Lipschitz Quotients

[S. Bates], W.B.J., J. Lindenstrauss, D. Preiss, G. Schechtman

Background

A mapping \( f : X \to Y \), is a co-Lipschitz map provided there is a constant \( C \) so that for all \( x \) in \( X \) and all \( r \),

\[
f[B_r(x)] \supset B_{r/C}(f(x)).
\]

co-Lip\((f)\) denotes the smallest such \( C \).

A co-Lipschitz map is open in a Lipschitz sense. A function is a **Lipschitz quotient map** if it is both Lipschitz and co-Lipschitz. Thus a one-to-one map is a Lipschitz quotient mapping iff it is bi-Lipschitz.

If there is a Lipschitz quotient map \( f \) from \( X \) onto \( Y \), we say \( Y \) is a Lipschitz quotient of \( X \) (\( \lambda \)-Lipschitz quotient if \( \text{Lip}(f) \cdot \text{co-Lip}(f) \leq \lambda \)).
A mapping \( f : X \to Y \), is a **co-Lipschitz** map provided there is a constant \( C \) so that for all \( x \) in \( X \) and all \( r \),

\[
f[B_r(x)] \supset B_{r/C}(f(x)).
\]

Related concept [David-Semmes]

\( T : X \to Y \) is ball non collapsing provided \( \exists \ \omega > 0 \) s.t. \( \forall x \in X \ \exists y \in Y \) s.t.

\[
TB_r(x) \supset B_{\omega r}(y).
\]

Example of a ball non collapsing Lipschitz map which is NOT a Lipschitz quotient: fold a sheet of paper.
Examples of Lipschitz quotients maps in $\mathbb{R}^n$

From $\mathbb{R}$ to $\mathbb{R}$ they must be bi-Lipschitz.

From $\mathbb{R}^2$ to $\mathbb{R}$, they carry considerable structure. For example, the number of components of $f^{-1}(t)$ is bounded and each component of $f^{-1}(t)$ separates the plane.

Define $f$ on $\mathbb{R}^2$ to be the homogenous extension to $\mathbb{R}^2$ of the mapping $z \mapsto z^n$ on the unit circle. This is a Lipschitz quotient mapping which is “typical” – EVERY Lipschitz quotient map on $\mathbb{R}^2$ can be written as $P \circ h$ where $P$ is a (complex) polynomial and $h$ is a homeomorphism of $\mathbb{R}^2$.

From $\mathbb{R}^3$ to $\mathbb{R}^2$, $f^{-1}(t)$ can contain a plane but cannot be a plane. [Csornyei]

References for non linear quotients in $\mathbb{R}^n$: [JLPS], [Csornyei], [Heinrich], [Randriantoanina], [Maleva].
A mapping $f : X \to Y$, is a **co-Lipschitz** map provided there is a constant $C$ so that for all $x$ in $X$ and all $r$,

$$f[B_r(x)] \supset B_{r/C}(f(x)).$$

Let $f : X \to Y$ be a surjective Lipschitz map. Then co-Lip$(f) < \lambda$ iff for all finite weighted trees $T$, $t_0 \in T$, $g : T \to Y$ with Lip$(g) \leq 1$, and $x_0 \in X$ with $f(x_0) = g(t_0)$, there exists a lifting $\tilde{g} : T \to X$ so that $\tilde{g}(t_0) = x_0$, Lip$(\tilde{g}) \leq \lambda$, and $g = f \circ \tilde{g}$. 
For Banach spaces, the fundamental question is:

If $Y$ is a Lipschitz quotient of $X$, [when] must $Y$ be a linear quotient of $X$?

In every case where we know “$Y$ is a Lipschitz quotient of $X$ $\implies$ $Y$ is a linear quotient of $X$” we also know that the existence of a ball non collapsing Lipschitz map from $X$ to $Y$ implies that $Y$ is a linear quotient of $X$.

