Coding Theorems for Reversible Embedding

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Outline

1. Gel'fand-Pinsker Coding Theorem
2. Noise-Free Embedding
3. Reversible Embedding
4. Robust and Reversible Embedding
5. Partially Reversible Embedding
6. Remarks
The Gelfand-Pinsker Coding Theorem

Rate: \( R = \log_2 \frac{N}{I} \).

Error probability: \( \{ M \neq M \} \Pr = 3^D \)

Channel: discrete memoryless \( \{ \mathcal{Z}, (x, y, |z) \Pr, \mathcal{X} \times \mathcal{Z} \} \)

Side information: \( \Pr \{ N x \in N X \} \)

Messages: \( \{ M, \cdots, 2, 1 \} \subseteq \{ m = M \} \Pr \)

\( (x)^D \)

\( (N Z)p = M \)

\( (x, y, z)^D \)

\( (N X, M)^D = N \lambda \)

\( N \lambda \)
Capacity

The side-information capacity $C_{si}$ is the largest such that for all large enough $N$, there exist $N$ encoders and decoders with $R - \epsilon < H$ for all $\epsilon > 0$ such that for all $\epsilon > 0$.

**Theorem (Gelfand-Pinsker [1980]):**

\[
\max_{d \geq \frac{1}{2} R} (X : \Omega) I - (Z : \Omega) I = C_{si}
\]

Achievability proof: Fix a test-channel $P_t(u | x)$. Consider sets $\mathcal{A}^\epsilon(x)$. Give these sequences $\{x_1, \ldots, x_N\}$ to the encoder. The decoder then attempts to decode the message.

For each message index $w$, choose a sequence $u_N$ such that $(u_N, x_N) \in \mathcal{A}^\epsilon$. Each sequence exists almost always if $R > I(U; X)$ (roughly).

Theorem (Gelfand-Pinsker [1980]):

\[
\max_{d \geq \frac{1}{2} R} (X : \Omega) I - (Z : \Omega) I = C_{si}
\]
that $P$ is achievable.

Observations

Conclusion is that $\mathcal{R} < I(\mathcal{U} : \mathcal{Z})$ is achievable.

A: As an intermediate result the decoder recovers the sequence $u^N$.
B: The transmitted $u^N$ is jointly typical with the side-info sequence $x^N$. Then $y^N$ is transmitted.

\[
(x | h^n, n)^T P(x)^T P(h^T \leq \Omega) = (x, n) P
\]

Note that $P$ is achievable.

\[
N^n (X | \Omega) > N^n \mathcal{R} + N^n \mathcal{R} \quad \text{i.e.} \quad (X, \Omega) \in \mathcal{A}_n \quad \text{for all } N^n \quad \text{such that } (Z | \Omega) > N^n \mathcal{R} + N^n \mathcal{R}.
\]

(d) The decoder upon receiving $z^N$ looks for the unique sequence $u^N$ such that $(u^N, z^N) \in \mathcal{A}_n$.

(c) The input sequence results from applying the "channel" $P$.

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II. Noise-free Embedding

- Embedding distortion: $\{A \in \mathcal{A} \mid x \in \mathcal{X}, (f_i' x)^{\fix} A\}$
- Rate: $\frac{\log N}{I} = H$
- Error probability: $\{M \neq M\}_P = 3^H$

Source (host): $\mathcal{X}$

Messages: $\{\mathcal{W}, \ldots, \mathcal{W}\}_P$ for $\frac{I}{W} = \{m = M\}_P$

Embedding distortion:

$E = \left[\sum_{n=1}^{N} D_{xy}(x;e_n)\right]$ for some distortion matrix $f_D(x;y)$.
A rate-distortion pair \((\tilde{X}, \tilde{Y})\) is said to be achievable if for all \(\xi > 0\), there exists enough large encoders and decoders such that

\[
\begin{align*}
R &\geq 2D \\
\sum_{x} \tilde{P}(x) \tilde{D}(\tilde{x}, y) &\geq \tilde{H}(\tilde{X}) \\
\sum_{x} \tilde{P}(x) \tilde{D}(\tilde{x}, y) - D &\leq \tilde{H}(\tilde{X})
\end{align*}
\]

Theorem (Chen [2000], Barron [2000]): The set of achievable rate-distortion pairs is equal to \(G_{nfe}\) which is defined as

\[
G_{nfe} = \{ (\tilde{x}, \tilde{y}) : 0 < \tilde{D} < 0 \} 
\]

which is called the test-channel.
Achievability:

