Universal Linkage and the Uniqueness of EDM Completions

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Introduction

Every configuration $p = (p^1, \ldots, p^n)$ in $\mathbb{R}^n$ defines EDM $D = (d_{ij} = ||p^i - p^j||^2)$. For example,
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for all $0 \leq x \leq 4$. 

EDM Completions

- Given a symmetric partial matrix $A$ and a graph $G$. Let $a_{ij} : \{i, j\} \in E(G)$ be specified, or fixed, and $a_{ij} : \{i, j\} \notin E(G)$ be unspecified, or free.

$D$ is an EDM completion of $A$ if $D$ is an EDM and $d_{ij} = a_{ij}$ for all $\{i, j\} \in E(G)$.

A free entry $d_{ij}$ is universally linked if $d_{ij}$ is constant in all EDM completions of $A$.

If all free entries $d_{ij}$ are universally linked, then $D$ is the unique completion of $A$.

The set $\{d_{ij} : \{i, j\} \notin E(G)\}$ for all EDM completions $D$ is called Cayley configuration space (CCS) of $A$.

CCS is a spectrahedron, i.e., intersection of psd cone with an affine space.
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- CCS is a spectrahedron, i.e., intersection of psd cone with an affine space.
Example

Consider $D = \begin{bmatrix} 0 & 1 & 4 & 2 & 2 \\ 1 & 0 & 1 & 1 & 1 \\ 4 & 1 & 0 & 2 & 2 \\ 2 & 1 & 2 & 0 & 4 \\ 2 & 1 & 2 & 4 & 0 \end{bmatrix}$. Let the free elements of $D$ be \{1, 4\}, \{3, 5\} and \{4, 5\}. 

The CCS of $D$ is $d_{14} = 2$, $d_{35} = 2$ and $0 \leq d_{45} \leq 4$. Thus $d_{14}$ and $d_{35}$ are universally linked, while $d_{45}$ is not universally linked.

The embedding dimension of EDM $D$ is the dimension of the affine span of its generating points. The embedding dimension of $D$ for $d_{45} = 0$ or 4 is 2, while it is 3 for $0 < d_{45} < 4$. 
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Bar-and-Joint Frameworks

\[ D = \begin{bmatrix}
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\end{bmatrix} \]

Think of the edges of \( G \) as rigid bars, and of the nodes of \( G \) as joints. Thus we have a bar-and-joint framework \((G, p)\).

Note that this \((G, p)\) folds across the \( \{1, 3\} \) edge.

The CCS of \( D \) is \( y_{14} = 0 \), \( y_{35} = 0 \) and \(-4 \leq y_{45} \leq 0\).

\{k, l\} is universally linked iff its CCS is contained in the hyperplane \( y_{kl} = 0 \) in \( \mathbb{R}^\bar{m} \), \( \bar{m} \) = num. of missing edges of \( G \).
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Universal Rigidity, Dimensional rigidity and Affine Motions

- Given framework $(G, p)$, let $H$ be the adjacency matrix of $G$. 
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Universal Rigidity, Dimensional rigidity and Affine Motions

- Given framework \((G, p)\), let \(H\) be the adjacency matrix of \(G\).
- \((G, p)\) is universally rigid if \(H \circ D_p = H \circ D_q\) implies that \(D_p = D_q\). \((\circ)\) denotes Hadamard product.
- \((G, p)\) is dimensionally rigid if \(\not\exists (G, q): H \circ D_p = H \circ D_q\) and embedd \((D_q) \succ embedd (D_p)\).
- \((G, p)\) has an affine motion if \(\exists (G, q):\)
  (i) \(H \circ D_p = H \circ D_q\),
  (ii) \(D_p \neq D_q\) and
  (iii) \(q^i = Ap^i + b\) for \(i = 1, \ldots, n\).
Geometric Characterizations

- Thus \((G, p)\) is universally rigid iff its CCS = \{0\}. 

