j, i = 1, \ldots, t. \\
&\sim \text{ Every } \mathcal{X}_i, i = 1, \ldots, t \text{ is an exclusive set and we can use it in decomposition lemma.} \\
&\sim \text{ Representation given by } \mathcal{X}_i \cap \varphi \text{ is sufficient for our needs.} \\
&\sim \text{ Now we only have to solve partial problem for each strong component } K_i \text{ of } D_f. \end{aligned} \]
Strong components

- We say, that an implicate \((A \lor u)\) of \(f\) is of
- type 0, if no element of \(A\) belong to the same strong component of \(G_f\) as \(u\), and it is of
- type 1, if one element of \(A\) belongs to the same strong component of \(G_f\) as \(u\).

- If \(K\) is a strong component of \(D_f\) and \(f\) is CQ, then all clauses belonging to \(K\) are of the same type.
- Therefore we can assign this type to \(K\) as well.
- If \(K\) is of type 0, we can leave the clauses in \(K \cap \varphi\) as they are, primality and irredundancy of \(\varphi\) is sufficient in this case.
Type 1 (example)

We shall demonstrate what we can do with strong components of type 1 on the following example:

\[ \varphi = (\overline{b} \lor c) \land (\overline{b} \lor e) \land (\overline{a} \lor \overline{c} \lor b) \land (\overline{a} \lor \overline{e} \lor b) \land (\overline{a} \lor \overline{d} \lor b) \land (\overline{a} \lor \overline{b} \lor d) \]

\[ G_{\varphi} \]
Type 1 (example)

$\sim$ $D_\varphi$ has two strong components:

$K_1 = \{ (\overline{b} \lor c), (\overline{b} \lor e) \}$

$K_2 = \{ (\overline{a} \lor \overline{c} \lor b), (\overline{a} \lor \overline{e} \lor b), (\overline{a} \lor \overline{d} \lor b), (\overline{a} \lor \overline{b} \lor d) \}$

$\sim$ $K_1$ is itself minimum (primality and irredundancy are sufficient for it).
We can find smaller representation of $K_2$ by finding a smaller representation of strong component of $G_\varphi$ containing $b$, $c$, $d$, and $e$, but blue arcs generated by clauses in $K_1$ cannot change.
Type 1 (example)

By this we get an equivalent minimum CNF:

\[ \varphi' = (\overline{b} \lor c) \land (\overline{b} \lor e) \land (\overline{a} \lor \overline{e} \lor d) \]
\[ \land (\overline{a} \lor \overline{d} \lor e) \land (\overline{a} \lor \overline{e} \lor b) \]

Smallest representation of a strong component with some fixed arcs can be found in polynomial time.
Essential sets

Based on the minimization algorithm, we can define the essential sets.

We have to distinguish, whether clause $C_i$ belongs to the strong component $K(C_i)$ of type 0, or 1.

We give only illustrative pictures of definitions of vectors defining the essential sets to give impression of their complexity.
Type 0

\[ A_i \subseteq FC_{\varphi_i \setminus K(C_i)}(A_i \cup X_i) \]

\[ Cone_G(u_i) \cap Q(u_i) \cap Q_i(u_i) \subseteq FC_{\varphi_i}(A_i \cup X_i) \]

\[ 1 \quad 0 \]
Type 1

$Cone_G(u_i)$

$A_i$

$FC_{\varphi_i \backslash K_{C_i}}(A_i)$

$Q_i(u_i)$

$X_i$

$u_i$

$FC_{\varphi_i}(A_i \cup X_i)$

$Q(u_i)$

1 1/0 0
Conclusions

- There are other classes, about which we can show, that they are coverable. (E.g. interval functions)
- Horn coverable functions form a nontrivial subclass of Horn functions.
- We still do not know, if
  - we can recognize, whether given Horn CNF represent a coverable function,
  - and what is the complexity of minimization of Horn coverable functions.