In the Gelfand-Pinsker achievability proof, note that

\[ Z = \mathcal{Q} \]

Achievability:

In the Gelfand-Pinsker achievability proof, note that

\[ Z = \mathcal{Q} \]
\( (\tilde{f}, x) \overset{\text{fix}}{\in} \mathcal{D}\{(\tilde{f}, x) = (X, x)\}, \mathcal{D} \mathcal{P} \mathcal{D} \quad \mathcal{=} \quad \mathcal{D} \mathcal{P} \mathcal{D} \quad \mathcal{D}_{\text{Distortion part}}: \)

\[
(\tilde{u}\tilde{f}, u x) \overset{\text{fix}}{\in} \mathcal{D} \mathcal{P} \mathcal{D} \sum_{\mathcal{I}} \frac{N}{\mathcal{I}} \{ (N, \tilde{f}, N x) = (N, X, X) \} \mathcal{P} \mathcal{D} \quad \mathcal{=} \quad \tilde{f} x \mathcal{D}
\]

\( (x)^{\mathcal{D}} = \{ x = X \} \mathcal{P} \mathcal{D} \)

\( \forall x \in X \) and \( y \in Y \), Note that for \( x \in X \) and \( y \in Y \),

\[
\{ (\tilde{f}, x) = (\tilde{u}X, u X) \} \mathcal{P} \mathcal{D} \sum_{\mathcal{I}} \frac{N}{\mathcal{I}} = \{ (\tilde{f}, x) = (X, X) \} \mathcal{P} \mathcal{D}
\]

where \( X \) and \( Y \) are random variables with
III. Reversible Embedding

for some $[(\mathcal{N}X, M)^{u \in u} X^h] \subseteq \mathcal{N}^r = \mathcal{N} \mathcal{D} = \mathcal{D}$

Embedding distortion: 

Rate: $R = \log_2 \frac{N}{I}$

Error probability: $P_e \leq \sum P_{r} = 3P$

Source (host): $\mathcal{N}X \in \mathcal{N}^{X}$ for $(x)^{s} \mathcal{P} \mathcal{N} = u = \{ \mathcal{N}X = \mathcal{N}X \}$

Messages: $P_{r} = \{ 1, \ldots, 2, \ldots \}$ for $1 = \{ m = M \}$

Inspired by Fridrich, Goljan, and Du, "Lossless data embedding for all image formats.

Note that \( \{(x|h)^{+}, P, X\} \) is the test channel.

\[
(x|h)^{+}, P, X 
\]

\[
\left\{(x|h)^{+}, P, X\right\} = (h^{x}, P) 
\]

\[
(x|h)^{+}, P, X \leq (h^{x}, P) 
\]

\[
(X) H - (X) H \geq d \geq 0 : (h^{x}, P) \right\} = \delta
\]

RESULT (Kaheer-Willems [2002]): The set of achievable rate-distortion pairs is equal to \( \delta \) which is defined as

\[
\begin{align*}
\epsilon & \geq 3d, \\
\epsilon + h^{x} & \leq h^{x} D, \\
\epsilon - d & \leq R
\end{align*}
\]

There exists for all large enough encoders and decoders such that there exists for all \( \epsilon > 0 \). A rate-distortion pair \((h^{x}, P, X, d)\) is said to be achievable if for all \( \epsilon > 0 \).
Proof:

Achievability:

In the Gelfand-Pinsker achievability proof, note that 

\[ Z = Y \] (noiseless channel) and take the auxiliary random variable \[ U = [X;Y] \].

Then \( x \in N \) can be reconstructed by the decoder, and the auxiliary random variable \( \lambda \) can be reconstructed by the decoder and the \( \{X^N\} \) noiseless channel (assuming \( X \) is a random variable that takes values in \( \mathcal{X} \)).

