Topology, Lattices, and Logic Programming

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Background

- How to assign meanings to a logic program $P$, e.g.
  - $\text{odd}(s(0))$,
  - $\text{odd}(x) \rightarrow \text{odd}(ss(x))$.

- Herbrand universe: $U_P = \{ s^i 0 \mid i \geq 0 \}$

- Herbrand base: all ground atomic formulas formed using terms from $U_P$ and predicates in $P$.

- $\text{ground}(P)$: the set of ground instances of $P$.
  - $\text{odd}(s0)$,
  - $\text{odd}(0) \rightarrow \text{odd}(ss0)$,
  - $\text{odd}(s0) \rightarrow \text{odd}(sss0)$, ...

- The meaning of logic programs reduces to the interpretation of a set of “implications” of the form
  - $X \rightarrow a$ or $X \rightarrow Y$
Motivation

Fixed-point semantics has a lot to do with topology. It is the topological property of the immediate consequence operator $T_P$ that determines the property of the semantics. (E.g. existence and uniqueness)
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- This leads to frames/locales, the abstract notion of topology which takes open sets as the starting point.

- Paradigm: open sets as propositions, points as models. Open sets first, points secondary.
Overview

Approach: generate an abstract topological space (frame) from primitive data \( \text{ground}(P) \), then recover “interpretations” (models) as “points” derived from the topology as completely prime filters.
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Outline of this talk: coverage relations, definite programs, logic programs with negation, disjunctive logic programs, other issues.
Given \((A, \vdash)\), where \(\vdash\) is a set of implications \(X \vdash a\), with \(X\) a finite subset of \(A\) and \(a\) a member of \(A\).

Interpretation: think of each element of \(A\) as an open set, and each implication as containment \(\bigcap X \subseteq a\).

Question: which topological space?

The “topological space” consists of all finite meets \(\land\) and arbitrary joins \(\lor\) generated from \(A\), subject to the interpretation of constraints \(\vdash\) given above.

How to generate a frame \(\text{Frm}(A)\) from \((A, \vdash)\)?
Frames and coverage relations

- A frame (locale) is a poset with finite meets and arbitrary joins which satisfies the infinite distributive law
  \[ x \land \bigvee Y = \bigvee \{ x \land y \mid y \in Y \}. \]

- A frame morphism is a function \( f : F \to G \) that preserves finite meets and arbitrary joins.

- Let \((S, \land, \leq)\) be a meet-semi-lattice. A coverage on \(S\) is a relation \( \succ \subseteq 2^S \times S \) satisfying
  - if \( Y \succ a \) then \( Y \subseteq \downarrow a \).
  - if \( Y \succ a \) then for any \( b \leq a \), \( \{ y \land b \mid y \in Y \} \succ b \).

- A coverage relation (or coverage) \( \succ \) is compact if
  \[ Y \succ a \text{ implies } X \succ a \text{ for some finite } X \subseteq^{\text{fin}} Y. \]
A meet-semi-lattice $S$ with a coverage $\succ$ is called a site. A frame $\text{Frm}(S)$ is generated from $(S, \succ)$ if there exists $i$ s.t.

- $i : S \rightarrow \text{Frm}(S)$ preserves finite meets,
- $i$ transforms covers to joins: $Y \succ a \Rightarrow i(a) = \bigvee i(Y)$, and
- $\text{Frm}(S)$, $i$ is universal, i.e., for any frame $F$ and any meet-preserving and cover-to-join transforming function $f : S \rightarrow F$, there exists a unique frame morphism $g : \text{Frm}(S) \rightarrow F$ s.t. the following diagram commutes:

$$
\begin{array}{ccc}
S & \xrightarrow{f} & F \\
\downarrow i & & \downarrow g \\
\text{Frm}(S) & \xrightarrow{?} & F \\
\end{array}
$$
Ideals and filters

- An *ideal* of a poset is a lower closed, directed subset of the poset.
  In a lattice, an ideal is a $\lor$-closed, lower set.

