

Hyperbolic Polynomials Approach to Van der
Waerden/Schrijver-Valiant like Conjectures :
Sharper Bounds , Simpler Proofs and Algorithmic
Applications

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Doubly Stochastic matrices and matrix tuples , Permanent , Mixed Discriminant

Doubly Stochastic $n \times n$ Matrix :

$$\Omega_n = \{A = A(i, j) : A(i, j) \geq 0, 1 \leq i, j \leq n; Ae = A^T e = e,$$

$\Omega_n =$ The set of $n \times n$ Doubly Stochastic matrices.

Doubly Stochastic n -tuple $\mathbf{A} = (A_1, \dots, A_n)$:

$$A_i \succeq 0 \text{ (PSD } n \times n \text{ complex hermitian) , } tr A_i = 1, 1 \leq i, j \leq n; \sum_{i=1}^n A_i = I .$$

$D_n =$ The set of Doubly Stochastic n -tuples.

$$\text{The permanent : } per(A) = \sum_{\sigma \in S_n} \prod_{i=1}^n A(i, \sigma(i))$$

The mixed discriminant :

$$D(A_1, A_2, \dots, A_n) = \frac{\partial^n}{\partial t_1 \dots \partial t_n} \det(t_1 A_1 + \dots + t_n A_n)$$

Determinantal Polynomial :

$$DET_{\mathbf{A}}(t_1, \dots, t_n) = \det(\sum_{1 \leq i \leq n} t_i A_i).$$

Multilinear Polynomial :

$$Mul_A(t_1, \dots, t_n) = \prod_{1 \leq i \leq n} \sum_{1 \leq j \leq n} A(i, j) t_j.$$

$$per(A) = \frac{\partial^n}{\partial t_1 \dots \partial t_n} Mul_A(t_1, \dots, t_n)$$

($per(A) = 2^{-n} \sum_{b_i \in \{-1, 1\}, 1 \leq i \leq n} Mul_A(b_1, \dots, b_n)$: Ryser's formula .)

Multilinear is commutative(solvable) case of Determinantal .

Van der Waerden Conjecture

The famous Van der Waerden Conjecture states that

$$\min_{A \in \Omega_n} D(A) = \frac{n!}{n^n} \text{ (VDW-bound)}$$

and the minimum is attained uniquely at the matrix J_n in which every entry equals $\frac{1}{n}$.

Van der Waerden Conjecture was posed in 1926 and proved only in 1981 :

D.I. Falikman proved the lower bound (**VDW-bound**) and the full conjecture , i.e. the uniqueness part , was proved by G.P. Egorychev . They shared Fulkerson Prize , 1982 .

Aleksandrov-Fenchel inequalities and many other ingredients , about 25 years of research.

Was used by N. Linial, A. Samorodnitsky and A. Wigderson (1998) to approximate the permanent of nonnegative matrices :

$$A = \text{Diag}(a_1, \dots, a_n) B \text{Diag}(b_1, \dots, b_n), B \in \Omega_n$$

Sinkhorn's Scaling .

As $\frac{n!}{n^n} \leq \text{per}(B) \leq 1$ thus

$$f(A) =: \prod_{1 \leq i \leq n} a_i b_i \Rightarrow 1 \leq \frac{f(A)}{\text{per}(A)} \leq \left(\frac{n!}{n^n}\right)^{-1} \approx e^n.$$

Strongly polynomial algorithms .

Bapat's Conjecture (Van der Waerden Conjecture for mixed discriminants)

One of the problems posed by R.V.Bapat (1989) is to determine the minimum of mixed discriminants of doubly stochastic tuples : $\min_{A \in D_n} D(A) = ?$

Quite naturally, R.V.Bapat conjectured that

$$\min_{A \in D_n} D(A) = \frac{n!}{n^n} \text{ (Bapat-bound)}$$

and that it is attained uniquely at $\mathbf{J}_n =: (\frac{1}{n}I, \dots, \frac{1}{n}I)$.