Theorem [A 2005] \((G, p)\) is universally rigid iff it is both dimensionally rigid and has no affine motions.
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- Thus \((G, p)\) has no affine motion iff affine hull of minimal face(0) = \{0\}.
- Theorem [A 2005] \((G, p)\) is universally rigid iff it is both dimensionally rigid and has no affine motions.
Example

\[ D = \begin{bmatrix}
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Obviously \((G, p)\) is not dimensionally rigid. It has an affine motion, and neither \{1, 3\} nor \{2, 4\} is universally linked.
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A stress of framework \((G, p)\) is \(\omega : E(G) \rightarrow \mathbb{R}\) such that

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\sum_j \omega_{ij}(p^i - p^j) = 0.
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Stress Matrix $\Omega$

- A stress of framework $(G, p)$ is $\omega : E(G) \rightarrow \mathbb{R}$ such that
  \[ \sum_j \omega_{ij}(p^i - p^j) = 0. \]

- A stress matrix $\Omega$ of framework $(G, p)$ is:
  \[ \Omega_{ij} = \begin{cases} 
  -\omega_{ij} & \text{if } \{i, j\} \in E(G) \\
  0 & \text{if } \{i, j\} \in E(G) \\
  \sum_k: \{i, k\} \in E(G) \omega_{ik} & \text{if } i = j 
  \end{cases} \]

- If $(G, p)$ is $r$-dimensional, then rank $\Omega \leq n - 1 - r$.

- $\Omega$ is optimal dual variable in a certain Semidefinite programming problem.
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- If $(G, p)$ is $r$-dimensional, then $\text{rank } \Omega \leq n - 1 - r$.
- $\Omega$ is optimal dual variable in a certain Semidefinite programming problem.
Theorem[A. ’05, Connelly ’82]: Let $\Omega$ be a stress matrix of $r$-dimensional framework $(G, p)$, $r \leq n - 2$. If $\Omega$ is psd and of rank $n - r - 1$, then $(G, p)$ is dimensionally rigid.
Theorem[A. ’05, Connelly ’82]: Let Ω be a stress matrix of $r$-dimensional framework $(G, p)$, $r \leq n - 2$. If Ω is psd and of rank $n - r - 1$, then $(G, p)$ is dimensionally rigid.

Theorem[A and Yinyu Ye ’13]: Let Ω be a stress matrix of $r$-dimensional framework $(G, p)$. $r \leq n - 2$. If rank $\Omega = n - r - 1$ and if $p$ is in general position, then $(G, p)$ has no affine motion.
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Theorem[A and Nguyen ’13]: Let $\Omega$ be a stress matrix of $r$-dimensional framework $(G, p)$. $r \leq n - 2$. If rank $\Omega = n - r - 1$ and if for each vertex $i$, the set $\{p^i\} \cup \{p^j : \{i, j\} \in E(G)\}$ is in general position, then $(G, p)$ has no affine motion.
Main Results

- Let $E_{ij}$: 1 in $ij$th and $ji$th entries and 0s elsewhere.
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- Let $E^{ij}$: 1 in $ij$th and $ji$th entries and 0s elsewhere.
- Let $\Omega$ be non-zero psd stress matrix of $r$-dimensional $(G, p)$, $r \leq n - 2$. 
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- Let $\Omega$ be non-zero psd stress matrix of $r$-dimensional $(G, p)$, $r \leq n - 2$.
- Theorem [A. ’16] If $\not\exists y_{kl} \neq 0$:

$$\Omega\left(\sum_{\{i,j\}\in E(G)} y_{ij}E^{ij}\right) = 0,$$

then $\{k, l\}$ is universally linked.
Main Results

- Let $E^{ij}$: 1 in $ij$th and $ji$th entries and 0s elsewhere.
- Let $\Omega$ be non-zero psd stress matrix of $r$-dimensional $(G, p)$, $r \leq n - 2$.
- Theorem [A. '16] If $\exists y_{kl} \neq 0$:
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  then $\{k, l\}$ is universally linked.
- Theorem [A. '16] If $\exists y=(y_{ij}) \neq 0$:
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  \Omega\left( \sum_{\{i,j\} \in E(G)} y_{ij} E^{ij} \right) = 0,
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  then $(G, p)$ is universally rigid.
e is the vector of all 1s.

