Throughput and Delay Optimal Resource Allocation in Multiple Access Fading Channels

DIMACS Network Information Theory Workshop
March 18, 2003

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Acknowledgments

Many thanks to Professor Robert Gallager and Professor Emre Telatar for their advice and encouragement.
Multiple Access Communications

- Multiple access (many to one): multiple senders transmit to one receiver (possibly) over fading channels.
- Ex: cellular telephony, satellite networks, local area networks.
Central Problems

- Contention/interference - resource sharing.
- Bursty sources ⇒ random number of active senders.
- Network/MAC layer QOS issues - throughput, delay.
- Physical layer issues - channel modelling, coding, detection.
Need for Cross-Layer Approach

- Multiple access network theory (ALOHA, CSMA) - concentrates on source burstiness and delay; poor modelling of noise and interference.

- Multiple access information theory - concentrates on channel modelling and coding; ignores random arrival of messages and delay.

- Need more unified cross-layer framework:
  - Random packet arrivals affect resource sharing.
  - Choice of modulation and coding affects QOS issues.
  - Random fading affects resource allocation.
  - Gallager (85), Ephremides and Hajek (98).
New Approach

• Goal:
  – Combine information-theoretic limits with QOS issues.
  – Establish fundamental bounds on throughput/delay performance.

• Implementation:
  – Random arrivals, information-theoretic optimal coding.
  – Power control and rate allocation as function of fading and queue states to optimize throughput and delay
Previous Work

- Telatar and Gallager (95)
  - Achievable multiple access scheme with feedback.
  - Poisson arrivals; no queueing; single-user decoding; processor sharing system.

- Telatar (95)
  - Analogy between MAC and multi-processor queue.
  - Each user has fixed pool of bits to send.
  - Optimal processor assignment to minimize average packet delay.

- Yeh (01)
  - Poisson arrivals; queueing.
  - Optimal rate allocation from $C$ to min. average packet delay.
  - Longer Queue Higher Rate (LQHR) policy strongly delay optimal.
Multiple Access Fading Channel

- Continuous-time $M$-user Gaussian multiple access fading channel with bandwidth $W$:
  $$Y(t) = \sum_{i=1}^{M} \sqrt{H_i(t)} X_i(t) + Z(t).$$

- \{Z(t)\}: white Gaussian noise, density $N_0/2$.

- Slowly-varying and flat-fading (under-spread) channel.
Multiple Access Fading Channel

- Block fading model, block length = \( T \).
- \( T \) large enough for reliable communication at a fixed fade.
- \( \{ \mathbf{H}(t) = (H_1(t), \ldots, H_M(t)) \} \) modulated by finite-state ergodic Markov chain.
- Transmitter \( i \) has (long-term) average power constraint \( \overline{P}_i \), and (short-term) peak power constraint \( \hat{P}_i \).
Information-theoretic Capacity Region $C(h, p)$

(Ahlswede, Liao, Cover, Wyner 1971-75)

- Fixed $h = (h_1, \ldots, h_M)$ and $p = (p_1, \ldots, p_M)$.
- $C(h, p) = \text{set of } r \in \mathbb{R}_+^M \text{ such that }$
  \[ \sum_{i \in S} r_i \leq W \log \left( 1 + \frac{\sum_{i \in S} h_i p_i}{N_0 W} \right), \quad \forall S \subseteq \{1, \ldots, M\}. \]
- Reliable communication possible inside $C(h, p)$, impossible outside $C(h, p)$, for any coding and modulation scheme.
- **Polymatroid** structure (Tse and Hanly 98).
Two-User Capacity Region $\mathcal{C}(h, p)$
Multiple Access Channel with Random Arrivals

\[ A_1(t) \rightarrow \quad \text{Transmitter 1} \quad \rightarrow H_1(t) \]

\[ A_2(t) \rightarrow \quad \text{Transmitter 2} \quad \rightarrow H_2(t) \]

\[ \vdots \quad \vdots \]

\[ A_M(t) \rightarrow \quad \text{Transmitter } M \quad \rightarrow H_M(t) \]

\[ H(t), U(t) \rightarrow \quad \text{Controller} \]
Arrivals and Unfinished Work