The original conjecture was formulated for real symmetric PSD matrices.

L.G. had proved it (1999 , 2006 in *Advances in Mathematics* for the complex case, i.e. for complex positive semidefinite and, thus, hermitian matrices .

Was motivated by the ellipsoid algorithm to approximate (deterministically) mixed discriminants/mixed volumes .

Schrijver-Valiant Conjecture

Let $\Lambda(k, n)$ denote the set of $n \times n$ matrices with nonnegative integer entries and row and column sums all equal to k (k -regular bipartite graphs) .

We define the following subset of rational doubly stochastic matrices : $\Omega_{k,n} = \{k^{-1}A : A \in \Lambda(k, n)\}$

.

Define

$$\lambda(k, n) = \min\{per(A) : A \in \Omega_{k,n}\} =$$

$$k^{-n} \min\{per(A) : A \in \Lambda_{k,n}\};$$

$$\theta(k) = \lim_{n \rightarrow \infty} (\lambda(k, n))^{\frac{1}{n}}.$$

$\lambda(2, n) = 2^{-n+1}$, Erdos-Renyi (1968) : $\theta(k) = ?$, even the case $k = 3$ was open until 1979-1980 .

M. Voorhoeve in (1979) : $\lambda(k, n) \geq (\frac{2}{3})^{2(n-3)} \frac{2}{9}$.

Schrijver-Valiant (1980) $\theta(k) \leq g(k) = (\frac{k-1}{k})^{k-1}$, which gives $\theta(3) = \frac{4}{9}$.

Schrijver-Valiant Conjecture (1980) : $\theta(k) = g(k) = \left(\frac{k-1}{k}\right)^{k-1}$.

Settled by Lex Schrijver in 1998 : $\min\{per(A) : A \in \Omega_{k,n}\} \geq \left(\frac{k-1}{k}\right)^{(k-1)n}$ (**Schrijver-bound**) .

remarkable result — unpassable proof .

I will present a vast and unifying generalization of those three results .

Homogeneous polynomials with nonnegative coefficients

Let $Hom(m, n)$ be a linear space of homogeneous polynomials $p(x)$, $x \in R^m$ of degree n in m variables ; correspondingly $Hom_+(m, n)$ ($Hom_{++}(m, n)$) be a subset of homogeneous polynomials $p(x)$, $x \in R^m$ of degree n in m variables and nonnegative(positive) coefficients .

Let

$$\begin{aligned} p \in Hom_+(n, n), p(x_1, \dots, x_n) &= \\ &= \sum_{r_1+\dots+r_n=n} a_{(r_1, \dots, r_n)} \prod_{1 \leq i \leq n} x_i^{r_i} . \end{aligned}$$

The support :

$$supp(p) = \{(r_1, \dots, r_n) \in I_{n,n} : a_{(r_1, \dots, r_n)} \neq 0\} .$$

The convex hull $CO(supp(p))$ of $supp(p)$ is called the Newton polytope of p .

For a subset $A \subset \{1, \dots, n\}$ we define

$$S_p(A) = \max_{(r_1, \dots, r_n) \in \text{supp}(p)} \sum_{i \in A} r_i.$$

Given a vector (a_1, \dots, a_n) with positive real coordinates, consider univariate polynomials

$$D_A(t) = p(t(\sum_{i \in A} e_i) + \sum_{1 \leq j \leq n} a_j e_j),$$

$$V_A(t) = p(t(\sum_{i \in A} e_i) + \sum_{j \in A'} a_j e_j).$$

$S_p(A)$ can be expressed as an univariate degree :

$$S_p(A) = \text{deg}(D_A) = \text{deg}(V_A)$$

Homogeneous polynomials with nonnegative coefficients

The following linear differential operator maps $Hom(n, n)$ onto $Hom(n - 1, n - 1)$:

$$p_{x_1}(x_2, \dots, x_n) = \frac{\partial}{\partial x_1} p(0, x_2, \dots, x_n).$$

