Preferences On Intervals: a general framework

(Extended Abstract)

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1 Introduction

Preferences are usually considered as binary relations applied on a set of objects, let's say A. Preference modelling is concerned by two basic problems (see [31]).

The first can be summarised as follows. Consider a decision maker replying to a set of preference queries concerning a the elements of the set A: "do you prefer a to b?", "do you prefer b to c?" etc.. Given such replies the problem is to check whether exists (and under which conditions) one or more real valued functions which, when applied to A, will return (faithfully) the preference statements of the decision maker. As an example consider a decision maker claiming that, given three candidates a, b and c, he is indifferent between a and b as well as between b and c, although he clearly prefers a to c. There are several different numerical representations which could account for such preferences. For instance we could associate to a the interval $[5, 10]$, to b the interval $[3, 6]$ and to c the interval $[1, 4]$. Under the rule that x is preferred to y iff the interval associated to x is completely to the right (in the sense of the reals) of the one associated to y and indifferent otherwise, the above numerical representation faithfully represents the decision makers preference statements.

The second problem goes the opposite way. We have a numerical representation for all elements of the set A and we would like to construct preference relations for a given decision maker. As an example consider three objects a, b and c whose cost is 10, 12 and 20 respectively. For a certain decision maker we could establish that a is better than b which is better than c. For another decision maker the model could be that both a and b are better than c, but they are indifferent among them since the difference is too small. In both cases the adoption of a preference model implies the acceptation of a number of properties the decision maker should be aware of.

In this paper we focus our attention on both cases, but with particular attention to the situations where the elements of the set A can or are actually represented by intervals (of the reals). In other terms we are interested on the one hand to the necessary and sufficient conditions for which the preference statements of a decision maker can be represented through the comparison of intervals and on the other hand on general models through which the comparison of intervals can lead to the establishment of preference relations.

The paper's subject is not that new. Since the seminal work of Luce ([16]) there have been several contributions in literature including the classics [10], [26] and [22], as well as some key papers: [1],[6], [9], [11], [12]. Our main contribution in this paper is to suggest a general framework enabling to clarify the different preference models that can be associated to the comparison of intervals including situations of crisp or continuous hesitation of the decision maker.

The paper is organised as follows. In Section 2 we introduce all basic notation and all hypotheses that hold in the paper. In section 3 we introduce the structure of the general framework we suggest, based on two dimensions: the type of preference structure to be used and the structure of the intervals. In section 4 we introduce some further conditions enabling to characterise well known preference structures in the literature. We conclude showing the future research directions of this work.

2 Notation and Hypotheses

In the following we consider a countable set of objects which we denote with A. Variables ranging within A will be denoted with $x, y, z, w \cdots$, while specific objects will be denoted $a, b, c \cdots$. Letters $P, Q, I, R, L \cdots$, possibly subscribed, will denote preference relations on A, that is binary predicates on the universe of discourse $A \times A$ (each binary relation being a subset of $A \times A$). Letters $f, g, h, r, l \cdots$, possibly subscribed, will denote real valued functions mapping A to the reals. Since we work with intervals we will reserve the letters r and l for the functions representing, respectively, the right and

left extreme of each interval. Letters $\alpha, \beta, \gamma \cdots$ will represent constants. The usual logical notation applies including its equivalent set notation. Therefore we will have:
- $P \cup R$ equivalent to $\forall x, y \;\; P(x,y) \wedge R(x,y)$;
- $P \subseteq R$ equivalent to $\forall x, y \;\; P(x,y) \rightarrow R(x,y)$.
We will add the following definitions:
- $P.R$ equivalent to $\forall x, y \exists z \;\; P(x,z) \wedge R(z,y)$;
- $I_o = \{(x,x) \in A \times A\}$, the set of all identities in $A \times A$.

As far as the properties of binary relations are concerned we will adopt the ones introduced in [23]. For specific types of preference structures such as total orders, weak orders etc. we will equally adopt the definitions within [23].

We introduce the following definition:

Definition 2.1 *A preference structure is a collection of binary relations P_j $j = 1, \cdots n$, partitioning the universe of discourse $A \times A$:*
- $\forall x, y, j \; P_j(x,y) \rightarrow \neg P_{i \neq j}(x,y)$;
- $\forall x, y \exists j \; P_j(x,y) \vee P_j(y,x)$

Further on we will often use the following proposition:

Proposition 2.1 *Any symmetric binary relation can be seen as the union of two asymmetric relations, the one being the inverse of the other, and I_o.*

Proof. Obvious.

