Power grid vulnerability analysis

Daniel Bienstock

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Dimacs 2010
Background: a power grid is three systems
Challenges to analysis

- Power grids follow the laws of physics, characterized by nonlinear, nonconvex equations that make fast computation difficult.
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- Furthermore, direct control is difficult: we cannot dictate how power will flow.
- Power grids are subject to “noise” which is difficult to model accurately.
- Power grids can exhibit non-monotone behavior as a result of control or adversarial actions.
- Power grids can cascade.
AC power flows – polar coordinates

→ Voltage at a node (“bus”) $k$ is of the form $U_k e^{j\theta_k}$, where $j = \sqrt{-1}$

→ Power flowing on edge (“line”) $\{k, m\}$ equals $p_{km} + jq_{km}$, where

\[
p_{km} = U_k^2 g_{km} - U_k U_m g_{km} \cos \theta_{km} - U_k U_m b_{km} \sin \theta_{km}
\]

\[
q_{km} = -U_k^2 (b_{km} + b_{km}^{sh}) + U_k U_m b_{km} \cos \theta_{km} - U_k U_m g_{km} \sin \theta_{km}
\]

Here, $\theta_{km} = \theta_k - \theta_m$

$g_{km}$, $b_{km}$, $b_{km}^{sh}$ are known parameters (series conductance, series reactance, shunt susceptance)
Voltage at \( k = U_k e^{j\theta_k} \); power on line \( \{k, m\} = p_{km} + jq_{km} \), where

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(\( \theta_{km} = \theta_k - \theta_m \))

\[
P_k = \sum_{\{k, m\}} p_{km} \text{ (active power)}, \quad Q_k = \sum_{\{k, m\}} q_{km} \text{ (reactive power)}
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**Power flow problem:** Choose the vectors $p$, $q$, $\theta$, $P$, $Q$ so as to satisfy all equations above, and
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**Power flow problem:** Choose the vectors \( p, q, \theta, P, Q \) so as to satisfy all equations above, and meet demand requirements and generator constraints.
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**Power flow problem:** Choose the vectors $p, q, \theta, P, Q$ so as to satisfy all equations above, and meet demand requirements and generator constraints and, ideally, meet thermal constraints (flow limits) on the power lines.
Research challenges

→ Do we have **fast** and **reliable** algorithm for the power flow problem?

- Should not require human input in order to terminate.
- When no “acceptable” solution exists, should produce a certificate that this is the case.
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What about the cases where multiple solutions exist?

- After a contingency has take place, or a control has been applied: which solution should be instantiated?

- What if all solutions are “bad”?
Solution methodologies

- Newton-Raphson (iterative) algorithms to solve system of equations
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- **New result:** Low et al (2010). Some (many?) optimal power flow problems can be solved using semidefinite programming.
Linearized ("DC") model:  $\sin(\theta_i - \theta_j) \approx (\theta_i - \theta_j)$ for $\theta_i \approx \theta_j$

A **power flow** is a solution $f, \theta$ to:

- $\sum_{ij} f_{ij} - \sum_{ij} f_{ji} = b_i$, for all $i$, where
  - $b_i > 0$ for each generator $i$,
  - $b_i < 0$ for demand node $i$.  

Lemma: Given a choice for $b$ with $\sum_i b_i = 0$ (a requirement), the system has a unique (in $f$) solution.
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2. \( x_{ij} f_{ij} - \theta_i + \theta_j = 0 \) for all \( (i, j) \). (\( x_{ij} = \text{“reactance”} \))
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A quote from:

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Cause 1 was “inadequate system understanding”
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Cause 1 was “inadequate system understanding” – stated 20 times

Cause 2 was “inadequate situational awareness” – stated 14 times

Cause 3 was “inadequate tree trimming” – stated 4 times

Cause 4 was “inadequate RC diagnostic support” – stated 5 times
Cascades

Load (demand)

Generator

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Cascades
Cascades
Cascades
Cascades
Cascades

= lost demand
Cascades
Cascades
Formal cascade model (Dobson et al)

→ Initial fault event takes place (an “act of God”).
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For $r = 1, 2, \ldots$,

1. Reconfigure demands and generator output levels.
Islanding

![Diagram showing the concept of islanding with supply greater than demand.]
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1. Reconfigure demands and generator output levels.