- A *filter* of a frame $F$ is a subset $u \subseteq F$ which is $\land$-closed, upper set. Ideals always contain the bottom element and filters always contain the top element.
  A filter $u$ of a frame $F$ is *completely prime* if $\bigvee P \in u \Rightarrow P \cap u \neq \emptyset$ for any $P \subseteq u$. 
Given a site $(S, \succ)$, a $\succ$-ideal is a subset $I$ of $S$ which is
- lower-closed: $a \in I \& b \leq a \Rightarrow b \in I$,
- covered: $U \succ a \& U \subseteq I \Rightarrow a \in I$.

Example. Let $D$ be a distributive lattice. Let the coverage be defined as $U \succ a$ if
- $U \subseteq \downarrow a$ and
- $\exists X \subseteq^\text{fin} U, a = \vee X$

A $\succ$-ideal is then exactly an ideal of $D$ in this case.

Definition. A frame (locale) is said to be spectral if it is isomorphic to the ideal completion of a distributive lattice.
Coverage theorem

- Theorem (Johnstone82). The collection of $\succ$-ideals under inclusion is the frame generated from a site $(S, \succ)$.
- Definition. A point of a frame is a completely prime filter.
- Fact. If $H$ is generated from $(S, \succ)$ (with $i$) then points are exactly filters $F$ of $S$ such that
  $$i(a) \in F \& Y \succ a \Rightarrow (\exists b \in Y) \ i(b) \in F$$
- Definition. Frame $H$ is spatial if for any $a, b \in H$,
  $$a \leq b \text{ iff } \forall \text{ point } F, \ a \in F \Rightarrow b \in F.$$  
- Fact. Spectral frames are spatial.
Recall: a coverage relation (or coverage) \( \succ \) is called compact if for every \( X \subseteq S \) and every \( a \in S \),
\[ X \succ a \text{ implies } Y \succ a \text{ for some finite } Y \subseteq^{\text{fin}} X. \]

Lemma. If \((S, \succ)\) is a site for which the coverage relation \( \succ \) is compact, then for any directed set \( F \) of \( \succ \)-ideals,
\[ \bigvee F = \bigcup F. \]

Lemma. Suppose \((S, \succ)\) is a site and \( \succ \) is compact. Then a \( \succ \)-ideal is a compact element in the generated frame if and only if it is generated by a finite subset of \( S \).

Compact Coverage Theorem (Z.03). A frame is spectral iff it can be generated from a compact coverage relation.
Definition. An information system is a pair \((A, \sqsubseteq)\) such that the relation \(\sqsubseteq \subseteq \text{Fin}(A) \times A\) is reflexive and transitive.

Definition. An ideal element of \(A\) is a subset \(x \subseteq A\) such that \(X \subseteq x \& X \sqsubseteq a \Rightarrow a \in x\).

Theorem (Scott82) For any information system \((A, \sqsubseteq)\), the set of ideal elements under inclusion \((|A|, \subseteq)\) is a complete algebraic lattice. Conversely, any complete algebraic lattice is order-isomorphic to one from some information system.
A definite logic program \((A, \models)\) gives rise to a site \((A^\wedge, \not\supset)\), with \(A^\wedge\) the freely generated meet-semi-lattice from \(A\), and \(\{a \land (\land X)\} \not\supset \land X \iff X \models a\).

(Note \(\bigcap X \subseteq a\) iff \(\bigcap X = a \cap \bigcap X\))

Proposition. This compact coverage relation generates a spectral frame \(\text{Frm}(A)\). The “points” of the frame are in 1-1 correspondence with ideal elements of \(|A|\).

The Compact Coverage Theorem implies that \(\leq\) is sound and complete with respect to these models. In particular, \(X \models a\) if for each point \(x\), \(x \models X \Rightarrow x \models a\).

Moreover, since \(\models\) is “embedded in” \(\leq\), we obtain the “derived rules”, e.g. reflexivity and transitivity.
Scott topology.

A set is *Scott open* if it is upwards closed and inaccessible by lubs of directed sets. Sets of the form 

$$[X] := \{ x \mid X \subseteq x \land x \in |A| \}$$

form a *basis* of the Scott topology over \((|A|, \subseteq)\).

