affine flexes, steven gortler

- joint with bob connelly. and louis theran.
- given a graph $G$ with $n$ vertices and $m$ edges.
- given a framework $(G, p)$ in $E^d$ with full span.
- def: the framework admits an affine flex if there is a $d$-dimensional affine, but not euclidean, transform $A$, such that $(G, p)$ is equivalent to $(G, A(p))$
  - here are some examples in 3d and 2d
  - note that when there is an affine flex, there will always be a continuum of them, as in our examples.
- this is obviously a very special situation
- given the coordinates of a specific $p$, one can, in fact check for an affine flex using linear algebra
- the goal of this work is to better understand when this can happen.
one motivating situation

• def: a framework is universally rigid if there is no equivalent framework in any dimension except for congruences.

• • represent frameworks that can be found using SDP

• a stronger property is that of super stability

• def: an equilibrium stress matrix for \((G, p)\) is an \(n\)-by-\(n\) symmetric matrix. with zero entries on all \(ij\) non-edge-pairs. with the all-ones vector in its kernel. with each of the coordinate \(n\)-vectors of \(p\) in its kernel.

• • so it can have rank at most \(n - d - 1\).

• def: a framework is super stable if it has an equilibrium stress matrix that is of rank \(n - d - 1\) that is PSD. plus the framework does not have an affine flex.

• t (con): super stability implies universal rigidity.

• so for super stability, we have to explicitly rule out the possibility of an affine flex.
alfakah’s thm

• thm: (alf) Suppose $(G, p)$ has an equilibrium stress matrix that is of rank $n - d - 1$. If each vertex nbhd has a full affine span then the framework does not have an affine flex.

• so we can get super stability with just the stress matrix and local affine span.
quadric stuff

• def: we say that \((G, p)\) has its edge directions on a conic at infinity if there is a non-zero d-by-d symmetric matrix \(Q\) such that, for each edge \(ij\), we have \((p_i-p_j)^t Q (p_i-p_j) = 0\).

• thm: (con) a framework has an affine flex iff its edge directions lie on a conic at infinity.
• in 2D, this means that the edges lie in at most 2 directions.

• def: we say that a framework is ruled by a quadric (or just ruled) if all of the points along all of the edges lie on a quadric.
• in 2D, this means that the framework lies on 2 lines.

• note: ruled \(\Rightarrow\) conic at infinity, since edge direction is just the intersection of the edge’s line with the plane at infinity.
• here are some examples and non examples in 2d and 3d.
**main thm and main cor**

- **thm:** Suppose that \((G, p)\) is “NAR” then it has an affine flex iff it is ruled.
  - Proof is very simple, and I may get to it.
- It will turn out that a max rank equilibrium stress matrix implies that a framework is NAR, giving us the following corollary.

- **cor:** Suppose \((G, p)\) has an equilibrium stress matrix that is of rank \(n - d - 1\). Then it has an affine flex iff it is ruled.
  - Note that a ruled framework cannot have \(d\) vertices in general position each with full affine span nbhds.
  - So this corollary is stronger than Alfakih’s thm.
SAP

• this corollary is also related to something called the strong arnold property of a matrix.

• indeed, the corollary can also be proven using a different recent theorem by alfakin on SAP together with an older theorem of Godcil on SAP
cone frameworks

• we can use our main cor to study the super stability of cone frameworks.

• def: we denote a cone framework of cone graph (in $E^{d+1}$) as $p_0 \ast (G, p)$. $G$ denotes the subgraph induced by removing vertex 0, which is connected to all of the vertices in $G$. (we assume $p_0$, the cone vertex position, is not coincident with any of the points in $p$.)

• note: universal rigidity of a cone framework is the same as the uniqueness of an PSD matrix completion problem with known diagonal entries.
operations

• we can take a framework in $E^d$ and cone it to create a cone framework in $E^{d+1}$.

• we can take a cone framework and slide it (avoiding $p_0$)

• we can take a cone framework $E^{d+1}$ and slice it by sliding the vertices of $G$ to lie in a hyperplane and then considering the framework of $G$ in $E^d$. 
what is known

• if we cone a universally rigid framework, the result is universally rigid
• if we cone a super stable framework, the result is super stable
• if we slide a universally rigid cone framework, the result is universally rigid
• if we slide a super stable cone framework, the result is super stable
• if we slice a universally rigid cone framework, the result might not be universally rigid
  • • this happens when a cone framework does not have its edge directions on a conic at infinity but the slice does have its edges directions on a conic at infinity.
what about super stability under slicing

• lem: if $p_0 \ast (G, p)$ is super stable, the sliced result must have a max rank PSD equilibrium stress matrix

• main observation: if $p_0 \ast (G, p)$ is not ruled, then neither is the slice.

• thm: if we slice a super stable cone framework, the result is super stable
projective transforms

- c: if a framework is super stable, then so is the result after any invertible projective transform.

- proof: the projective transform can be modeled using coning, affine transforms in $E^{d+1}$ followed by slicing.
**NAR**

- **def:** \((G, p)\) is nbhd affine equivalent to \((G, q)\) if for each vertex, there is an affine transform that maps its nbhd in \(p\) to its nbhd in \(q\).

- **def:** \((G, p)\) is affine congruent to \((G, q)\) if there is a single affine transform that maps \(p\) to \(q\).

- **def:** \((G, p)\) is NAR if for any framework \((G, q)\) to which \((G, p)\) is nae to, we always have that \((G, p)\) is ac to \((G, q)\).
proof of main thm

• We will do the hard direction. If NAR and affine flex with conic \( Q \), then ruled.
perturbation

• suppose that \((G, p)\) has an affine flex, so that its edge directions are on a conic at infinity defined by \(Q\).

• def: let \(m(x) := x + [x^t Q x] v\)

• lem: \((G, p)\) is nae to \((G, m(p))\).

• the proof is just a two line calculation.

• proof: we have assumed

\[
0 = (p_j - p_i)^t Q (p_j - p_i)
\]

we get

\[
p_j^t Q p_j = -p_i^t Q p_i + 2 p_i^t Q p_j
\]

• Treating \(p_i\) as a constant, we see that \(p_j^t Q p_j\) can be expressed as an affine function of \(p_j\).

• Thus the action of \(m\) on the neighborhood of \(p_i\) can be modeled with an affine transform.
now add in the NAR assumption

•lem: suppose that \((G, p)\) is NAR and has an affine flex, so that its edge directions are on a conic at infinity defined by \(Q\). then \((G, p)\) is ac to \((G, m(p))\).
what does this congruence imply

*lem: if \((G, p)\) is to \((G, m(p))\) then all of the vertices must lie on a quadric with quadratic terms defined by \(Q\).

*the proof is another 2 line calcuation

*proof: the ac means

\[
[p_i^t Q p_i] v = A p_i + t
\]

where \(A\) is a \(d \times d\) matrix.

*multiplying on the left by \(v^t\) we get

\[
p_i^t Q p_i = [v^t A] p_i + v^t t
\]

*which places \(p_i\) on a quadric
three linear points on a quadric

• for each edge, we have its two endpoints on a quadric.
• the edge direction \((p_i - p_j)\) is a point at infinity on this same line.
• and it is on the same quadric.
• since we have 3 colinear points on the conic, the entire line must be on the quadric.
• this gives us a ruled framework.
• QED.