We finally make the following hypotheses:

- H1 We consider only intervals of the reals. Therefore there will be no incomparability in the preference structures considered.
- H2 If necessary we associate to each interval a flat uncertainty distribution. Each point in an interval may equally be the "real value".
- H3 Without loss of generality we can consider only asymmetric relations.
- H4 We consider only discrete sets. Therefore we can consider only strict inequalities.

Remark 2.1 *Hypothesis 3 is based on proposition 2.1. The reason for eliminating symmetric relations from our models will become clear later on in the paper. However, we can anticipate that the use of asymmetric relations allows to better understand the underlying structure of intervals comparison.*

Remark 2.2 *Hypothesis 4 makes sense only when the purpose is to establish a representation theorem for a certain type of preference statements. The basis idea is that, since numerical representations of preferences are not unique, A being countable, is always possible to choose a numerical representation for which it never occurs that any of the extreme values of the intervals associated to two elements of A are the same. However, in the case the numerical representation is given and the issue is to establish the preference structure holding, the possibility that two extreme values coincide cannot be excluded.*

3 General Framework

In order to analyse the different models used in the literature in order to compare intervals for preference modelling purposes we are going to consider two separate dimensions.

1. *The type of preference structure.* We basically consider the following cases.

   - Use of two asymmetric preference relations P_1 and P_2. Such a preference structure is equivalent to the classic preference structure (in absence of incomparability) considering only strict preference (P_2 in our notation) and indifference ($P_1 \cup P_1^{-1} \cup I_o$ in our notation). For more details see [23].
   - Use of three asymmetric preference relations P_1, P_2 and P_3. Such structures are known under the name of PQI preference structures (see [30]), allowing for a strict preference (P_3 in our notation), a "weak preference" (P_2 in our notation), representing an hesitation between strict preference and indifference and an indifference ($P_1 \cup P_1^{-1} \cup I_o$ in our notation).
   - Use of n asymmetric relations $P_1, \cdots P_n$. Usually P_n to P_2 represent $n-1$ preference relations of decreasing "strength", while $P_1 \cup P_1^{-1} \cup I_o$ is sometimes considered as indifference. For more details the reader can see [9].

- Use of a continuous valuation of hesitation between strict preference and indifference. In this case we consider valued preference structures, that is preference relations are considered fuzzy subsets of $A \times A$. The reader cas see more in [20].

2. *The structure of the numerical representation of the interval.* We consider the following cases:

   - Use of two values. Such two values can be equivalently seen as the left and the right extreme of each interval associated to each element of A or as a value associated to each element of A and a threshold allowing to discriminate any two values.
   - Use of three values. Again several different interpretations can be considered. For instance the three values can be seen as the two extremes of each interval plus an intermediate value aiming to represent a particular feature of the interval. They can be seen as a value associated to each element of A and two thresholds aiming to describe two different states of discrimination. They can also be seen as representing an extreme value of the interval, while the other extreme is represented by an interval.
   - Use of four or more values. The reader will realise that we are extending the previous structures. The four values can be seen as the two extremes and two "special" intermediate values or as two imprecise extremes such that each of them is represented by an interval. The use of n values can be seen as a value associated to each element of A and $n-1$ thresholds representing different intensities of preference. Possibly we can extend such a structure to the whole length of any interval associated to each element of A such that we may obtain a continuous valuation of the preference intensity.

In table 1 we summarise the possible combinations of preference structures and interval structures.

The reader can see more details in the following references:
- Interval Orders and Semi Orders: [16], [10], [22], [11];
- Split Interval Orders and Semi Orders: [13], [2];
- Tolerance and Bi-tolerance orders: [14], [15], [4], [3], [5];
- *PQI* Interval Orders and Semi Orders: [28], [17], [18];

<table>
<tr><th></th><th>2 values</th><th>3 values</th><th>> 3 values</th></tr>
<tr><td>2 asymmetric relations</td><td>Interval Orders and Semi Orders</td><td>Split Interval Orders and Semi Orders</td><td>Tolerance and Bi-tolerance orders</td></tr>
<tr><td>3 asymmetric relations</td><td>PQI Interval Orders and Semi Orders</td><td>Pseudo orders and double threshold orders</td><td>-</td></tr>
<tr><td>n asymmetric relations</td><td>-</td><td>-</td><td>Multiple Interval Orders and Semi Orders</td></tr>
<tr><td>valued relations</td><td colspan="3">Valued Preferences
Fuzzy Interval Orders and Semi Orders
Continuous PQI Interval Orders</td></tr>
</table>

Table 1: A general framework for interval comparison

- Pseudo Orders and Double Threshold Orders: [24], [25], [30], [27];
- Multiple Interval Orders and Semi Orders: [6], [9], [8];
- Valued Preference Structures: [21], [20], [7], [29], [19].