From \(\triangleright\)-ideals \(u\) to Scott opens: 

$$u \longmapsto \bigcup \{ [X] \mid \wedge X \in u \}$$

From Scott opens \(O\) to \(\triangleright\)-ideals: 

$$O \longmapsto \{ \wedge X \mid [X] \subseteq O \}$$

Consistent with Fitting85, Fitting87, Seda-Hitzler95, 99, Batarekh-Subrahmanian89, Rounds-Z.01, Z.-Rounds01
Coverage for negation

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- Coverage relation: for each rule $X, \neg Y \rightarrow a$, for each $x$ s.t. $x \land y \leq 0$ for all $y \in Y$, put
  $$\{a \land x \land z \land (\land X) \mid \forall y \in Y (z \land y \leq 0)\} \succ x \land (\land X)$$
**Coverage for negation**

- $P$ consists of implications of the form $X, \neg Y \rightarrow a$

- Coverage relation: for each rule $X, \neg Y \rightarrow a$, for each $x$ s.t. $x \land y \leq 0$ for all $y \in Y$, put
  \[
  \{a \land x \land z \land (\land X) \mid \forall y \in Y \ (z \land y \leq 0)\} \supset x \land (\land X)
  \]

- By the Coverage Theorem, we have (in the frame)
  \[
  \forall x (. ) x \land (\land X) = \lor \{a \land x \land z \land (\land X) \mid \forall y \in Y \ (z \land y \leq 0)\}
  \]
  iff $\lor \{x \land (\land X) \mid x \land (\forall Y) \leq 0\}$ (Simplifying the right $\lor \lor$ to $\lor$)
  \[
  = \lor \{a \land z \land (\land X) \mid z \land (\forall Y) \leq 0\}
  \]
  iff $(\land X) \land \lor \{x \mid x \land (\forall Y) \leq 0\}$
  \[
  = a \land (\land X) \land \lor \{z \mid z \land (\forall Y) \leq 0\}
  \]
  iff $(\land X) \land \land \{-b \mid b \in Y\} \leq a$

  Note that $\neg b := b \rightarrow 0 := \lor \{x \mid x \land b \leq 0\}$
Strong negation

- Parameter: the underlying semi-lattice, corresponding intuitively to a “basis” of the topology we want to generate.

\[(\text{Fin}(A \cup \bar{A}), \cup, \supseteq)\] subject to \(a \land \bar{a} \leq 0\) for all \(a \in A\).

- Coverage relation: for each rule \(X, \neg Y \rightarrow a\), for each \(x\) s.t. \(x \land y \leq 0\) for all \(y \in Y\), put
  \[
  \{a \land x \land z \land (\land X) \mid \forall y \in Y \ (z \land y \leq 0)\} \supset x \land (\land X)
  \]

- But now there is a unique largest element \(\land \bar{Y}\) for which \(\land \bar{Y} \land y \leq 0\) for each \(y \in Y\). So the coverage relation reduces to \(\{a \land \land \bar{Y} \land (\land X)\} \supset \land \bar{Y} \land (\land X)\) for each rule \(X, \neg Y \rightarrow a\).

- By the Coverage Theorem, we have \((\land X) \land \land \bar{Y} \leq a\) for each \(X, \neg Y \rightarrow a\), in the generated frame.
Weak negation (patch/Lawson topology)

- \((\text{Fin}(A \cup \overline{A}), \cup, \supseteq)\) subject to \(a \lor \overline{a} = 1\) and \(a \land \overline{a} = 0\) for all \(a \in A\).

- Coverage relation: for each rule \(X, \neg Y \rightarrow a\), for each \(x\) s.t. \(x \land y \leq 0\) for all \(y \in Y\), put
  \[\{a \land x \land z \land (\land X) \mid \forall y \in Y (z \land y \leq 0)\} \supset x \land (\land X)\]

- \(\land \overline{Y}\) the largest element for which \(\land \overline{Y} \land y \leq 0\) for each \(y \in Y\). So the coverage relation reduces to
  \[\{a \land \land \overline{Y} \land (\land X)\} \supset \land \overline{Y} \land (\land X)\] for each rule \(X, \neg Y \rightarrow a\).

