Combinatorial Auctions

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October 2007
Outline

- Introduction
- Single-Minded Bidders
- Challenges
Combinatorial Auctions: Allocate $K$ items to $N$ people.

The allocation to $i$ is $x^i \in \{0, 1\}^K$ where $x^i_k = 1$ if and only if $i$ gets item $k$.

Feasibility: $x = (x^1, ..., x^N) \in F$ if and only if $x^i \in \{0, 1\}^K$ and $\sum_i x^i_k \leq 1$ for all $k$.

Utility for $i$: $v^i(x^i, \theta^i) - y^i$ where $\theta^i \in \Theta^i$.
[For reverse auctions, use $y^i - c^i(x^i, \theta^i)$.]
Is there a combinatorial auction problem?

If agents are obedient and infinitely capable, and if the mechanism is infinitely capable, then to maximize revenue or to achieve efficiency:

Have each $i$ report $v^i(x^i, \theta^i)$ for all $x^i \in \{0, 1\}^K$.

Let $x^* = \arg\max \sum v^i(x^i, \theta^i)$ subject to $x \in F$.

Allocate $x^* i$ to each $i$.

Charge each $i$, $y^i = v^i(x^* i, \theta^i)$.

This is efficient and revenue maximizing.

Note: If $y^i = 0$ for each $i$, then you get buyer efficiency.
Is there a problem?

Have each i report \( v^i(x^i, \theta^i) \) for all \( x^i \in \{0, 1\}^K \).
Communication: \( 2^K \) can be a lot of numbers.

Let \( x^* = \text{argmax} \sum v^i(x^i, \theta^i) \) subject to \( x \in F \).
Computation: Max problem isn’t polynomial.

Charge each i, \( y^i = v^i(x^{i*}, \theta^i) \)
Incentives: So, why should I tell you \( \theta^i \)?

Subject to Communication, Computation, Voluntary Participation, and Incentive Compatibility Constraints,

What is the Best Auction Design?
Some Design Features to Consider

Bids allowed - single items, all packages, some (which?)

Timing - synchronous, asynchronous

Pricing - pay what you bid, uniform (second price), incentive pricing

Feedback - all bids, provisional winning bids only, number of bids for each item, item prices (which?), ...

Others - minimum increments, activity rules, withdrawals, reserve prices (secret or known), retain provisional losing bids, XOR, proxies, ...
Example Practical Questions

- Public sector - Spectrum Auctions
  Use Design #1 (single item bids, synchronous, iterative) or use Design #2 (package bids, synchronous, iterative)?

- Private sector - Logistics Acquisitions
  Use Design #1 (package bids, synchronous, iterative) or use Design #2 (package bids, one-shot sealed bid)?

  How Should we Decide? What about Other Designs?
Combinatorial Auctions: The Art of Design - the 1st generation

Sealed bid, IC pricing
- Vickrey-Clarke-Groves (1963, 71, 73)

Sealed bid, pay what you bid
- Rasenti-Smith-Bulfin (1982)

Iterative, asynchronous,
- Banks, Ledyard, Porter 1989 - AUSM

Iterative, synchronous,
- Ledyard, Olson, Porter, etc. 1992 - Sears

Iterative, synchronous, no package bids, activity rules
- McMillan, Milgrom 1994 - FCC-SMR
Combinatorial Auctions: The Art of Design - the 2nd generation

Iterative, synchronous, Proxies
- Parkes 1999 - iBEA

Iterative, synchronous, price feedback
- Kwasnica, Ledyard, Porter 2002 - RAD

Clock auction, packages, synchronous
- Porter, Rassenti, Smith 2003

CC, proxies
- Ausubel, Milgrom 2005

How should we decide
Which Design is Best for which Goals in which Situations?

DIMACS-9
Combinatorial Auction Design: Three approaches

- Experimental: the economist’s wind tunnel
- Agent-based: the computer scientist’s wind tunnel
- Theoretical: the analyst’s wind tunnel

<table>
<thead>
<tr>
<th>approach</th>
<th>behavioral model</th>
<th>mechanism complexity</th>
<th>environmental coverage</th>
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<tbody>
<tr>
<td>experimental</td>
<td>correct (naive?)</td>
<td>not stressed</td>
<td>costly</td>
</tr>
<tr>
<td>agent-based</td>
<td>open? (not str.for.)</td>
<td>can stress</td>
<td>moderate</td>
</tr>
<tr>
<td>theoretical</td>
<td>stylized</td>
<td>open?</td>
<td>complete</td>
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A Taste of the Experimental Approach:
(Brunner-Goeree-Holt-Ledyard)