4 Further Conditions

The general framework discussed in the previous section suggests that there exist several different ways to compare intervals in order to model preferences. Each of such preference models could correspond to different interpretations associated to the values representing each interval. A first general question is the following:

- given a set A, if it is possible to associate to each element x of A n functions $f_i(x),\ i = 1, \cdots n$, such that $f_n(x) > \cdots > f_1(x)$, how many preference relations can be established?

In order to reply to this question we consider different conditions which may apply to the values of each interval and their differences. For notation purposes, given an interval to which n values are associated, we denote the i-th sub-interval of any element $x \in A$ (from value $f_i(x)$ to value $f_{i+1}(x)$) as x_i. When there is no risk of confusion x_i will also represent the "length" of the same sub-interval (the quantity $f_{i+1}(x) - f_i(x)$). We are now ready to consider the following cases:

	free	coherent	weak monotone	monotone
2 values:	3	2	2	2
3 values:	10	5	4	3
4 values:	35	14	8	4
n values:	$\frac{(2n)!}{2(n!)^2}$	$\frac{1}{n+1}\binom{2n}{n}$	$?(2^{n-1})?$	n

Table 2: Number of possible relations comparing intervals

1. No conditions. We consider that the functions describing the intervals are free to take any value.

2. Coherence conditions. We impose that $\forall i \; f_1(x) > f_1(y) \rightarrow f_i(x) > f_i(y)$. This is equivalent to claim that $\forall i \; x_i > y_i$.

3. Weak monotonicity conditions. We now impose that $\forall i, j, i \geq j \; x_i \geq y_j$. In other terms we demand that there are no sub-intervals of x included to any sub-interval of y. Such a condition implies coherence (but not vice-versa).

4. Monotonicity conditions. We now impose that $\forall i \; x_i \geq y_i \geq x_{i-1} \geq y_{i-1}$ (sub-intervals of x or y are never included and they increase as the index i increases). Such a conditions implies weak monotonicity (but not viceversa). The reader can easily check that a representation which satisfies such a condition is the one where all sub-intervals have the same constant length.

In table 2 we summarise the situation for all the above cases.

A second question concerns the existence of a general structure among the possible relations that the comparison of intervals allow. Consider for instance the ten possible relations allowed by the use of three values associated to each interval. Is there any relation among them?

In order to reply to this question we consider any preference relation as a vector of $2n$ elements. Indeed, since $P_j(x, y)$ compares two vectors (x and y) of n elements each ($\langle f_1(x), \cdots f_n(x)\rangle$ and $\langle f_1(y), \cdots f_n(y)\rangle$), there is a unique sequence of such $2n$ values which exactly describes each relation P_j. Consider the case of three values and the ten possible relations. These can be described as follows:

$P_1(x,y) : \langle f_1(y), f_1(x), f_2(x), f_3(x), f_2(y), f_3(y)\rangle$
$P_2(x,y) : \langle f_1(y), f_1(x), f_2(x), f_2(y), f_3(x), f_3(y)\rangle$
$P_3(x,y) : \langle f_1(y), f_1(x), f_2(y), f_2(x), f_3(x), f_3(y)\rangle$
$P_4(x,y) : \langle f_1(y), f_1(x), f_2(x), f_2(y), f_3(y), f_3(x)\rangle$
$P_5(x,y) : \langle f_1(y), f_1(x), f_2(y), f_2(x), f_3(y), f_3(x)\rangle$
$P_6(x,y) : \langle f_1(y), f_2(y), f_1(x), f_2(x), f_3(x), f_3(y)\rangle$
$P_7(x,y) : \langle f_1(y), f_1(x), f_2(y), f_3(y), f_2(x), f_3(x)\rangle$
$P_8(x,y) : \langle f_1(y), f_2(y), f_1(x), f_2(x), f_3(y), f_3(x)\rangle$
$P_9(x,y) : \langle f_1(y), f_2(y), f_1(x), f_3(y), f_2(x), f_3(x)\rangle$
$P_{10}(x,y) : \langle f_1(y), f_2(y), f_3(y), f_1(x), f_2(x), f_3(x)\rangle$

We now introduce the following definition.