We define p_{x_i} , $2 \leq i \leq n$ in the same way for all polynomials $p \in Hom(n, n)$. Notice that

$$p(x_1, \dots, x_n) = x_i p_{x_i}(x_2, \dots, x_n) + q(x_1, \dots, x_n); q_{x_i} = 0.$$

The following inequality follows straight from the definition :

$$S_{p_{x_1}}(A) \leq \min(n - 1, S_p(A)) : \\ A \subset \{2, \dots, n\}, p \in Hom_+(n, n).$$

Consider $p \in Hom_+(n, n)$ We define the **Capacity**

as

$$Cap(p) = \inf_{x_i > 0, \prod_{1 \leq i \leq n} x_i = 1} p(x_1, \dots, x_n).$$

It follows that if $p \in Hom_+(n, n)$ then

$$Cap(p) \geq \frac{\partial^n}{\partial x_1 \cdots \partial x_n} p(0, 0, \dots, 0)$$

($p(x_1, \dots, x_n) = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} p(0, 0, \dots, 0) x_1 \dots x_n +$ nonnegative stuff .)

Notice that

$$\log(Cap(p)) = \inf_{\sum_{1 \leq i \leq n} y_i = 0} \log(p(e^{y_1}, \dots, e^{y_n})),$$

and if $p \in Hom_+(n, n)$

then the functional $\log(p(e^{y_1}, \dots, e^{y_n}))$ is convex .

EXAMPLE

Let $A = \{A(i, j) : 1 \leq i \leq n\}$ be $n \times n$ matrix with nonnegative entries . Assume that $\sum_{1 \leq j \leq n} A(i, j) > 0$ for all $1 \leq i \leq n$. Define the following homogeneous polynomial :

$$Mul_A(t_1, \dots, t_n) = \prod_{1 \leq i \leq n} \sum_{1 \leq j \leq n} A(i, j)t_j .$$

$$Mul_A \in Hom_+(n, n) \text{ and } Mul_A \neq 0 .$$

It is easy to check that

$$S_{Mul_A}(\{j\}) = |\{i : A(i, j) \neq 0\}|$$

($S_{Mul_A}(\{j\})$ is equal to the number of non-zero entries in the j th column of A) .

Notice that if $A \in \Lambda(k, n)$ (or $A \in \Omega(k, n)$) then $S_{Mul_A}(\{j\}) \leq k, 1 \leq j \leq n$.

More generally , consider a n -tuple $\mathbf{A} = (A_1, A_2, \dots, A_n)$, where the complex hermitian $n \times n$ matrices are positive semidefinite and $\sum_{1 \leq i \leq n} A_i \succ 0$ (their sum is positive definite).

Then the homogeneous polynomial

$DET_{\mathbf{A}}(t_1, \dots, t_n) = \det(\sum_{1 \leq i \leq n} t_i A_i) \in Hom_+(n, n)$
and $DET_{\mathbf{A}} \neq 0$.

Similarly to polynomials Mul_A :

$$S_{DET_{\mathbf{A}}}(\{j\}) = Rank(A_j), 1 \leq j \leq n.$$

As Van Der Waerden conjecture on permanents as well Bapat's conjecture on mixed discriminants can be equivalently stated in the following way (notice the absence of doubly stochasticity):

$$\frac{n!}{n^n} \text{Cap}(q) \leq \frac{\partial^n}{\partial x_1 \dots \partial x_n} q(0, \dots, 0) \leq \text{Cap}(q)(*)$$

The van der Waerden conjecture on the permanents corresponds to polynomials $Mul_A \in Hom_+(n, n) : A \geq 0$, the Bapat's conjecture on mixed discriminants corresponds to $DET_{\mathbf{A}} \in Hom_+(n, n) : \mathbf{A} \succeq 0$. The connection between inequality (*) and the standard forms of the van der Waerden and Bapat's conjectures is established with the help of the scaling .