We do not know whether in general the existence of a ball non collapsing Lipschitz map from $X$ to $Y$ implies that $Y$ is a Lipschitz quotient of $X$. 
$f$ admits **affine localization** if for every $\varepsilon > 0$ and every ball $B \subset X$ there is a ball $B_r \subset B$ and an affine function $L : X \to Y$ so that
\[
\| f(x) - Lx \| \leq \varepsilon r, \quad x \in B_r.
\]

The couple $(X, Y)$ has the **approximation by affine property (AAP)** if every Lipschitz map from $X$ into $Y$ admits affine localization.

AAP is enough to ensure that if $f$ is a Lipschitz quotient map from $X$ to $Y$ then (for $\varepsilon$ small enough) the linear approximant is a linear quotient map; and if $f$ is a $\lambda$-Lipschitz quotient, (i.e., $\text{Lip}(f) \cdot \text{co-Lip}(f) \leq \lambda$) the linear approximant is a $\lambda + \varepsilon$ linear quotient.
$f$ admits $\delta$-affine localization if for every $\varepsilon > 0$ and every ball $B \subset X$ there is a ball $B_r \subset B$ and an affine function $L : X \to Y$ so that
\[ \|f(x) - Lx\| \leq \varepsilon r, \quad x \in B_r \]
and $r \geq \delta(\varepsilon) \text{radius}(B)$ ($\delta(\varepsilon) > 0$ $\forall \varepsilon > 0$).

The couple $(X, Y)$ has the uniform approximation by affine property (UAAP) if there is a function $\delta(\varepsilon) > 0$ so that every Lipschitz map with constant one from $X$ into $Y$ admits $\delta$-affine localization.

This notion (not the terminology) was introduced by [David-Semmes]. They proved that $(X, Y)$ has the UAAP if both spaces are finite dimensional.

**Theorem.** The couple $(X, Y)$ has the UAAP iff one of the spaces is super-reflexive and the other is finite dimensional.

A Banach space is super-reflexive iff it is isomorphic to a uniformly convex space iff it is isomorphic to a uniformly smooth space.
Repeat:

(1) If \((X, Y)\) has the AAP and \(Y\) is a \(\lambda\)-Lipschitz quotient of \(X\) then \(Y\) is a \((\lambda + \epsilon)\)-isomorphic to a linear quotient of \(X\).

(2) If \(X\) is super-reflexive and \(Y\) is finite dimensional, then \((X, Y)\) has the AAP.

Therefore:

(3) If \(X\) is super-reflexive and \(Z\) is a \(\lambda\)-Lipschitz quotient of \(X\), then every finite dimensional quotient of \(Z\) is \((\lambda + \epsilon)\)-isomorphic to a linear quotient of \(X\) (\(\iff\) every finite dimensional subspace of \(Z^*\) is \((\lambda + \epsilon)\)-isomorphic to a subspace of \(X^*\)).

(4) If \(Z\) is a \(\lambda\)-Lipschitz quotient of a Hilbert space, then \(Z\) is \(\lambda\)-isomorphic to a Hilbert space.

(5) If \(Z\) is a \(\lambda\)-Lipschitz quotient of \(L_p\), \(1 \leq p < \infty\), then \(Z\) is \(\lambda\)-isomorphic to a quotient of \(L_p\).
The classification of Lipschitz quotients of $\ell_p$, $1 < p \neq 2 < \infty$ is open. A Lipschitz quotient of $\ell_p$ is a Lipschitz quotient of $L_p$. For $2 \leq r < p < \infty$, the space $\ell_r$ is linear quotient of $L_p$ but is not a Lipschitz quotient of $\ell_p$.

There are known to exist non separable Banach spaces $X$ and $Y$ which are bi-Lipschitz equivalent but not isomorphic [Aharoni-Lindenstrauss]. It turns out that $Y$ is not even a isomorphic to a linear quotient of $X$.