- By the Coverage Theorem, we have \((\land X) \land \land \overline{Y} \leq a\) for each \(X, \neg Y \rightarrow a\), in the generated frame.
Disjunctive logic programs

- **Definition.** A sequent structure \((A, \vdash)\) is a set of implications \(X \vdash Y\), with \(X, Y\) finite subsets of \(A\).

- **Interpretation:** think of each element of \(A\) as an open set, and each implication as containment \(\bigcap X \subseteq \bigcup Y\).

- **Coverage:** \(Y \succ \bigwedge X\)?

- **Not quite.** Here is the fix: \(\{b \land (\bigwedge X) \mid b \in Y\} \succ \bigwedge X\).

- **\succ**-ideals are subsets \(U \subseteq \text{Fin}(A)\) such that
  - if \(X \in U\) and \(Y \supseteq X\), then \(Y \in U\);
  - if \(\{a_1 \land (\bigwedge X), \ldots, a_n \land (\bigwedge X)\} \subseteq U\) and \(X \vdash a_1, \ldots, a_n\), then \(\bigwedge X \in U\).
For $U \subseteq \text{Fin}(A)$, write $cU$ for the $\triangleright$-ideal generated by $U$.

Corollary. For any sequent structure $(A, \vdash)$, the set of its $\triangleright$-ideals $H_0$ is a frame under inclusion. Moreover, the interpretation $m_0 : A \rightarrow H_0$ mapping $a$ to $c\{a\}$ is universal. Furthermore we have $X \vdash Y$ if and only if $\wedge m_0(X) \leq \vee m_0(Y)$ for all finite subsets $X, Y$ of $A$.

Definition. $x \subseteq A$ is an ideal element if for each instance $X \vdash Y$ of $\vdash$, $X \subseteq x$ implies $x \cap Y \neq \emptyset$.

Ideal elements corresponds to completely prime filters in $H_0$. Therefore, if for each ideal element $x$, $X \subseteq x$ implies $x \cap Y \neq \emptyset$, then $X \vdash Y$. 


With respect to a sequent structure \((A, \vdash)\), a *clause* is a finite subset of \(A\). A *clause set* is a collection of clauses.

An ideal element \(x\) is a *model of a clause* \(u\) if \(x \cap u \neq \emptyset\). \(x\) is a *model of a clause set* \(W\) if it is a model of every clause in \(W\).

Define

\[
W \models u \quad \text{if any model of } W \text{ is a model of } u
\]

\[
W \vdash^*_{hr} u \quad \text{if either } \emptyset \in W, \text{ or } u \text{ can be deduced from } W
\]

by the HR rule

\[
\frac{a_1, X \quad \ldots \quad a_n, X}{X} \quad \text{(if } a_1, \ldots, a_n \vdash X)\]

\[
\{X_1, \ldots, X_n\} \rightarrow u \quad \text{if for any choice}
\]

\[
a_1 \in X_1, \ldots, a_n \in X_n, \{a_i \mid 1 \ldots n\} \vdash u.
\]

Theorem. \(\models = \vdash^*_{hr} = \rightarrow u\)
Generating \( \models \)-closed clause sets

- Given sequent structure \((A, \models)\), consider the freely generated distributive lattice from \(A\) as the underlying meet-semi-lattice.

- Coverage relation:
  - \( X \triangleright \forall X \),
  - \( X \triangleright (a_1 \lor (\forall X)) \land \ldots \land (a_n \lor (\forall X)) \) if \( a_1, \ldots, a_n \models X \).

- Theorem (Coquand-Z.01). \( u \) is a \( \triangleright \)-ideal iff \( u \) is a \( \models \)-closed clause set.
Concluding remarks

- A general topological approach to construct semantic models.
- Completeness ensured by spatiality.
- Inference rules derived equationally.
- Treated definite logic programs, disjunctive logic programs, and negation.
- Stable model semantics and other semantics?
- Metrics?