Vertices and Facets of the Semiorder Polytope

Examples and Preliminary Results

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Outline

1. Geometric and Analytical Representations of Interval Orders
   - Introduction and Definitions
   - Interval Orders and Interval Representations

2. Semiorders and Scales
   - Examples
   - Minimal Representations: Coarsening the Scale

3. The Semiorder Polytope
   - Noses and Hollows: Saving Work
   - Our Project
   - The Motivating Question
   - Other Questions
A (strict) partial ordering $P$ on a set $X$ is a binary relation which satisfies

- **Irreflexivity:** $x \nless x$
- **Asymmetry:** If $x \less y$ then $y \nless x$
- **Transitivity:** If $x \less y$ and $y \less z$, then $x \less z$

If $x \nless y$ and $y \nless x$, we write $x \parallel y$
We seek an easier way of representing our poset than a list of comparabilities between elements.

An *Interval Representation* of a poset \((X, P)\) assigns to each element \(x \in X\) an interval \(I_x\).

\[ a \prec b \text{ if } I_a \cap I_b = \emptyset \text{ and } I_a \text{ is to the left of } I_b. \]

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Interval Orders and Semiorders

Semiorders and Scales

The Semiorder Polytope

Introduction and Definitions

Interval Orders

Vertices and Facets of the Semiorder Polytope
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but interestingly.....

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Fishburn and Mirkin Theorem

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Any Order that has no $2 + 2$ is an Interval order and has an interval representation.
A *unit interval representation* is an interval representation in which all intervals have the same length.

A *unit interval order* (or *semiorder*) is a poset that has a unit interval representation.
Scott-Suppes Theorem

No unit interval order can contain a $3 + 1$ suborder

and, strikingly...

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Any interval order that has no $3 + 1$ has a unit interval representation and is a unit interval order.
We can think of an interval order as a pair of functions, $f, g : X \to \mathbb{R}$

For all $x \in X$, $f(x) \leq g(x)$

$x \prec y$ iff $g(x) < f(y)$
For a semiorder, we need only one function, $f : X \to \mathbb{R}$, together with a predetermined interval length $r$.

$x \prec y$ iff $f(x) + r < f(y)$. 
A Semiorder
Making The Scale-Numerical Representations
We Can Do Better
How Good Can We Do?

Is there a minimal representation that preserves the semiorder relations?

What differentiation relations does this minimal representation preserve? Is everything ‘fixed’ on the scale?

Is the scale the same for all of these ‘minimal’ representations?
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Is the scale the same for all of these ‘minimal’ representations?
Let \((X, R)\) be a semiorder, and let \(x, y \in X\).

- (Preference) We say that \(xPy\) if \(x > y\) in \((X, R)\).

- (Incomparability) We say that \(x \parallel y\) if \(x \parallel y\) in \((X, R)\) (neither \(x > y\) nor \(y > x\)).

- (Trace) We say that \(xTy\) if \(f(x) > f(y)\).
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Other Assumptions

- Any two elements must be represented with distinct intervals.

- Every element is incomparable with its predecessor in the trace.

- Any two comparable elements are separated by at least one unit.
Inequalities

- If $xPy$ then $f(x) \geq f(y) + r + 1$.
- If $x \not\succ y$ then $|f(x) - f(y)| \leq r$
- $r > 0$ and $f(x_0) = 0$, where $x_0$ is our minimum element.
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- $r > 0$ and $f(x_0) = 0$, where $x_0$ is our minimum element.
\[ f(a) \geq f(c) + r + 1 \]
\[ |f(d) - f(e)| \leq r \]
\[ f(g) \geq f(q) + r + 1 \]
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The set of all representations forms a convex polytope in $n$-dimensional space, where $n$ is the number of elements in our semiorder.

Note that we fix the minimum element at zero, and that we have one dimension for the length of intervals ($r$).
Too Many Inequalities

Our example on 17 elements gives us 158 inequalities to deal with....

Many of these are implied by the others, saving us some work.
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Many of these are implied by the others, saving us some work.
We actually need only need a subset of these inequalities to imply the others (implications by transitivity and order in the trace).
Noses and Hollows

- Noses: We say that $xNy$ if $xPy$ and any element $z$ such that $xTzTy$ satisfies $zIy$ and $xlz$.

- Hollows: We say that $xHy$ if $xIy$, $yTx$ and for all $w$ and $z$ such that $zTyTxTw$, $zPw$.
Noses and Hollows

- Noses: Two elements in a ‘nose’ relation are comparable, but only just barely so.