Notice that the functional $\log(p(e^{y_1}, \dots, e^{y_1}))$ is convex if $p \in Hom_+(n, n)$. Thus the inequality (*) allows a convex relaxation of the permanent of nonnegative matrices and the mixed discriminant of semidefinite tu-

ples . This observation was implicit in [LSW, 1998] and crucial in [GS 2000 , 2002] .

VDW-FAMILIES

Consider a stratified set of homogeneous polynomials :
 $F = \bigcup_{1 \leq n < \infty} F_n$, where $F_n \in Hom_+(n, n)$. We call such set **VDW-FAMILY** if it satisfies the following properties :

1. If a polynomial $p \in F_n, n > 1$ then for all $1 \leq i \leq n$ the polynomials $p_{x_i} \in F_{n-1}$.
- 2.

$$Cap(p_{x_i}) \geq g(S_p(\{i\}))Cap(p) :$$

$$p \in F_j, 1 \leq i \leq j; g(k) = \left(\frac{k-1}{k}\right)^{k-1}, k \geq 1.$$

Meta-Theorem , main idea :

*Let $F = \bigcup_{1 \leq n < \infty} F_n$ be a **VDW-FAMILY** and the homogeneous polynomial $p \in F_n$. Then the following inequality holds :*

$$\prod_{1 \leq i \leq n} g(\min(S_p(\{i\}), i)) \text{Cap}(p) \leq \\ \leq \frac{\partial^n}{\partial x_1 \dots \partial x_n} p(0, \dots, 0) \leq \text{Cap}(p).$$

Corollaries :

1. If the homogeneous polynomial $p \in F_n$ then

$$\frac{n!}{n^n} \text{Cap}(p) \leq \frac{\partial^n}{\partial x_1 \dots \partial x_n} p(0, \dots, 0) \leq \text{Cap}(p).$$

2. If the homogeneous polynomial $p \in F_n$ and $S_p(\{i\}) \leq k \leq n, 1 \leq i \leq n$ then

$$\left(\frac{k-1}{k}\right)^{(k-1)(n-k)} \frac{k!}{k^k} \text{Cap}(p) \leq \frac{\partial^n}{\partial x_1 \dots \partial x_n} p(0, \dots, 0) \leq \leq \text{Cap}(p).$$

What is left now is to present a **VDW-FAMILY** which contains all polynomials $DET_{\mathbf{A}}$, where the n -tuple $\mathbf{A} = (A_1, \dots, A_n)$ consists of positive semidefinite hermitian matrices (and thus contains all polynomials Mul_A , where A is $n \times n$ matrix with nonnegative entries).

If such **VDW-FAMILY** set exists than Van der Waerden, Bapat, Schrijver-Valiant conjectures would follow (without any extra work, see Example) from Meta-Theorem and its Corollaries.

One of such **VDW-FAMILY** , consisting of *POS*-hyperbolic polynomials , is defined below.

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Hyperbolic polynomials

The following concept of hyperbolic polynomials was originated in the theory of partial differential equations **I.G. Petrowsky (1937 , in german) , L. Garding (1950) , L. Hormander**

It recently became "popular" in the optimization literature.

$$p \in \text{Hom}(m, n), X, e \in R^m : p(X - te) = 0$$

The polynomial p is e -hyperbolic

If all the roots $\lambda_n(X) \geq \dots \geq \lambda_1(X)$ are real .

Hyperbolic (convex) Cones :

$$N_e(p) = \{X \in R^m : \lambda_1(X) \geq 0 \text{ (closed) ,}$$

$$C_e(p) = \{X \in R^m : \lambda_1(X) > 0 \text{ (open) .}$$

$p \in \text{Hom}(m, n)$ is *POS*-hyperbolic if it is $(1, 1, \dots, 1) = e$ -hyperbolic , $p(e) > 0$
and the nonnegative orthant $R_+^m \subset N_e(p)$.