Theorem [Schoenberg '35, Young and Householder '38]: Let $D$ be a real symmetric matrix with zero diagonal. Then $D$ is EDM iff

$$\mathcal{T}(D) = -\frac{1}{2}(I - \frac{ee^T}{n})D(I - \frac{ee^T}{n}) \succeq 0.$$ 

Moreover, the embedding dimension of $D$ is equal to rank $\mathcal{T}(D)$. 
Characterizing EDMs

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  Moreover, the embedding dimension of $D$ is equal to rank $\mathcal{T}(D)$.

- $B = \mathcal{T}(D)$ is the **Gram matrix** of the generating points of $D$.
- $B$ is not invariant under translations. Thus impose $Be = 0$. 

Characterizing CCS

- Let $V$ be $n \times (n - 1)$ matrix: $V^T e = 0$ and $V^T V = I$. 

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- Let \( X = V^T B V = -VDV^T / 2 \) or \( B = VXV^T \). Thus \( X \) is called the projected Gram matrix of \( D \).
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- Thus there is a one-to-one correspondence between $n \times n$ EDMs $D$ and psd matrices of order $n - 1$. 
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- Thus there is a one-to-one correspondence between $n \times n$ EDMs $D$ and psd matrices of order $n - 1$.
- The CCS of $(G, p)$ is given by

$$\{y = (y_{ij}) : X + \sum_{ij : \{i,j\} \notin E(G)} y_{ij} M^{ij} \succeq 0\},$$

where $X$ is the projected Gram matrix of $(G, p)$ and $M^{ij}$s are universal matrices.
Facial Structure of CCS

Let $\mathcal{X}(y) = X + \sum_{ij: \{i,j\} \not\in E(G)} y_{ij} M_{ij}$. Thus CCS is given by

$$\mathcal{F} = \{ y : \mathcal{X}(y) \succeq 0 \}.$$
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- Let $\mathcal{X}(y) = X + \sum_{ij: \{i,j\} \notin E(G)} y_{ij} M_{ij}$. Thus CCS is given by

$$\mathcal{F} = \{ y : \mathcal{X}(y) \succeq 0 \}.$$

- **Theorem:** Let $U$ be the matrix whose columns form an orthonormal basis of $\text{null}(\mathcal{X}(y))$. Let $\Omega$ be a non-zero psd stress matrix of $(G, \rho)$. Then

$$\minface(y) = \{ x \in \mathcal{F} : \text{null}(\mathcal{X}(y)) \subseteq \text{null}(\mathcal{X}(x)) \}$$

$$\text{relint}(\minface)(y) = \{ x \in \mathcal{F} : \text{null}(\mathcal{X}(y)) = \text{null}(\mathcal{X}(x)) \}$$

$$\text{aff}(\minface)(y) = \{ x \in \mathbb{R}^m : \mathcal{X}(x)U = 0 \}$$

$$\Omega V \mathcal{X}(x) V^T = 0 \text{ for all } x \in \mathcal{F}.$$
Strong Arnold Property (SAP)

Given graph $G$, let $A$ be an $n \times n$ symmetric matrix $A$ such that $A_{ij} = 0$ for all $\{i,j\} \in E(G)$. Then $A$ satisfies SAP if $Y = 0$ is the only symmetric matrix satisfying: (i) $Y_{ij} = 0$ if $i = j$ or $\{i,j\} \in E(G)$ and (ii) $AY = 0$. Thus our sufficient condition for universal rigidity is equivalent to the assertion that stress matrix $\Omega$ satisfies SAP.
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Thus our sufficient condition for universal rigidity is equivalent to the assertion that stress matrix $\Omega$ satisfies SAP.
Given graph $G$, let rank $\Omega = k$ and let $S_k = \{ A \text{ is symm : rank } A = k \}$. Further, let $T_\Omega$ be the tangent space to $S_k$ at $\Omega$. Thus $\Omega \in S_k \cap L$. We say $S_k$ transversally intersects $L$ at $\Omega$ if $T_\Omega \perp \Omega \cap S_k \perp k = \{ 0 \}$. Thus our sufficient condition for universal rigidity is equivalent to the assertion that $S_k$ transversally intersects $L$ at $\Omega$. 
Transversal Intersections

