On Networks of Two-Way Channels

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\begin{itemize}
  \item Outline
    \begin{itemize}
      \item 1. Network model
      \item 2. Cut set bounds
      \item 3. Implications for network coding
      \item 4. Disconnecting set bounds
    \end{itemize}
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1. Network Model

- **Network**
  - graph $G=(V,E)$
  - vertices $V$: “terminals”
  - edges $E$: “channels”

- **Channels:**
  - directed/undirected
  - capacity restrictions

- **Demand (sources and destinations)**
  - multi-commodity flow
  - multi-casting
Communication Networks

- Edges: cables, wireless channels, etc.
  - two-way channels (TWCs)
    edge bc: $P(y_b, y_c|x_b, x_c)$
- Capacity Region
  - the set of rate pairs $(R_1, R_2)$ achievable with coding
  - convex if time-sharing permitted
  - consider $\varepsilon$–error capacity region
- Network capacity:
  what can the vertices can do?
Networks of TWCs

- Model:
  - messages $W_1, ..., W_M$ available at $s_1, ..., s_L$, where $L \leq M$
  - network is clocked, i.e., a universal clock ticks $N$ times
  - vertex $v$ can transmit one symbol into its TWCs after clock tick $n$ and before clock tick $n+1$ for all $n = 1, 2, ..., N$
  - symbols are received at clock tick $n+1$ for all $n$
  - flow or routing: vertices can collect, store and forward symbols (including local message symbols)
  - here: network coding is allowed, i.e., for all clock ticks $n$, vertex $v$ transmits (let $W_{M(v)}$ be the set of messages at $v$)

$$X_v[n] = f_n(W_{M(v)}, Y_v[1,2,...,n-1])$$
Network Coding Gains

- A standard example (Ahlswede, Cai, Li, Yeung, 2000):
  - a two-flow problem with directed, unit capacity edges
  - max flow: 1
  - max coded sum rate: 2
    - can even decode both messages at both nodes
  - avg. resources used:
    - flow: 3 edges/clock tick
    - coding: 7 edges/clock tick
2. Cut Set Bounds

- Cut set $E'$: edges that disconnect each of a set of sources from (one of) its sinks, and that divide $V$ into $(X,X')$
- $R_X$: sum of rates of flows starting in $X$ with a sink in $X'$
- $C_{X \rightarrow X'}$: sum of capacities of edges in $E'$ going from $X$ to $X'$
- $C_X$: sum of capacities of edges in $E'$
- Bounds: $R_X \leq C_{X \rightarrow X'}$
  $R_X + R_{X'} \leq C_X$
Information Theory (IT) Cut Set Bound

- Cut set: same as above
- Need bound to apply to network coding
- Optimization of a standard IT cut set bound:
  1) convert every edge (TWC) into a pair of directed edges (one-way channels) whose rate pair is a boundary point of the capacity region of this edge
  2) apply the flow cut set bound
  3) repeat 1) and 2) for all boundary points on all edges
- IT cut set bound implies the above flow cut set bound
Example 1: undirected edges

- unit capacity, undirected edges, multi-casting with two sinks
- flow cut set bound: $R \leq 2$
- IT cut set bound: $0 \leq R_{ij}, R_{ij} + R_{ji} \leq 1$
  \[ R \leq R_{ab} + R_{ac}, R_{ab} + R_{cb}, R_{ac} + R_{bc} \]
  The last two bounds give
  \[ 2R \leq R_{ab} + R_{ac} + 1 \leq 3 \]
- IT bound is stronger and tight
- rings with 1 source and $K$ separate sinks: $R = (K+1)/K$ is best
Example 2: symmetric TWCs

– suppose capacity regions are the set of \((R_1, R_2)\) satisfying

\[0 \leq R_1^2 + R_2^2 \leq 1\]

– flow cut set bound: \(R \leq 2\)

– IT cut set bound: \(R_{ij}^2 + R_{ji}^2 = 1\)

\[R \leq R_{ab} + R_{ac}, \quad R_{ab} + R_{cb}, \quad R_{ac} + R_{bc}\]

The last two bounds give

\[2R \leq R_{ab} + R_{ac} + (R_{cb} + R_{bc}) \leq 2 + 2^{1/2}\]

– IT bound is again stronger and tight
Example 3: bidirected edges

- suppose capacity region is the set of $(R_1, R_2)$ satisfying 
  \[ 0 \leq R_1 \leq 1, \ 0 \leq R_2 \leq 1 \]
- flow cut set bound: $R \leq 2$
- IT cut set bound: $R_{ij} = 1, R_{ji} = 1$
  \[ R \leq 2 \]
- Flow and IT cut set bounds are the same for networks with directed edges
- multi-casting capacity is known for directed graphs (Koetter, Médard 2003)
3. Implications for Network Coding

- If max-flow=flow-min-cut, routing is optimal
  - single commodity flow (Ford-Fulkerson, 1956)
  - two commodities in an undirected graph (Hu, 1963)
  - not true more generally (see standard example)
  - undirected planar graphs, multi-commodity flow, sources and sinks on boundary of infinite region (Okamura, Seymour, 1981)

- Flow/routing questions:
  - when is max-flow=IT-min-cut for undirected networks?
  - when is max-flow=IT-min-cut for mixed networks?
  - do there exist, e.g., disconnecting set bounds for coding?
4. A Disconnecting Set Bound

- Example: directed triangle
  - unit capacity edges
  - two commodities
  - max-flow is 1
- Disconnecting set: edge bc
  - IT cut set bound permits sum rate of 2!
  - Is this rate achievable with coding?
An improved IT bound

- We have the IT inequalities:
  \[ N(R_1+R_2) \leq I(W_1;X_{bc}) + I(W_2;X_{ca|W_1}) = I(W_1;X_{bc}) + I(W_2;X_{bc|W_1}) \leq I(W_1W_2;X_{bc}) \leq H(X_{bc}) \leq N \]

- A simple disconnecting set bound.
  Can one generalize it?
  Yes, but in a limited way.
Summary and Some Open Problems

- **Summary**
  - model: network of TWCs
  - IT cut set bound needed for network coding

- **Open Problems**
  - what can flow/routing achieve for TWC edges?
  - when is max flow=flow-min-cut for TWC edges?
  - when is max flow=IT-min-cut (even for basic TWCs)?
  - what kinds of network codes are needed for general TWC capacity regions? Linear/nonlinear?
  - does a symmetric TWC capacity region simplify things?