Equivalent definitionS :

$$|p(x_1 + iy_1, \dots, x_m + iy_m)| > 0 \text{ if } x_i > 0, 1 \leq i \leq n$$

So called wide sense stability in **CONTROL THEORY** .

Or : $p(1, \dots, 1) > 0$ and all the roots of the univariate equation $p(x_1 - t, \dots, x_n - t) = 0$
are real positive numbers if $x_i > 0, 1 \leq i \leq n$.

bf p -Mixed Forms

Let $p \in \text{Hom}(m, n)$ Khovanskii defined the p -mixed form of an n -vector tuple $\mathbf{X} = (X_1, \dots, X_n) : X_i \in C^m$ as

$$M_p(\mathbf{X}) =: M_p(X_1, \dots, X_n) = \frac{\partial^n}{\partial \alpha_1 \dots \partial \alpha_n} p\left(\sum_{1 \leq i \leq n} \alpha_i X_i\right)$$

The following polarization identity is well known

$$M_p(X_1, \dots, X_n) = 2^{-n} \sum_{b_i \in \{-1, 1\}, 1 \leq i \leq n} p\left(\sum_{1 \leq i \leq n} b_i X_i\right) \prod_{1 \leq i \leq n} b_i$$

Associate with any vector $r = (r_1, \dots, r_n) \in I_{n,n}$ an n -tuple of m -dimensional vectors \mathbf{X}_r consisting of r_i copies of $x_i (1 \leq i \leq n)$. It follows from the Taylor's formula that

$$p\left(\sum_{1 \leq i \leq n} \alpha_i X_i\right) = \sum_{r \in I_{n,n}} \prod_{1 \leq i \leq n} \alpha_i^{r_i} M_p(\mathbf{X}_r) \frac{1}{\prod_{1 \leq i \leq n} r_i!}$$

POS-Hyperbolic polynomials , basic facts

FACT 1 . $p(X) = p(e) \prod_{1 \leq i \leq n} \lambda_i(X)$.

FACT 2 . If p is e -hyperbolic polynomial and $p(e)$ is a real nonzero number then the coefficients of p are real.

If p is e -hyperbolic polynomial and $p(e) > 0$ then $p(X) > 0$ for all e -positive vectors $X \in C_e(p) \subset R^m$.

FACT 3 . Let $p \in Hom(m, n)$ be e -hyperbolic polynomial and $d \in C_e(p) \subset R^m$. Then p is also d -hyperbolic and $C_d(p) = C_e(p), N_d(p) = N_e(p)$.

FACT 4 . Let $p \in Hom(m, n)$ be e -hyperbolic polynomial . Then the polynomial $p_e(X) =: \frac{d}{dt}p(X + te)|_{(t=0)}$; $p_e \in Hom(m, n - 1)$ is also e -hyperbolic and $C_e(p) \subset C_e(p_e)$ (Rolle's theorem).

FACT 5 . Let $p \in Hom(m, n)$. Then the p -mixed form $M_p(X_1, \dots, X_n)$ is linear in each vector argument $X_i \in C^m$. Let $p \in Hom(m, n)$ be e -hyperbolic and $p(e) > 0$. Then $M_p(X_1, \dots, X_n) > 0$ if the vectors $X_i \in R^m, 1 \leq i \leq n$ are e -positive (proved by induction using FACT 4) .

POS-Hyperbolic polynomials form VDW-FAMILY

Define $Rank_q(X)$ as $|\{i : \lambda_i(X) \neq 0\}|$.

Then $S_q(A) = Rank_q(\sum_{i \in A} e_i)$.