- Given graph $G$, let rank $\Omega = k$ and let $S_k = \{A \text{ is symm: rank } A = k\}$. Further, let $T_\Omega$ be the tangent space to $S_k$ at $\Omega$.
- Let $L = \{A \text{ is symm: } A_{ij} = 0 \text{ if } \{i, j\} \in E(\overline{G})\}$.
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- Thus our sufficient condition for universal rigidity is equivalent to the assertion that $S_k$ transversally intersects $\mathcal{L}$ at $\Omega$. 
SDP Non-degeneracy (Alizadeh et al ’97)

Consider the pair of dual SDPs:

(P) \max_y \quad 0^T y \\
subject to \quad \mathcal{X}(y) = X + \sum_{ij} y_{ij} M^{ij} \succeq 0

(D) \min_Y \quad \text{trace}(XY) \\
subject to \quad \text{trace}(M^{ij} Y) = 0 \\
Y \succeq 0.

Let $L'_E = \text{span}\{M^{ij}: \{(i, j)\} \in E(G)\}$ and let $T_Y$ be the tangent space at $Y$ to the set of symmetric matrices of order $n-1$.

$Y$ is non-degenerate if $T_Y \cap L'_E = \{0\}$.

Theorem [Alizadeh et al ’97]: If (D) has an optimal non-degenerate $Y$, then $y$ in (P) is unique.
Consider the pair of dual SDPs:

\[(P) \text{ max}_y \quad 0^T y \quad \text{subject to} \quad x(y) = X + \sum_{ij} y_{ij} M^{ij} \succeq 0\]

\[(D) \text{ min}_Y \quad \text{trace}(XY) \quad \text{subject to} \quad \text{trace}(M^{ij} Y) = 0 \quad Y \succeq 0.\]

Let \( \mathcal{L}' = \text{span} \{ M^{ij} : \{i, j\} \in E(\bar{G}) \} \) and let \( T_Y \) be the tangent space at \( Y \) to the set of symm matrices of order \( n - 1 \).
Consider the pair of dual SDPs:

\[(P) \quad \max_y \quad 0^T y \]
subject to \[x(y) = X + \sum_{ij} y_{ij} M^{ij} \succeq 0\]

\[(D) \quad \min_Y \quad \text{trace}(XY) \]
subject to \[\text{trace}(M^{ij} Y) = 0 \]
\[Y \succeq 0.\]

Let \(L' = \text{span} \{ M^{ij} : \{i, j\} \in E(G) \} \) and let \(T_Y \) be the tangent space at \(Y\) to the set of symm matrices of order \(n - 1\).

\(Y\) is non-degenerate if \(T_Y \perp L' = \{0\}\).
Consider the pair of dual SDPs:

(P) \[ \max_y y^T 0 \]
subject to \[ x(y) = X + \sum_{ij} y_{ij} M_{ij} \succeq 0 \]

(D) \[ \min_Y \text{trace}(XY) \]
subject to \[ \text{trace}(M_{ij} Y) = 0 \]
\[ Y \succeq 0. \]

Let \( \mathcal{L}' = \text{span} \{ M_{ij} : \{i,j\} \in E(G) \} \) and let \( \mathcal{T}_Y \) be the tangent space at \( Y \) to the set of symm matrices of order \( n - 1 \).

\( Y \) is non-degenerate if \( \mathcal{T}_Y^\perp \cap \mathcal{L}' = \{0\} \).

Theorem[Alizadeh et al '97]: If (D) has an optimal non-degenerate \( Y \), then \( y \) in (P) is unique.
Thank You