- 12 licenses, 8 subjects (experienced - trained)
  - 6 regional bidders: 3 licenses each, $v \in [5, 75]$
  - 2 national bidders: 6 licenses each, $v \in [5, 45]$
  - 13,080,488 possible allocations

- 0.4 cents per point, (upto $1.25 for 3, $1.30 for 6)
  - with a synergy factor $\alpha$ per license of 0.2 (national) and 0.125 (regional)

- Earnings averaged $50/2 hour session incl $10 show-up fee.
  - 48 sessions of 8 subjects each. 10 auctions/session.
  - 120 auctions /design.
Economic Experiment Results

<table>
<thead>
<tr>
<th></th>
<th>SMR</th>
<th>CC</th>
<th>RAD</th>
<th>FCC*</th>
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</thead>
<tbody>
<tr>
<td>Average Efficiency</td>
<td>90.2%</td>
<td>90.8%</td>
<td>93.4%</td>
<td>89.7%</td>
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<tr>
<td>Average Revenue</td>
<td>37.1%</td>
<td>50.2%</td>
<td>40.2%</td>
<td>35.1%</td>
</tr>
<tr>
<td>Average Profits</td>
<td>53.1%</td>
<td>40.6%</td>
<td>53.3%</td>
<td>54.6%</td>
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Efficiency_{output} = \frac{E_{actual} - E_{random}}{E_{maximum} - E_{random}}.

Revenue = \frac{R_{actual} - R_{random}}{R_{maximum} - R_{random}}.

Profits = Efficiency - Revenue

Is Revenue of 50% big or small?

Are these the result of Behavior, Environment, or Design?
Outline

• Introduction

• Single-Minded Bidders

• Challenges
A Taste of the Theoretical Approach

An auction design is \( \gamma = \{N, S^1, ..., S^N, g(s)\} \).

Bidders behavior is \( b^i : \{(I^i, v^i, \gamma)\} \rightarrow S^i \).

The Design Problem is:

- Choose \( \gamma \) so that \( g(b(I, v, \gamma)) = [x(v), y(v)] \) is desirable.
The Economist's approach:
(1) Get an upper bound on performance; ignore Computation and Communication Constraints.
(2) Use all information available; Assume the seller has a prior
\[ \pi(\theta)d\theta = d\Pi(\theta) = d\Pi^1(\theta^1)...d\Pi^N(\theta^N). \]

Using the revelation principle, choose \((x, y) : \Theta^N \to \{(x, y)\}\) to maximize expected revenue
\[
\max \int \sum_i y^i(\theta)d\Pi(\theta)
\]
subject to
\[(x(\cdot), y(\cdot)) \in F^* \cap IC \cap VP.\]

Question: Interim or ex-post? Bayesian or Dominance?
Answer: Will see it doesn't matter.
Consider a special class of environments

**Single-Minded Bidders**

- Each bidder has a preferred package $x^*_i$ that is common knowledge (including the auctioneer).

$$v^i(x, \theta^i) = \theta^i q^i(x)$$ where

$$q^i(x) = 1 \quad \text{if} \quad x^i \geq x^*_i$$

$$q^i(x) = 0 \quad \text{otherwise}$$
Probability of winning is $Q^i(\theta^i) = \int q^i(x(\theta))d\Pi(\theta|\theta^i)$

Expected payment is $T^i(\theta^i) = \int y^i(x(\theta))d\Pi(\theta|\theta^i)$

Expected Utility is $\theta^i Q^i(\theta^i) - T^i(\theta^i)$

Incentive compatibility is $T(\theta) = T_0 + \int_{\theta_1}^{\theta} s dQ(s)$ and $dQ/d\theta \geq 0$

Voluntary participation is $\theta_1^i Q^i(\theta_1^i) - T^i(\theta_1^i) \geq 0$

Combine these with revenue maximization and get that $T = \theta Q - \int_{\theta_1}^{\theta} Q(s) ds$

So Expected revenue from i is $\int [\theta^i - \frac{1-\Pi(\theta^i)}{\pi(\theta^i)}] q^i(\theta) d\Pi(\theta)$
The optimal *interim* mechanism for single minded-bidders (where \( \Pi(\theta) \) is common-knowledge) solves

\[
x(\theta) \in \arg \max_{x \in F^*} \sum w_i(\theta^i)q^i(x)
\]

\[
y^i(\theta) = \theta^iQ^i(\theta^i) - \int_{\theta_1}^{\theta^i} Q^i(s)ds
\]

where \( w_i(\theta^i) = \frac{\theta^i - 1 - \Pi^i(\theta^i)}{\pi^i(\theta^i)} \)

 Requires \( dw^i/d\theta^i \geq 0 \), for incentive compatibility SOC. An increasing hazard rate is sufficient.