Converse: Rate part:

\[
\log_2 \binom{M}{N} \leq H(W;X^N) - H(W;X^N|Y^N) + \text{Fanoterm}.
\]

Hence is OK. For the embedding rate we obtain

\[
(X)H - (\lambda)H = (X'[\lambda'X])I - (\lambda'[\lambda'X])I = (X';\Omega)I - (Z';\Omega)I = \mathcal{R}
\]
where \( X \) and \( Y \) are random variables with \( \Pr(X;Y) = (x;y) \).

\[
\begin{align*}
\{ (\tilde{f}, x) \} \quad & \Pr \sum_{N}^{u} \Pr \left\{ \left( (\tilde{f}, x) \right) = (\tilde{\lambda}, x) \right\} = \\
\{ (\tilde{u}, \tilde{x}) \} \quad & \Pr \sum_{I}^{\frac{N}{\tilde{f}}} \Pr \left\{ \left( (\tilde{u}, \tilde{x}) \right) = (\tilde{\lambda}, \tilde{x}) \right\} = \tilde{x} \Delta
\end{align*}
\]

Distortion part:

\[
\{(x) \} \quad \Pr \sum_{I=0}^{N} \{ (\tilde{f}, x) \} = \{(x) \} \quad \Pr
\]

Let \( \tilde{P}, etc. \)

\[
\begin{align*}
\{ (\tilde{f}, x) \} \quad & \Pr \sum_{N}^{u} \Pr \left\{ \left( (\tilde{f}, x) \right) = (\tilde{\lambda}, x) \right\} = \\
\{ (\tilde{u}, \tilde{x}) \} \quad & \Pr \sum_{I}^{\frac{N}{\tilde{f}}} \Pr \left\{ \left( (\tilde{u}, \tilde{x}) \right) = (\tilde{\lambda}, \tilde{x}) \right\} = \tilde{x} \Delta
\end{align*}
\]

Distortion part:

\[
\{(x) \} \quad \Pr \sum_{I=0}^{N} \{ (\tilde{f}, x) \} = \{(x) \} \quad \Pr
\]

For \( x \in \chi \) and \( \tilde{\chi} \in \chi \), note that for \( x \in \chi \) and \( \tilde{\chi} \in \chi \),

\[
\begin{align*}
\{(\tilde{f}, x) \} \quad & \Pr \sum_{I=0}^{N} \{ (\tilde{f}, x) \} = \{(\tilde{f}, x) \} \quad \Pr
\end{align*}
\]

Where \( X \) and \( Y \) are random variables with...
\[
\frac{1}{2} \leq xd + \overline{\overline{\text{x}}_y} \leq \frac{1}{2}
\]

or \( xd - \frac{1}{2} \leq \overline{\overline{\text{x}}_y} \leq xd + \frac{1}{2} \).

Assume w.l.o.g. that \( \overline{\overline{\text{x}}_y} \) be such that \( xd - \frac{1}{2} \leq \overline{\overline{\text{x}}_y} \leq xd + \frac{1}{2} \).

First let \( xd - \frac{1}{2} \leq \overline{\overline{\text{x}}_y} \leq xd + \frac{1}{2} \).

We can write

\[
0p(xd - \frac{1}{2}) + (1p - \frac{1}{2})xd = \frac{1}{2}
\]

\[
0p(xd - \frac{1}{2}) + 1p xd < \overline{\overline{\text{x}}_y}
\]

Since

\[
\begin{array}{ccc}
0 & 0p & 0 \\
1 & 1p & 1 \\
\hline
\hline
\hline
\end{array}
\]

Example: Binary source, Hamming distortion.
and hence

\[ h(p_y) \geq h(p_x + xy) > \frac{xd - 1}{xd - 2/3} \geq \frac{xd - 1}{h_x \nabla} = 0p \]

Note that the test channel is not symmetric and that

\[ \frac{xd - 1}{h_x \nabla} = 0p \quad \text{and} \quad 0 = 1p \]

However, by taking \( h_x \nabla \), the rate is bounded as

\[ (xd)y - (h_x \nabla + xd)y = d \]

and hence

\[ (xd)y - (h_x \nabla + xd)y \geq (xd)y - (h_d)y \geq d \]
Plot of rate-distortion region

Horizontal axis $x$, vertical axis $y$, for $p = 0.2$. Maximum embedding rate $I - h(x) \approx 0.278$. Maximum embedding
Consider a blocked system with blocks of length \( N \).

The resulting embedding rate is

\[
(H(Y|X) - H(X)) = (H(Y|X) - H(X) - H(X|Y)) = R
\]

This requires given \( (y)_{N^k} \) (y)_{N^x} and corresponding distortion.

Then in block \( k \) message bits can be (noise-free) embedded with rate \( H(Y|X) \).

In block \( k + 1 \) message bits are embedded that allow for recon-

In a blocked system with blocks of length \( N \).
IV. Robust and Reversible Embedding

Messages: $\Pr\{W = w\} = \frac{1}{M}$ for $w \in \{1, 2, \ldots, M\}$.