- $\{A_i(t)\} = \text{ergodic packet arrival process to transmitter } i$.
- User $i$ packets i.i.d. $\sim F_{Z_i}(\cdot)$, $\mathbb{E}[Z_i] < \infty$.
- $U_i(t) = \text{number of untransmitted bits in queue } i \text{ at time } t$. 
Power Control and Rate Allocation

- Controller: \((H(t), U(t)) \mapsto (P(t), R(t))\).

- Two stages:
  1. **Power control policy** \(\mathcal{P}\):

     \[ p = \mathcal{P}(h, u) \]

     s.t. for all \(i\), \(E[\mathcal{P}_i(H, U)] \leq \bar{P}_i\), \(\mathcal{P}_i(h, u) \leq \hat{P}_i\) for all \((h, u)\).

  2. **Rate allocation policy** \(\mathcal{R}\):

     \[ r = \mathcal{R}(h, p, u) \in C(h, p). \]
Main Results

- Stability region $S$ of all bit arrival rates for which all queues can be kept finite.
- For given power control policy, find throughput optimal rate allocation policy.
- In symmetric scenario, find delay optimal rate allocation policy for any symmetric power control policy.
**Stability Region $S$**

- $\lambda_i = \lim_{t \to \infty} \frac{A_i(t)}{t} = \text{packet arrival rate to queue } i$.
- $\rho_i = \lambda_i E[Z_i] = \text{bit arrival rate to queue } i$.
- Define $f_i(\xi) = \limsup_{t \to \infty} \frac{1}{t} \int_0^t 1\{U_i(\tau) > \xi\} d\tau$.
- System **stable** if $f_i(\xi) \to 0$ as $\xi \to \infty$ for all $i$.
- $S = \text{set of all } \rho = (\rho_1, \ldots, \rho_M) \text{ for which can stabilize system.}$
Assume \( \{A_i(t)\} \) modulated by finite-state ergodic Markov chain.

**Theorem 1** \( S = C(\bar{P}, \hat{P}) = \) information-theoretic capacity region under power control (Tse and Hanly 98).

- \( C(\bar{P}, \hat{P}) = \bigcup_{\mathcal{P} \in \mathcal{F}} C(\mathcal{P}) \).
- \( \mathcal{F} = \{ \mathcal{P} : \mathbb{E}[P_i(H)] \leq \bar{P}_i, \forall i; \mathcal{P}_i(h) \leq \hat{P}_i, \forall h, \forall i \} \).
- \( C(\mathcal{P}) = \mathbb{E}[C(H, \mathcal{P}(H))] \).
Stability Theorem

- **Achievability**: \( \rho \in \text{int}(S) \): knowing \( \rho \) and statistics of \( \{H(t)\} \), can stabilize system using stationary \( P, R \) depending only on current channel state.

- **Converse**: \( \rho \notin S \): cannot stabilize system, even with non-stationary policy with knowledge of queue state and/or knowledge of future events, so long as

\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t p_i(\tau) d\tau \leq \overline{P}_i \ \forall i; \quad p_i(\tau) \leq \hat{P}_i, \forall \tau, \forall i.
\]
Throughput Optimal Resource Allocation

- Find “universal” power/rate policy to stabilize system even if $\rho$ not known, as long as $\rho \in \text{int}(S)$.
- Must use both $H(t)$ and $U(t)$.
- Suppose know $\rho \in \mathcal{C}(\mathcal{P}) = \mathbb{E}[\mathcal{C}(H, \mathcal{P}(H))]$.
- Assume $\{H_i(kT)\}$ i.i.d. for each $i$, $\{A_i((k + 1)T) - A_i(kT)\}$ i.i.d. for each $i$.
- Assume $\mathbb{E}[(A_i((k + 1)T) - A_i(kT))^2] < \infty$. 
No Work Conservation