This is a (very slight) generalization of Myerson (1981). Only \( F^* \) is different.
Using Mookherjee and Reichelstein (1992), monotonicity implies one can convert the *interim* mechanism to an *ex-post* mechanism with the same interim payoffs to everyone.

\[ x^*(\theta) \in \arg \max_{x \in F} \sum w_i(\theta^i)q^i(x) \]

\[ y^i(\theta) = \theta^i q^i(x^*(\theta)) - \int_{\theta_1}^{\theta^i} q^i(x^*(\theta/s^i)) ds^i \]

This mechanism is the optimal *ex post* mechanism because

*ex-post* \( F^* \cap IC \cap VP \subset *interim* \( F^* \cap IC \cap VP \)
Note that $q^i(x^*(\theta)) = 1$ if

$$\max_{x \in F} \sum_{j=1}^{N} w^j(\theta^j)q^j(x) > \max_{x \in F} \sum_{j \neq i} w^j(\theta^j)q^j(x)$$

Let

$$\theta^{*i}(\theta_{-i}) = \inf\{\theta^i | q^i(x^*(\theta)) = 1\}$$

The optimal ex-post mechanism is:

$$q^i(x^*(\theta)) = 1 \text{ iff } \theta^i \geq \theta^{*i}(\theta_{-i})$$

and

$$y^{*i}(\theta) = \theta^{*i}(\theta_{-i})q^i(x^*(\theta))$$
The optimal ex-post mechanism is not VGC.

It is closely related. They both look like

\[ q^i(x(\theta)) = \text{iff } \theta^i \geq \theta^i(\theta_{-i}) \]

and

\[ y^i(\theta) = \theta^i(\theta_{-i}) q^i(x(\theta)) \]

but the Optimal \( \theta^*_i(\theta_{-i}) \neq \text{VCG } \tilde{\theta}^i(\theta_{-i}) \)

\[ x^*(\theta) \in \arg\max_{x \in F} \sum_i \left( \theta^i - \frac{1 - \Pi^i(\theta)}{\pi^i(\theta)} \right) q^i(x) \]

\[ \hat{x}(\theta) \in \arg\max_{x \in F} \sum_i \theta^i q^i(x) \]

The optimal ex post mechanism is not output-efficient.

Even if conditioned on participation (as in Myerson).
The optimal \textit{ex post} optimal mechanism is \textbf{VCG} with preferences.

- **Request sealed bids** for packages: \( b^i \)

- **Subtract an \textit{individual} “preference”:** \( p^i = \frac{1 - \Pi^i(b^i)}{\pi^i(b^i)} \)

- **Maximize \textit{adjusted} bid revenue:** \( \max \sum_i (b^i - p^i)\nu^i \)
  subject to \( \nu^i \in \{0, 1\} \) and \( (\nu^1, ..., \nu^N) \) feasible

- **Charge \textit{pivot prices}:** \( y^i = \inf\{b^i | \nu^i = 1\} \)
Interesting Special Case

If values are uniformly distributed, then

$$\theta^i \sim U[m^i, M^i]$$, then $$p^i(b^i) = M^i - b^i$$ and $$b^i - p^i(b^i) = 2b^i - M^i$$.

In this case, the optimal auction is equivalent to:

- Charge a reserve price of: $$r^i = M^i / 2$$

- Maximize the reserve-adjusted surplus: $$\sum(b^i - r^i)\nu^i$$.
Example: $K = 2, N = 3$

$x^1 = (1, 0), x^2 = (0, 1), x^3 = (1, 1)$

$\theta^1, \theta^2$ are uniformly distributed on [0, 1]

$\theta^3$ is uniformly distributed on [0, $a$]

<table>
<thead>
<tr>
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<tr>
<td><strong>OA</strong></td>
<td>0.585</td>
<td>0.625</td>
<td>0.613</td>
</tr>
<tr>
<td><strong>VGC</strong></td>
<td>0.240</td>
<td>0.452</td>
<td>0.426</td>
</tr>
<tr>
<td><strong>Random</strong></td>
<td>0.480</td>
<td>0.465</td>
<td>0.413</td>
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OA & VCG highest for $a = 2$, the most competitive situation.