Source (host): $\Pr\{X^N = x^N\} = \prod_{n=1}^N P_s(x_n)$ for $x^N \in \mathcal{X}^N$.

Channel: discrete memoryless $\{Y, P_c(z|y), \mathcal{Z}\}$.

Error probability: $P_e = \Pr\{\hat{W} \neq W \vee \hat{X}_1^N \neq X^N\}$.

Rate: $R = \frac{1}{N} \log_2(M)$.

Embedding distortion: $D_{xy} = E\left[\sum_{n=1}^N D_{xy}(x_n, e_n(W, X, X^N))\right]$ for some distortion matrix $\{D_{xy}(x, y), x \in \mathcal{X}, y \in \mathcal{Y}\}$.
Achievable region for robust and reversible embedding

A rate-distortion pair \((\hat{r}, d)\) is said to be achievable if for all

\[ 0 \leq \epsilon \leq 3d \]

\[ \hat{r} + \epsilon \geq \hat{r} - \epsilon < \hat{r} \]

there exists for all large enough encoders and decoders such that

\[ \text{RESULT (Willems-Kailker [2003]): The set of achievable rate-distortion pairs is equal to } \gamma, \text{ which is defined as} \]

\[ \gamma = \{(\hat{r}, d) \mid (\hat{r}, d) \in \mathbb{R}^2, \text{ and } \hat{r} + \epsilon \geq \hat{r} - \epsilon < \hat{r} \} \]
Achievability: In the Gelfand-Pinsker achievability proof, we take the auxiliary random variable $\mathcal{X}$ and then the decoder can be reconstructed by $\mathcal{X}$. Then $\mathcal{X}$ can be reconstructed by $\mathcal{X}$. The advantage is that for the embedding rate we obtain

\[ (X)H - (Z : X)I = (X : [\mathcal{X}, X])I - (Z : [\mathcal{X}, X])I = (X : \mathcal{X})I - (Z : \mathcal{X})I = \beta \]

Converse: Rate part:

\[ \log_2 \left( \frac{(X)H - (Z : X)I}{(X)H - (Z : X)I} \right) \geq (W) \]
where $X;Y$ and $Z$ are random variables with

$$\Pr((X;Y;Z) = (x;y;z)) = 1.$$ 

$$\frac{\sum_{n=1}^{N} \{(N \cdot x; N \cdot y) = (\lambda; \mu)\} \cdot \Pr((X;Y) = (x;y))}{\sum_{n=1}^{N} \frac{\Pr((X;Y;Z) = (x;y;z))}{1}} = \Pr((X;Y;Z) = (x;y;z)),$$

**Distortion part:**

$$(\hat{h}|z)^2 \Pr \{\hat{h} = \lambda \mid z = Z\} \Pr \{X \in \lambda \text{ and } Z \in \lambda \text{ and } \hat{h} \in \hat{h}\}.$$ 

Note that for $X \in \lambda$, $X \in \lambda$, and $Z \in \lambda$, $\forall \in \lambda$, $\forall \in \hat{h}$. 

$$\forall x \in X, \forall x \in X, \forall z \in Z, \forall \hat{h} \in \hat{h}, \forall X \in X, \forall X \in X, \forall \hat{h} \in \hat{h}.$$

$$\forall x \in X, \forall x \in X, \forall z \in Z, \forall \hat{h} \in \hat{h}, \forall X \in X, \forall X \in X, \forall \hat{h} \in \hat{h}.$$
Similar analysis as before.

Example: Binary source, Hamming distortion, binary symmetric channel.
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\[ 0.218 \approx 0.062 \]

Minimal distortion for minimal embedding rate \( d \) for \( x = 0.1 \). Vertical axis, horizontal axis, and vertical axis \( x \).
The zero-rate case: Robustification

\[
\begin{align*}
\{ \mathcal{A} \oplus f_{\mathcal{A}} | \chi \in \chi, (f_{\mathcal{A}} x)_{\text{fix}} \} \quad & \text{detr. distortion} \\
\{ N X \neq N X \} \quad & \text{error probability:} \quad \Pr = 3^D \\
\{ \mathcal{Z} \cup (f_{\mathcal{Z}} z) \} \quad & \text{channel: discrete memoryless} \\
N \chi \oplus N x \quad & \text{SOURCE (host):} \quad \Pr \{ N x = N X \} \\
\end{align*}
\]

\[
\begin{align*}
(\mathbb{N}Z) p = \mathbf{1} N X \\
(\mathbb{N}Z) \mathbb{P} = \mathbb{P} N X \\
\mathbb{N}X \oplus N x \quad & \text{FOR} \quad (u x)^s \mathbb{P} N X = u \cup = \{ N x = N X \} \Pr \\
\end{align*}
\]
Achievable distortions for robustification

Related to Shannon's separation principle.