Theorem

1. Let $q \in Hom_+(n, n)$ be *POS*-hyperbolic polynomial . If $1 \leq Rank_q(e_1) = k \leq n$ then

$$Cap(q_{x_1}) \geq g(k)Cap(q), g(k) = \left(\frac{k-1}{k}\right)^{k-1}.$$

2. Let $q(x_1, x_2, \dots, x_n)$ be a *POS*-hyperbolic (homogeneous) polynomial of degree n . Then either the polynomial $q_{x_1} = 0$ or q_{x_1} is *POS*-hyperbolic . If $Cap(q) > 0$ then q_{x_1} is (nonzero) *POS*-hyperbolic .

Corollary

Let $PHP(n) \subset Hom_+(n, n)$ be a set of *POS*-hyperbolic polynomial of degree n in n variables ; define $PHP_+(n) = \{p \in PHP(n) : Cap(p) > 0\}$. Then as $\cup_{n \geq 1} (PHP(n) \cup \{0\})$ as well $\cup_{n \geq 1} PHP_+(n)$ is **VDW-FAMILY** .

Second Part was almost known : Rolle's theorem + a bit of pertubrations .

First is ... a particularly easy case of the Van der Waerden Conjecture .

$$q(t, x_2, \dots, x_n) = q(0, x_2, \dots, x_n) + tq_{x_1}(x_2, \dots, x_n) + \dots c_k t^k = R(t).$$

Fix a positive $n-1$ -dim. vector (x_2, \dots, x_n) , $x_2 \dots x_n = 1$.

We know that $R(t) \geq \text{Cap}(q)t$, $t \geq 0$ and want to prove

$$\text{that } q_{x_1}(x_2, \dots, x_n) = R'(0) \geq \left(\frac{k-1}{k}\right)^{k-1} \text{Cap}(q).$$

The fact that q is *POS*-hyperbolic implies that

$$R(t) = \prod_{1 \leq i \leq k} (a_i t + b_i) : a_i, b_i > 0.$$

Consider $k \times k$ matrix

$$A = [a|d|d|, \dots, |d], a = (a_1, \dots, a_n)^T, d = \frac{1}{n-1}(b_1, \dots, b_n)^T$$

.

Then $\frac{(k-1)!}{(k-1)^{k-1}}R'(0) = \text{per}(A)$ and $\text{Cap}(\text{Mul}_A) \geq \text{Cap}(q)$.

Sinkhorn's Scaling : $A = \text{Diag}_1 B \text{Diag}_2, B \in \Omega_n :$
 $\text{Cap}(q) \leq \text{Cap}(\text{Mul}_A) = \det(\text{Diag}_1 \text{Diag}_2),$

$$\text{per}(A) = \det(\text{Diag}_1 \text{Diag}_2) \text{per}(B) \geq \frac{(k)!}{(k)^k}$$

Which gives that

$$R'(0) = \left(\frac{(k-1)!}{(k-1)^{k-1}}\right)^{-1} \text{per}(A) \geq$$

$$\geq \frac{(k)!}{(k)^k} \left(\frac{(k-1)!}{(k-1)^{k-1}}\right)^{-1} \text{Cap}(q) = \left(\frac{k-1}{k}\right)^{k-1} \text{Cap}(q)$$

Another proof based on the Newton's inequalities :

Let $R(t) = \sum_{0 \leq i \leq n} d_i t^i$ be an univariate polynomial with real coefficients. If such polynomial R has all real roots then its coefficients satisfy the following Newton's inequalities :

$$NIs : d_i^2 \geq d_{i-1} d_{i+1} \frac{\binom{n}{i}^2}{\binom{n}{i-1} \binom{n}{i+1}} : 1 \leq i \leq n - 1.$$