Random (5 allocations possible) looks as good as VCG.
New Experiments

* 2 items, 3 subjects
* **Tested SMR, RAD, and SB**
* 1 session for each auction
* 9 subjects per session
* Randomly matched into groups of 3 at beginning
* 10 rounds for each group (the first 2 were practice rounds).
* Before round, bidders randomly assigned to role.
* Values for 1 and 2 are in \([0,100]\), values for 1,2 are in \([0,200]\)
* No withdrawals, no activity rules
Experiment Results (24 auctions of each type)

Mean (Std. Dev.)

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<td>SMR</td>
<td>58.13 (43.16)</td>
<td>0.90 (0.20)</td>
<td>0.46 (0.33)</td>
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<tr>
<td>RAD</td>
<td>66.71 (46.99)</td>
<td>0.97 (0.09)</td>
<td>0.53 (0.30)</td>
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RAD > SMR in revenue.

# rounds for RAD (5.65) < SMR (7.46).

But OA > RAD
Experiment Results (24 auctions of each type)

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<tr>
<td>SB</td>
<td>89.79 (36.99)</td>
<td>0.96 (0.19)</td>
<td>0.74 (0.19)</td>
</tr>
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SB > OA > RAD > SMR.

No reserve price used in SB.
For combinatorial auctions with single minded bidders

We find the DSIC design that maximizes expected revenue.
- It is neither VGC nor output efficient.
- It is VCG with individualized bid preferences.

In a small experiment, SB > OA > RAD > SMR,
- RAD gets 85% of the revenue of the theoretical upper bound.
- SB gets 116% of the revenue of the theoretical upper bound.
Outline

- Introduction
- Single-Minded Bidders
- Challenges
Combinatorial Auctions:

- The auction design: $\gamma = \{N, S^1, ..., S^N, g(\cdot)\}$.

- Bidders behavior: $b^i : \{(I^i, \theta^i, \gamma)\} \rightarrow S^i$

- Choose a feasible $\gamma$ so that $g(b(I, \theta, \gamma))$ is desirable.

The tension is between theory and practice.
Choose a feasible $\gamma$ so that $g(b(I, \theta, \gamma))$ is desirable.

- Which $\gamma$ are feasible?

Need pliable communication and computation constraints
- A finer grid than NP-hard, polynomial, etc.
- An analytic version that can be used as constraints in a maximization problem.

Need a revelation principle for feasible mechanisms, $G^F \subset G$.
- Usual: $\forall \gamma \in G^F$, $\exists \gamma^* \in G^D$ with $\gamma^* = \{N, \Theta, h(\cdot)\}$ such that $h(\theta) = g(b(\theta, \gamma))$ and $b(\theta, \gamma^*) = \theta$.
- But inverse is now a problem. Need to characterize $G^{D*}$ such that if $\gamma^* \in G^{D*}$ then $\exists \gamma \in G^F \exists h(b(\theta, \gamma^*)) = g(b(\theta, \gamma))$. 
Choose a feasible $\gamma$ so that $g(b(I, \theta, \gamma))$ is desirable.

- What is the "right" theory of behavior?

Need better theory of behavior in iterative auctions
- Game theoretic equilibria such as Dominance & Bayes make sense for simple, direct revelation auctions but are "wrong."
- With iteration, straight-forward bidding tempting, but "wrong."
- Incorporate behavioral learning models (agents) into optimal auction methodology?

Need behavior model to be more sensitive to details
- Designing to prevent collusion often involves information issues finessed by direct mechanisms.
- Reveal bids and bidders? Reveal only winning bids? Endogenous sunshine?
Choose a feasible $\gamma$ so that $g(b(I, \theta, \gamma))$ is desirable.

- What does desirable mean?
  Need to consider all costs and benefits
  - Tradeoff between mechanism and bidder computations
  - Iteration may reduce costs of determining values but increase costs of bidding?

- How do we choose?
  Can we always reduce to an optimization problem?
  - Need to deal with multi-dimensional incentive constraints
  - Need to find a simple characterization for feasible $\gamma$.
  - Or do we just need to generate a lot of experiments?