\( r_2 \)

\( r_1 \)

Dominant face

\( r_A \)

\( r_B \)
Theorem 2  Given $P \in \mathcal{F}$, throughput optimal rate allocation policy is

$$r^* = R^*(h, P(h, u), u) = \arg \max_{r \in C(h, P(h, u))} \sum_{i=1}^{M} u_i r_i$$

(1)

• Idea appeared in Tassiulas and Ephremides ’92; McKeown, et al. ’96; Tassiulas ’97; Neely et al. ’02.

• Here, motivated by delay optimality results.
Longest Queue receives Highest Possible Rate (LQHPR)

- Due to polymatroidal nature of $C(h, P(h, u))$, solution to (1) has special form.
  \[
  r^*_i = W \log \left( 1 + \frac{h[i]P[i](h, u)}{\sum_{j<i} h[j](t)P[j](h, u) + N_0 W} \right)
  \]
- Longest Queue receives Highest Possible Rate (LQHPR).
Two-User Rate Allocation

\[ r_A, r_B \]

- \( u_1 \geq u_2 : r_B \)
- \( u_1 < u_2 : r_A \)
Proof of Stability Theorem

- Stability of Markov chains based on negative Lyapunov drift.
- \( V(\mathbf{U}) = \sum_i U_i^2 \).
- Show there exists compact set \( \Gamma \subset \mathbb{R}^M \) s.t. for some \( \epsilon > 0 \),
  \[
  \mathbb{E}[V(\mathbf{U}(t+T)) - V(\mathbf{U}(t))|\mathbf{U}(t)] \leq -\epsilon
  \]
  whenever \( \mathbf{U} \notin \Gamma \).
Delay Optimal Resource Allocation

- Beyond stabilization, keep queues as short as possible.
- Find feasible $\mathcal{P}$ and $\mathcal{R}$ to minimize $\lim_{t \to \infty} E[\sum_{i=1}^{M} U_i(t)]$ (average bit delay) for $\rho \in \text{int}(S)$. 

\[\]
Delay Optimal Rate Allocation

- Focus on symmetric Poisson/exponential case.
- \( \{A_i(t)\} = \text{Poisson}(\lambda) \) for each \( i \).
- All packets i.i.d. \( \sim \exp(\mu) \).
- Queue state \( Q(t) = (Q_1(t), \ldots, Q_M(t)) \) - number of packets.
- For fixed \( P \), find \( R \) to minimize \( \lim_{t \to \infty} E \left[ \sum_{i=1}^{M} Q_i(t) \right] \) (average packet delay).
- Yeh '01: non-faded symmetric MAC.
Delay Optimal Rate Allocation

• Symmetric fading process $\mathbf{H}(t)$:
  For any $\mathbf{a} = (a_1, \ldots, a_M)$, $\Pr(H_1(t) = a_1, \ldots, H_M(t) = a_M) = \Pr(H_1(t) = a_{\pi(1)}, \ldots, H_M(t) = a_{\pi(M)})$ for any permutation $\pi$.  
  e.g. for every $t$, $H_1(t), \ldots, H_M(t)$ i.i.d.

• Symmetric power control $\mathcal{P}(\mathbf{h}, \mathbf{q}) = \mathcal{P}(\mathbf{h})$:
  $\mathcal{P}_i(a_1, \ldots, a_M) = \mathcal{P}_{\pi^{-1}(i)}(a_{\pi(1)}, \ldots, a_{\pi(M)})$ for all $\pi$.  
  e.g. $M = 2$ and $a_1 > a_2$: $\mathcal{P}_1(a_1, a_2) = \mathcal{P}_2(a_2, a_1)$.  
  e.g. Knopp and Humblet (’95).