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3. The next set of faults takes place.
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   (Stochastic or history-dependent criterion)
Outage mechanism

\[ f_e = \text{flow on line } e \]

\[ u_e = \text{flow “limit” (threshold) on } e \]
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\[ \text{Prob( } e \text{ fails)} = F(|f_e|/u_e), \text{ where } F(x) \rightarrow 1 \text{ as } x \rightarrow +\infty. \]
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- \( \text{Prob( } e \text{ fails) } = F(\lfloor f_e \rfloor / u_e) \), where \( F(x) \rightarrow 1 \) as \( x \rightarrow +\infty \).

- Set \( \tilde{f}_e^r = \alpha_e |f_e^r| + (1 - \alpha_e) \tilde{f}_{e}^{r-1} \), where \( 0 \leq \alpha_e \leq 1 \) is given.
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\( \rightarrow \tilde{f}_e^r = \text{running average of } |f_e|. \)

\( \rightarrow r = \text{round (time)}. \)
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  - \( \tilde{f}_e^r = \text{running average of } |f_e| \).
  - \( r = \text{round (time)} \).
  - \( e \text{ fails if } \tilde{f}_e > u_e \).
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\[ \text{→ } \tilde{f}_e^r = \text{running average of } |f_e|. \]
\[ \text{→ } r = \text{round (time).} \]
\[ \text{→ } e \text{ fails if } \tilde{f}_e > u_e. \text{ or: } e \text{ fails if } \tilde{f}_e \geq u_e \]
Outage mechanism

\( f_e = \text{flow on line } e \)

\( u_e = \text{flow “limit” (threshold) on } e \)

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- Set \( \tilde{f}_e^r = \alpha_e |f_e^r| + (1 - \alpha_e) |f_e^{r-1}| \), where \( 0 \leq \alpha_e \leq 1 \) is given.

  \( \to \tilde{f}_e^r = \text{two-round average of } |f_e| \).

  \( \to r = \text{round (time)} \).

  \( \to e \text{ fails if } \tilde{f}_e > u_e, \)
Outage mechanism

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\[ \tilde{f}_e^r = \text{two-round average of } |f_e|. \]

\[ r = \text{round (time).} \]

\[ e \text{ fails if } \tilde{f}_e > u_e, \text{ (or, } e \text{ fails if } \tilde{f}_e \geq u_e) \]
Stochastic faults

\( e \) fails if \( u_e < \tilde{f}_e^r \).
Stochastic faults

\[ e \text{ fails if } u_e < \tilde{f}_e^r, \]

\[ e \text{ does not fail if } (1 - \gamma)u_e > \tilde{f}_e^r, \quad (\gamma = \text{tolerance}) \]
Stochastic faults

\begin{align*}
e \text{ fails if } & \ u_e < \tilde{f}_e^r, \\
e \text{ does not fail if } & \ (1 - \gamma) u_e > \tilde{f}_e^r, \quad (\gamma = \text{tolerance}) \\
\text{if } & \ (1 - \gamma) u_e \leq \tilde{f}_e^r \leq u_e \text{ then } e \text{ fails with probability } 1/2
\end{align*}
Formal cascade model (Dobson et al)

→ Initial outage event takes place (an “act of God”).