Robustification is not possible if

\[
(Z; X) I^P < (X) H
\]

\[
\text{for all large} N
\]

\[
\{(Z; X) I > (X) H \text{ such that}
\]

\[
(f|z)^P (x|f)^P p(x) = (z, f, x) P_f
\]

\[
\text{for} \ (f^x, x) \text{ is } p(f^x) \text{ such that}
\]

\[
\sum f^x P(x) \left\{ f^x \nabla : f^x \nabla \right\} = \delta^R
\]

\[
\text{RESULT: The set of achievable distortions is equal to } \delta^R
\]

\[
\epsilon \supseteq \mathcal{D}
\]

\[
\epsilon + f^x \nabla \supseteq f^x \mathcal{D}
\]

all large enough encoders and decoders such that for all \( \epsilon > 0 \) there exists for

\[
\text{A distortion } \nabla \text{ is said to be achievable if for all } \epsilon > 0 \text{ there exists for}
\]

\[
\text{Achievable distortions for robustification}
\]
V. Partially Reversible Embedding

For a dis-

\[ \{(N \lambda)^{u \in \lambda} \lambda x \} \text{ distortion matrix} \]

Restoration distortion:

\[ \{(N \lambda)^{u \in \lambda} \lambda x \} \text{ distortion matrix} \]

Embedding distortion:

\[ (W)^{2 \log_2 \frac{N}{T}} = H \quad \text{Rate:} \]

\[ \{M \neq \lambda\} \text{ Error probability:} \]

Source (host): \[ \{N \lambda \in N \lambda^2 \} \text{ Messages:} \]

\[ \{W, \cdots, 2, 1\} \text{ for} \left\{N \lambda \right\} = \{m = \lambda\} \text{ Pr:} \]

\[ (N \lambda)^{f = \lambda \lambda} \text{ Pr:} \]

\[ (N \lambda)^{p = \lambda \lambda} \text{ Pr:} \]

\[ (N X, M)^{e = \lambda \lambda} \text{ Pr:} \]

\[ (x)^s p \]
Achievable region for partially reversible embedding

A rate-distortion triple \((x; x' ; x\hat{v})\) is said to be achievable if for all \(\epsilon > 0\) there exists for all large enough encoders and decoders such that

\[
\{(x|\alpha'\hat{x})D(x)^{\epsilon}d = (\alpha'\hat{x}, x)D \text{ for }
\]
\[
\forall x' \sum_{x'\hat{x}} (\alpha'\hat{x})D(x') < \alpha x \triangledown
\]
\[
\forall x' \sum_{x'\hat{x}} (\hat{x}\hat{v})D(x') < \hat{x} \triangledown
\]
\[
\forall x' \sum_{x'\hat{x}} (\alpha x \triangledown, \hat{x} \triangledown, d) = \delta
\]

RESULT (Willems-Kalker [2002]): The set of achievable rate-distortion triples is given by \(\delta_{\text{pre}}\) which is defined as

\[
\{(L, X)I - (X)H \geq d \geq 0 : (\alpha x \triangledown, \hat{x} \triangledown, d)\} = \delta_{\text{pre}}
\]

\[
\begin{align*}
\epsilon & > 2d \\
\alpha x \triangledown & > \alpha x D \\
\hat{x} \triangledown & > \hat{x} D \\
\epsilon - d & < R
\end{align*}
\]
Proof:

Achievability:

In the Gelfand-Pinsker achievability proof, note again that \( Z = Y \) (noiseless channel) and take the auxiliary random variable \( U = [Y;V] \). Then, \( v_N \) can be reconstructed by the decoder and since \((x_N; y_N; v_N) \in A\) both \( \Lambda 'X \) and \( \Lambda 'Y \) are OK for the embedding. For the decoder, we obtain

\[
\Lambda 'X I = (x \mid \Lambda 'X) I - (x \mid \Lambda 'Y) I - (y \mid \Lambda 'X) I - (y \mid \Lambda 'Y) I = R
\]

Then can be reconstructed by the decoder and since \( Z = Y \) (noiseless channel) and take the auxiliary random variable \( \Lambda = Z \) that

Achievability:

Proof:
Let $P \rightarrow 0$, etc.