The following weak Newton's inequalities $WNIs$ follow from NIs if the coefficients are nonnegative:

$$WNIs : d_i d_0^{i-1} \leq \frac{d_1^i}{n} \binom{n}{i} : 2 \leq i \leq n.$$

Lemma

Let $R(t) = \sum_{0 \leq i \leq n} d_i t^i$ be an univariate polynomial with real nonnegative coefficients satisfying weak Newton's inequalities $WNIs$. If for some positive real number C the inequality $R(t) \geq Ct$ holds for all $t \geq 0$ then

$$d_1 \geq C \left(\frac{n-1}{n} \right)^{n-1}.$$

Proof:

If $d_0 = 0$ then $d_1 \geq C > C((\frac{n-1}{n})^{n-1})$. Thus we can assume that $d_0 = 1$. It follows from weak Newton's inequalities *WNIs* that

$$d_i \leq \left(\frac{d_1}{n}\right)^i \binom{n}{i} : 2 \leq i \leq n.$$

Therefore for nonnegative values of $t \geq 0$ we get the inequality

$$R(t) \leq 1 + \left(\frac{d_1 t}{n}\right) \binom{n}{1} + \left(\frac{d_1 t}{n}\right)^2 \binom{n}{2} + \dots + \left(\frac{d_1 t}{n}\right)^n \binom{n}{n} = \left(1 + \frac{d_1 t}{n}\right)^n.$$

Which gives the inequality $(1 + \frac{d_1 t}{n})^n \geq Ct$. The inequality $d_1 \geq C((\frac{n-1}{n})^{n-1})$ follows now easily. Indeed consider the following optimization problem $\min_{t>0} \log((1 + \frac{d_1 t}{n})^n) - \log(t)$. Its only minimizer is $t = \frac{n}{d_1(n-1)}$. Which gives the next inequality :

$$d_1 \left(\frac{n}{n-1}\right)^{n-1} = \min_{t>0} \left(1 + \frac{d_1 t}{n}\right)^n t^{-1} \geq$$

$$\geq \inf_{t>0} \frac{R(t)}{t} \geq C.$$

We finally get that $d_1 \geq C\left(\left(\frac{n-1}{n}\right)^{n-1}\right)$.

Consider the class of the Minkowski polynomials :

$Vol_{\mathbf{C}}(x_1, \dots, x_n) = Vol(x_1C_1 + \dots + x_nC_n)$, where C_i are convex compact subsets of R^n .

The Minkowski polynomials are not generally hyperbolic if $n \geq 3$. But the previous Lemma allows to prove that there exists a **VDW-FAMILY** containing the Minkowski polynomials .

This leads to a randomized (we need to evaluate $Vol(x_1C_1 + \dots + x_nC_n)$) poly-time algorithm to approximate the mixed volume

$$M(C_1, \dots, C_n) = \frac{\partial^n}{\partial x_1 \dots \partial x_n} Vol_{\mathbf{C}}(0, \dots, 0)$$

within a multiplicative factor e^n . The best result up to date .

Algorithmic Applications

Theorem

1. Let $p \in Hom_+(n, n)$ be *POS*-hyperbolic polynomial . Then the function $Rank_p(\sum_{i \in A} e_i) = S_p(A)$ is submodular , i.e. $S_p(A \cup B) \leq S_p(A) + S_p(B) - S_p(A \cap B) : A, B \subset \{1, 2, \dots, n\}$.
2. Consider a nonnegative integer vector

$$r = (r_1, \dots, r_n), \sum_{1 \leq i \leq n} r_i = n .$$

Then

$$r \in \text{supp}(p) \text{ iff } r(S) = \sum_{i \in S} r_i \leq S_p(S) : S \subset \{1, 2, \dots, n\} .$$

Corollary

Let $p \in Hom_+(n, n)$ be *POS*-hyperbolic polynomial. Associate with this polynomial p the following bounded convex polytope :

$$SUB_p = \{(x_1, \dots, x_n) : \sum_{i \in S} x_i \leq S_p(S) : S \subset \{1, 2, \dots, n\}\};$$

$$\sum_{1 \leq i \leq n} x_i = n; x_i \geq 0, 1 \leq i \leq n \}$$

$$\sum_{1 \leq i \leq n} x_i = n; x_i \geq 0, 1 \leq i \leq n \}.$$

Then SUB_p is equal to the Newton polytope of p , i.e.

$$SUB_p = CO(supp(p)).$$

Corollary

Given POS -hyperbolic polynomial $p \in Hom_+(n, n)$ as an oracle, there exists strongly polynomial-time oracle algorithm for the membership problem as for $supp(p)$ as well for the Newton polytope $CO(supp(p))$.