It may be that separable Banach spaces that are bi-Lipschitz equivalent must be isomorphic. The results on quotients suggest that if $X$ is separable and $Y$ is a Lipschitz quotient of $X$, then $Y$ is isomorphic to a linear quotient of $X$ (at least if $X$ is one of the classical examples of Banach spaces). However,....
A metric space $X$ is a *metric tree* provided it is complete, metrically convex, and there is a unique arc (which then by metric convexity must be a geodesic arc) joining each pair of points in $X$. There is an equivalent constructive definition of a separable metric tree, which we term an SMT because the equivalence to separable metric tree is not needed. Using the constructive definition, it is more-or-less clear that every metric tree is obtained by starting with a (possibly infinite) weighted tree and filling in each edge with an interval whose length is the distance between the vertices of the edge.

**The $\ell_1$ union of two metric spaces**

If $X \cap Y = \{p\}$, the $\ell_1$ union is $(X \cup Y, d)$, where the metric $d$ agrees with $d_X$ on $X$, $d$ agrees with $d_Y$ on $Y$, and if $x \in X$, $y \in Y$, then $d(x, y)$ is defined to be $d_X(x, p) + d_Y(p, y)$. 

**Metric trees and Lipschitz Quotients of spaces containing $\ell_1$ [JLPS]**
Construction of an SMT

Let $I_1$ be a closed interval or a closed ray and define $T_1 := I_1$. The metric space $T_1$ is the first approximation to our SMT. Having defined $T_n$, let $I_{n+1}$ be a closed interval or a closed ray whose intersection with $T_n$ is an end point, $p_n$, of $I_{n+1}$, and define $T_{n+1} := T_n \cup_1 I_{n+1}$. The completion, $T$, of $\bigcup_{n=1}^{\infty} T_n$ is an SMT. If each $I_n$ is a ray with end point $p_{n-1}$ for $n > 1$ and the set $\{p_n\}_{n=1}^{\infty}$ of nodal points is dense in $T$, then we call $T$ an ‘$\ell_1$ tree’ and say that $\{I_n\}_{n=1}^{\infty}$, $\{T_n\}_{n=1}^{\infty}$, $\{p_n\}_{n=1}^{\infty}$ describe an allowed construction of $T$.

**Proposition.** Let $T$ be an $\ell_1$ tree. Then every separable, complete, metrically convex metric space is a 1-Lipschitz quotient of $T$. 
Proposition. Let $T$ be an $\ell_1$ tree. Then every separable, complete, metrically convex metric space is a $1$-Lipschitz quotient of $T$.

Let $Y$ be a separable, complete, metrically convex metric space. Build the desired Lipschitz quotient map by defining it on $T_n$ by induction (where $\{I_n\}_{n=1}^{\infty}$, $\{T_n\}_{n=1}^{\infty}$, $\{p_n\}_{n=1}^{\infty}$ describe an allowed construction of $T$).

Suppose you have a $1$-Lipschitz map $f : T_n \to Y$, and $y$ is taken from some countable dense subset $Y_0$ of $Y$. Extend $f$ to $T_{n+1}$ by mapping $I_{n+1}$ to a geodesic arc $[f(p_n), y]$ which joins $f(p_n)$ to $y$; $f$ is an isometry on $\{z \in I_{n+1} : d(p_n, z) \leq d(f(p_n), y)\}$ and $f$ maps points on $I_{n+1}$ whose distance to $p_n$ is larger than $d(f(p_n), y)$ to $y$. This makes $f$ act like a Lipschitz quotient at $p_n$ relative to $[f(p_n), y]$. Since the nodal points are dense in $T$, a judicious selection of the points from $Y_0$ will produce a $1$-Lipschitz quotient mapping.
Lemma. Assume that $X$ and $Y$ are 1-absolute Lipschitz retracts which intersect in a single point, $p$. Then $X \cup_1 Y$ is also a 1-absolute Lipschitz retract.

A metric space $X$ is a 1-absolute Lipschitz retract if and only if $X$ is metrically convex and every collection of mutually intersecting closed balls in $X$ have a common point.

Corollary. Let $T$ be an SMT. Then $T$ is a 1-absolute Lipschitz retract.
**Proposition.** Every SMT is a 1-Lipschitz quotient of $C(\Delta)$, where $\Delta$ is the Cantor set $\{-1, 1\}^\mathbb{N}$.