- Hollows: Two elements in a ‘hollow’ relation are incomparable, but only just barely so.
- \(aNc, \ cNe, \ eNg \) etc.
- \(bHa, \ dHb, \ fHd, \) etc.
- \(hNj \ iHh, \ jHi\)
Why Noses and Hollows?

We can list the preferences in a matrix.

The rows and columns correspond to the semiorder elements.

The matrix has only 0’s and 1’s, with a 1 indicating that the element corresponding to the row is preferred to the element in the column.
Interval Orders and Semiorders
Semiorders and Scales
The Semiorder Polytope

Noses and Hollows: Saving Work
Our Project
The Motivating Question
Other Questions

Vertices and Facets of the Semiorder Polytope
A semiorder always forms this type of step matrix. Our connectivity conditions keep this matrix away from the main diagonal and keeps differences between the rows and columns.

- The Noses are places where we could remove a 1 and still have a step-matrix.

- The Hollows are places where we could remove a 0 and still have a step-matrix.
Using Fewer Inequalities

The only relevant inequalities correspond to the noses and hollows of the preference matrix.

This reduces our large example of 158 inequalities to 19 inequalities.
The Super-Synthetic Graph: Noses and Hollows

Vertices and Facets of the Semiorder Polytope
Pirlot (1990) proved the existence of a minimal representation of a given semiorder. This representation is minimal in the sense that

- The function values are as small as possible (the intervals are as far left as possible)
- The scale cannot be any smaller
Minimal Representation(s)

There exist other ‘minimal’ representations in the sense that the scale is the shortest possible and the function values are the smallest possible to satisfy a maximal number of noses and hollows.

These minimal representations are exactly the vertices of the semiorder polytope.
One Vertex
Two Vertices

OR

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Vertices and Facets of the Semiorder Polytope
Finding the vertices of the semiorder polytope corresponds to finding cycles in the SSG.

Those inequalities on the cycles found are satisfied with equality.

A minimal representation will satisfy a maximal number of noses and hollows with equality.
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Finding the vertices of the semiorder polytope corresponds to finding *cycles* in the SSG.

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A minimal representation will satisfy a maximal number of noses and hollows with equality.
We used *PORTA* to compute the solution space to the linear system $Ax \leq b$, where $A$ is derived from the incidence matrix of the SSG.

Formally, we must augment $A$ by a column vector corresponding to $r$, the cycle length.

*PORTA* returns the vertices and extremal rays of each system.
Our natural (and seemingly innocuous) question was ‘Do all minimal scales have the same interval length?’

In other words, do all vertices have the same value for $r$? Do they all lie in the $r = k$ plane?
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One Semiorder, Two Scales

\[ r = 4 \]

\[ r = 7 \]
One Semiorder, Two Scales

Balof, Doignon, and Fiorini

Vertices and Facets of the Semiorder Polytope
A semiorder polytope has all of its vertices on the same \( r = k \) plane if all cycles of the corresponding SSG ‘behave’.
Two Cycle Types

Vertices and Facets of the Semiorder Polytope
More Pathological Examples

This example is part of a larger family of examples which have exactly two vertices, one at \( r = k \) and one at \( r = 2k + 1 \).

We have constructed more examples with 3, 4, and more different \( r \) values for the vertices.
Some Results

- Any vertex of the semiorder polytope must satisfy at least one \textit{cycle} of the SSG with equality.

- The minimal interval length $r^*$ is equal to the length of the longest conformal cycle in the SSG. Any cycles of this length will be satisfied in any vertex in the plane $r = r^*$.

- The extremal rays of the semiorder polytope correspond exactly to ‘cycle-breaking’ subsets of the edges.
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- The extremal rays of the semiorder polytope correspond exactly to ‘cycle-breaking’ subsets of the edges.
Can we classify the minimal representations of a semiorder by examining the SSG and the Nose and Hollow Inequalities?

Can we determine which semiorders have certain desirable properties? (e.g., all vertices with the same $r$ value, only one vertex, a specified number of vertices, etc.)

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References

- Doignon, J-P, and Falmagne, J-C. *Knowledge Spaces*. Springer-Verlag Berlin Heidelberg 1999


- Balof, Doignon, and Fiorini, “Vertices and Facets of the Semiorder Polytope", in preparation
THANK YOU