The membership problem for $supp(p)$ is **NP-HARD** for general $p \in Hom_+(n, n)$:

$$\text{Define } Bar(x_1, \dots, x_n) = tr((Diag(x_1, \dots, x_n)A)^n),$$

then

$$\frac{1}{n} \frac{\partial^n}{\partial x_1 \dots \partial x_n} \text{Bar}(0, \dots, 0) =$$

the number of Hamiltonian circuits in the graph defined by a boolean matrix A .

Or , let $F = \{S_1, \dots, S_m\} : S_i \subset \{1, 2, \dots, n\}, |S_i| = k; \frac{n}{k} \in \mathbb{Z}$.

Define $COV_F(x_1, \dots, x_n) = (\sum_{S_j \in F} \prod_{i \in S_j} x_i)^{\frac{n}{k}}$.

Then $(k!)^{-1} \frac{\partial^n}{\partial x_1 \dots \partial x_n} COV_F(0, \dots, 0) =$

the number of exact coverings .

There exists a deterministic polynomial-time oracle algorithm which computes for given as an oracle indecomposable *POS*-hyperbolic polynomial $p(x_1, \dots, x_n)$ a number $F(p)$ satisfying the inequality

$$\begin{aligned} \frac{\partial^n}{\partial x_1 \dots \partial x_n} p(0, \dots, 0) &\leq F(p) \leq \\ &\leq 2 \left(\prod_{1 \leq i \leq n} g(\min(S_p(\{i\}), i)) \right)^{-1} \frac{\partial^n}{\partial x_1 \dots \partial x_n} p(0, \dots, 0) \leq \\ &\leq 2 \frac{n^n}{n!} \frac{\partial^n}{\partial x_1 \dots \partial x_n} p(0, \dots, 0) . \end{aligned}$$

The prev. result can be (slightly) improved . I.e. it can be applied to the polynomial

$$p_k(x_{k+1}, \dots, x_n) = \frac{\partial^k}{\partial x_1 \dots \partial x_k} p(0, \dots, 0, x_{k+1}, \dots, x_n).$$

Notice that the polynomial p_k is a homogeneous polynomial of degree $n - k$ in $n - k$ variables .

If $p = p_0$ is *POS*-hyperbolic and $Cap(p) > 0$ then for all $0 \leq k \leq n$ the polynomials p_k are also *POS*-hyperbolic and $Cap(p_k) > 0$.

Also , if $p = p_0$ is indecomposable then p_k is indecomposable as well (Theorem 4.6).

The trick is that if $k = m \log_2(n)$ then (using the polarizational formula) the polynomial p_k can be evaluated using $O(n^{m+1})$ oracle calls of the (original) polynomial $p = p_0$.

This observations allows to decrease the worst case multiplicative factor from e^n to $\frac{e^n}{n^m}$ for any fixed m . If the polynomial $p = p_0$ can be explicitly evaluated in de-

deterministic polynomial time , this observation results
 in deterministic polynomial time algorithms to approx-
 imate $\frac{\partial^n}{\partial x_1 \dots \partial x_n} p(0, \dots, 0)$ within multiplicative factor $\frac{e^n}{n^m}$
 for any fixed m . Which is an improvement of results
 in [LSW] (permanents , p is a multilinear polynomial)
 and in [GS] (mixed discriminants, p is a determinantal
 polynomial) .