Let $r_n$ be the $n$th coordinate projection on $\Delta$. In the space $C(\Delta)$, the sequence $\{r_n\}_{n=1}^\infty$ is isometrically equivalent to the unit vector basis of $\ell_1$. For $n = 1, 2, \ldots$, let $E_n$ be the functions in $C(\Delta)$ which depend only on the first $n$ coordinates. Notice that if $x$ is in $E_n$ and $m > n$ then for all real $t$, $\|x + tr_m\| = \|x\| + |t|$. In other words, if $I$ is a ray in the direction of $r_m$ emanating from a point $p$ in $E_n$, then, in $C(\Delta)$, the set $E_n \cup I$ is an $\ell_1$ union of $E_n$ and $I$.

That $\{r_n\}_{n=1}^\infty$ acts like the $\ell_1$ basis over $C(\Delta)$ is the key to proving the above proposition. The lemma is used to extend a 1-Lipschitz mapping from $E_n \cup I$ into the SMT to a 1-Lipschitz mapping from $E_m$ into the SMT.
Corollary. If $Y$ is a separable, complete, metrically convex metric space, then $Y$ is a 1-Lipschitz quotient of $C(\Delta)$.

In particular, every separable Banach space is a 1-Lipschitz quotient of $C(\Delta)$, but it is well known that e.g. $\ell_1$ is NOT isomorphic to a linear quotient of $C(\Delta)$.

From known results in the linear theory it then follows:

Theorem. Let $X$ be a separable Banach space which contains a subspace isomorphic to $\ell_1$ and let $\varepsilon > 0$. Then every separable, complete, metrically convex metric space is a $(1 + \varepsilon)$-Lipschitz quotient of $X$. (Moreover, the Gateaux derivative of the Lipschitz quotient mapping has rank at most one wherever it exists.)
$f : X \to \mathbb{R}^n$ is measure non collapsing provided $\mu f(B_r(x)) \geq \delta r^n$; $f$ is ball non collapsing if $f(B_r(x)) \supset B_{\delta r}(y)$ ($\mu = \text{Lebesgue measure}$).

[David-Semmes] if $f : \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz and measure non collapsing then it is ball non collapsing. $\mathbb{R}^m$ can be replaced by any super-reflexive space [BJLPS].

If $X$ is a separable Banach space containing an isomorph of $\ell_1$ then $\exists f : X \to \mathbb{R}^2$ Lipschitz, measure non collapsing, but $f(X)$ is closed and has empty interior (hence $f$ is NOT ball non collapsing).
Problems and concluding remarks

(1) Classify the metric spaces which are Lipschitz quotients of a Hilbert space. In particular, must each such space bi-Lipschitz embed into a Hilbert space?

(2) Classify the metric spaces which are Lipschitz quotients of a subset of a Hilbert space.

We know only:

(2.1) There are metric spaces which are not Lipschitz quotients of any subsets of a Hilbert space.

(2.2) There are metric spaces which are Lipschitz quotients of subsets of a Hilbert space but which do not bi-Lipschitz embed into a Hilbert space.

Quantitative versions of problem 2 might be interesting.

(3) Estimate, in terms of $\lambda$ and $N$, the largest Euclidean distortion of an $N$-point metric space which is a $\lambda$-Lipschitz quotient of a subset of a Hilbert space.
Recall the definition of $\delta$-affine localization:

$f$ admits $\delta$-affine localization if for every $\varepsilon > 0$ and every ball $B \subset X$ there is a ball $B_r \subset B$ and an affine function $L : X \to Y$ so that

$$\|f(x) - Lx\| \leq \varepsilon r, \quad x \in B_r$$

and $r \geq \delta(\varepsilon) \text{radius}(B)$.

(4) Is there an analogue of $\delta$-affine localization for Lipschitz mappings between other classes of metric spaces? Maybe metric groups? Are there conditions which will guarantee that a pair $(X, Y)$ has the analogue of UAAP?