Definition 4.1 *For any two relations $P_l, P_k, l, k \in I$ we write $P_l \rhd P_k$ and we read "relation P_l is stronger than relation P_k" iff relation P_k can be obtained from P_l by a single shift of values of x and y or it exists a sequence of P_i such that $P_l \rhd \cdots P_i \rhd \cdots P_k$.*

The reader will easy verify the following proposition.

Proposition 4.1 *Relation $\rhd$ is a partial order defining a complete lattice on the set of possible preference relations.*

In figure 1 we show the lattice for the cases where $n = 2$ (3 relations) and $n = 3$ (10 relations).

How do well known in the literature preference structures fit the above presentation? The reader can easily check the following equivalences.

Interval orders:
$P = P_3$, $I = P_1 \cup P_2 \cup I_o \cup P_1^{-1} \cup P_2^{-1}$
Partial Orders of dimension. 2:
$P = P_3 \cup P_2$, $I = P_1 \cup I_o \cup P_1^{-1}$
Semi Orders:
$P = P_3$, $I = P_2 \cup I_o \cup P_2^{-1}$, P_1 empty
PQI Interval orders:
$P = P_3$, $Q = P_2$, $I = P_1 \cup I_o \cup P_1^{-1}$
PQI Semi orders:
$P = P_3$, $Q = P_2$, $I = I_o$, P_1 empty
Split Interval orders:

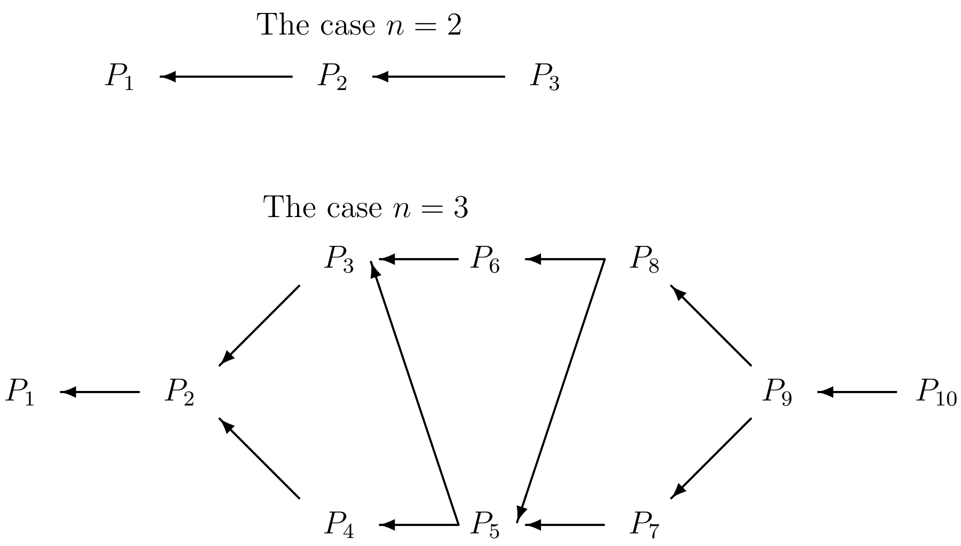

Figure 1: Partial Order among Preference Relations

$P = P_{10} \cup P_9$, I the rest
Double Threshold orders:
$P = P_{10}$ $Q = P_9 \cup P_8 \cup P_6$, I the rest
Pseudo Orders:
$P = P_{10}$ $Q = P_9 \cup P_8$, $I = P_5 \cup P_7 \cup I_o \cup P_7^{-1} \cup P_5^{-1}$,
P_1, P_2, P_3, P_4, P_6 empty
Constant thresholds:
$P = P_{10}$ $Q = P_9$, $I = P_5 \cup I_o \cup P_5^{-1}$,
$P_1, P_2, P_3, P_4, P_6, P_7, P_8$ empty

5 Conclusions

In this paper we introduce a general framework for the comparison of intervals under preference modelling purposes. Two possible extensions of such a framework can be envisaged. The first concerns the comparison of intervals for other purposes such as comparing time intervals. The second concerns the possibility to derive a general structure for representation theorems concerning any preference structure which can be conceived within the above framework.

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