\[(a ' x) \alpha x \mathcal{D} \{(a ' x) = (\Lambda ' X)\} \uparrow \mathcal{D} \mathcal{C} = \]
\[
(ua ' u x) \alpha x \mathcal{D} \bigcup_{\frac{N}{u}} \{ (N ' a ' N x) = (N \Lambda ' N X) \} \uparrow \mathcal{D} \mathcal{C} = \alpha x \mathcal{D}
\]

\[
(h ' x) \beta x \mathcal{D} \{(h ' x) = (\Lambda ' X)\} \uparrow \mathcal{D} \mathcal{C} = \]
\[
(uh ' u x) \beta x \mathcal{D} \bigcup_{\frac{N}{u}} \{ (N h ' N x) = (N \Lambda ' N X) \} \uparrow \mathcal{D} \mathcal{C} = \beta x \mathcal{D}
\]

Distortion parts:

\[
(x) \gamma \mathcal{D} = \{ x = X \} \uparrow \mathcal{D}
\]

\[
\forall \mathcal{C} \in \mathcal{C}, \text{Note that for } x \in \mathcal{C} \text{ and } a ' \Lambda ' \mathcal{C} \in \mathcal{C} ' \text{, and } x \in \mathcal{C} ' \text{, the random variables } \Lambda \text{ and } \Lambda ' \text{ are random variables with}
\]
\[
\{ (a ' h ' x) = (u \Lambda ' u \Lambda ' u X) \} \uparrow \mathcal{D} \bigcup_{\frac{N}{u}} \{ (a ' h ' x) = (\Lambda ' \Lambda ' X) \} \uparrow \mathcal{D}
\]
Consider a blocked system with blocks of length $N$. In block $k$ a message can be (noise-free) embedded with rate $R$. Then in block $k$ + 1 data is embedded that specifies a restoration sequence $v_N(k)$ given $y_N(k)$. Therefore the remaining embedding rate is $\Lambda|\Lambda'|X)I_N \Lambda|X)H + (X)H - (X|\lambda)H = R$

\[ (\Lambda|\lambda|X)H + (X|\lambda)H - (X|\lambda)H = R \]

The other perspective again
The zero-rate case: Self-Embedding

Embedding distortion:
$$D_{xy} = \mathbb{E} \left[ \sum_{n=1}^{N} D_{xy}(X^n; e^n(W; X)) \right]$$
for some distortion matrix $D_{xy}$. For $(u_x)^s p^{N' = u} \cup \{N x = N X\}$

Restoration distortion:
$$D_{xv} = \mathbb{E} \left[ \sum_{n=1}^{N} D_{xv}(X^n; f^n(Y)) \right]$$
for some distortion matrix $D_{xv}$. For $N x \in N X$.
Achievable distortions for self-embedding mean.

$\{ (\lambda \downarrow \lambda )I \geq (\lambda )H \} \forall x \forall \alpha \text{ such that }$

$(x|\alpha) \sigma d (x|\alpha) \text{ for } \alpha \text{ such that }$

$(\alpha x \sigma d (\alpha x)) \preceq \alpha x \downarrow$

$(\alpha x \sigma d (\alpha x)) \preceq \alpha x \downarrow$ : $(\alpha x \downarrow, \alpha x \downarrow)$} = \gamma_\delta

The set of achievable distortion pairs is equal to $\gamma_\delta$ which is defined as

RESULT (Willems-Kalker [2002]):

$\alpha x \downarrow + \epsilon \geq \alpha x \downarrow$

$\alpha x \downarrow + \epsilon \geq \alpha x \downarrow$

A distortion pair $(\alpha x \downarrow, \alpha x \downarrow)$ is said to be achievable if for all $\epsilon > 0$ there exists $N$ encoders and decoders such that making an abstract index to a restoration vector vector quantizer into a scalar quantizer. Or self-embedding is putting a vector quantizer into a scalar quantizer.
I. Our results are related to results of Sutivong, Cover, et al. Slightly different setups however. Embedding distortion.

II. Remarks

1. Our results are related to results of Sutivong, Cover, et al. Slightly different setups however. Embedding distortion.

2. We cannot do the partially reversible AND robust case. An achievable region. No converse.

3. Coding techniques for the reversible case have been studied (with Deen Maas [2002]).

4. Open problems: (A) Ariimoto-Blahut methods to compute the rate-distorion functions, (B) Coding techniques, especially for the zero-rate cases.

Sutivong, Cover, et al. Slightly different setups however. Embedding distortion.