• For this case, $\max \sum u_i r_i$ (LQHPR) policy is delay optimal.
Majorization and Weak Majorization

- Need to quantify load balancing.
- For $u \in \mathbb{R}^M$, let
  \[ u[1] \geq \cdots \geq u[M]. \]
- For $u, v \in \mathbb{R}^M$,
  \[ u \prec_w v \quad \text{if} \quad \sum_{i=1}^k u[i] \leq \sum_{i=1}^k v[i], \quad k = 1, \ldots, M. \]
  Say $u$ weakly majorized by $v$. If equality holds for $k = M$, say $u$ majorized by $v$: $u \prec v$.
- Ex: $(1 \ 1) \prec_w (3 \ 0), \ (1 \ 1) \prec (2 \ 0)$.
- See Marshall and Olkin (79).
Stochastic Weak Majorization

- Use **stochastic coupling** to show weak majorization on queue vectors, in a **stochastic** sense.

- $\mathbf{U} = (U_1, \ldots, U_M), \mathbf{V} = (V_1, \ldots, V_M)$ random vectors. $\mathbf{U}$ is said to be **stochastically weak-majorized** by $\mathbf{V}$, $\mathbf{U} \prec_{st}^{w} \mathbf{V}$, if there exist random vectors $\tilde{\mathbf{U}}$ and $\tilde{\mathbf{V}}$ such that
  (a) $\mathbf{U}$ and $\tilde{\mathbf{U}}$ are identically distributed.
  (b) $\mathbf{V}$ and $\tilde{\mathbf{V}}$ are identically distributed.
  (c) $\tilde{\mathbf{U}} \prec_{w} \tilde{\mathbf{V}}$ a.s.
Strong Delay Optimality of LQHPR

Theorem 3 Let $q_0$ be initial queue state. Let $Q(t)$ be queue evolution under $g_{\text{LQHPR}}$ for $t \geq 0$. Let $Q'(t)$ be corresponding quantity under any policy $g \in G_D$. Then under all symmetric $\mathcal{P}$,

$$Q(t) \mathrel{\prec_w}^{st} Q'(t) \; \forall t \geq 0.$$

- Proof: generalize stochastic coupling argument for non-faded symmetric MAC.
Consequences

Corollary 1

\[ E[\varphi(Q(t))] \leq E[\varphi(Q'(t))] \quad \forall t \geq 0 \]

for all \( \prec_w \)-preserving \( \varphi : \mathbb{R}^M \mapsto \mathbb{R} \) for which expectations exist.

- \( \varphi \) is \( \prec_w \)-preserving if \( x \prec_w y \Rightarrow \varphi(x) \leq \varphi(y) \) for \( x, y \in \mathbb{R}^M \).
- \( \prec_w \)-preserving \( \iff \) Schur-convex, increasing.
- Includes all symmetric, convex and increasing real functions on \( \mathbb{R}^M \).
- Examples:
  \[ \varphi(x) = \max_{i_1 < i_2 < \cdots < i_k} (|x_{i_1}| + \cdots + |x_{i_k}|), \quad 1 \leq k \leq M; \]
  \[ \varphi(x) = \sum_{i=1}^{M} |x_i|^r \] for \( r \geq 1 \) or \( r \leq 0; \)
  \[ \varphi(x) = (\sum_{i=1}^{M} |x_i|^r)^{1/r} \] for \( r \geq 1. \]
Summary and Conclusions

• General framework for resource allocation in fading MAC with random arrivals.

• Stability region \( S = \mathcal{C}(\overline{P}, \hat{P}) \).

• \( \max \sum_i u_i r_i \) (LQPHR) policy throughput optimal for given \( P \).

• LQHPR minimizes average packet delay for any symmetric \( P \) in symmetric scenario.

• LQHPR implements adaptive successive decoding at physical layer.
Summary and Conclusions

• “Converse”: LQHPR establishes fundamental throughput/delay performance limit for any multiple access coding scheme which meets any given required $P_e$ (Fano).

• “Achievability”: To approach rates in $\mathcal{D}$, need sufficiently large $T$ and code over large number of bits.