For \( r = 1, 2, \ldots, \)

1. Reconfigure demands and generator output levels.

2. New power flows are instantiated.

3. The next set of outages takes place.
   (Stochastic or history-dependent criterion)
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→ If no more faults occur or too much demand has been lost, STOP
Online control

→ Initial outage event takes place.
Online control

→ Initial outage event takes place. **Compute control algorithm.**
Online control

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For $r = 1, 2, \ldots, R - 1$

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3a. **Take measurements and apply control to shed demand.**
Online control

→ Initial outage event takes place. Compute control algorithm.

For \( r = 1, 2, \ldots, R - 1 \)

1. Reconfigure demands and generator output levels.
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3b. Reconfigure generator outputs;
Online control

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For $r = 1, 2, \ldots, R - 1$

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3a. **Take measurements and apply control to shed demand.**
3b. **Reconfigure generator outputs; get new power flows.**

4. The next set of outages takes place.
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→ Initial outage event takes place. **Compute control algorithm.**

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3b. **Reconfigure generator outputs; get new power flows.**

4. The next set of outages takes place.

At round $R$, reduce demands so as to remove any line overloads.
Deterministic, no history model

“Optimal” control via integer programming formulation
Deterministic, no history model

“Optimal” control via integer programming formulation?
Deterministic, no history model

“Optimal” control via integer programming formulation?

- $f_{jr}^r$ = flow on arc $j$ at round $r$
- $y_{jr}^r = 1$, if arc $j$ fails in round $r$, 0 otherwise
- $d_{jr}^r = $ demand at node $i$ in round $r$
- and many other variables
\[
\begin{align*}
\text{max } & \sum_{i \in D} d_i^R \\
\text{Subject to: } & \sum_{j \in \delta^+(i)} f_j^r - \sum_{j \in \delta^-(i)} f_j^r = \begin{cases} 
s_i^r & i \in G \\
-d_i^r & i \in D \\
0 & \text{otherwise}
\end{cases} \forall 1 \leq r \leq R \\
f_j^r &= \pi_j^r - \nu_j^r \quad \forall j \in A \text{ and } 1 \leq r \leq R \\
\pi_j^r &\leq \tilde{D}p_j^r, \quad \nu_j^r \leq \tilde{D}n_j^r, \quad \forall j \in A \text{ and } 1 \leq r \leq R \\
p_j^r + n_j^r &= 1 - \sum_{h=1}^{r-1} y_j^h, \quad \forall j \in A \text{ and } 1 \leq r \leq R \\
\pi_j^r + \nu_j^r - u_j &\leq \tilde{D}y_j^r \quad \forall j \in A \text{ and } 1 \leq r \leq R \\
\pi_j^r + \nu_j^r &\geq u_j y_j^r \quad \forall j \in A \text{ and } 1 \leq r \leq R - 1 \\
\pi_j^r + \nu_j^r &\leq u_j \quad \forall j \in A \\
|\phi_i^r - \phi_j^r - x_j f_j^r| &\leq M_j \sum_{h=1}^{r-1} y_j^h \quad \forall j \in A \\
0 &\leq s_i^r \leq \bar{s}_i \quad \forall i \in G, \quad 0 \leq d_i^r \leq \bar{d}_i \quad \forall i \in D, \\
p_j^r, n_j^r, y_j^r &= 0 \text{ or } 1, \quad \forall j \in A \text{ and } 1 \leq r \leq R \\
0 &\leq \pi_j^r, \quad 0 \leq \nu_j^r, \quad \forall j \in A \text{ and } 1 \leq r \leq R.
\end{align*}
\]
What’s bad about the formulation

- probably can’t solve it for medium to large networks
- stochastic variant probably needed, harder
What’s bad about the formulation

- probably can’t solve it for medium to large networks
- stochastic variant probably needed, harder
- optimal solutions = complex policies
Adaptive affine controls

For each demand $v$, and round $r$, control $c^r_v$, $b^r_v$, $s^r_v$ to be computed.
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$\rightarrow$ Parameterized by integer $r > 0$. 
Adaptive affine controls

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At round $r$,

- Let $\kappa =$ maximum overload of any line within radius $r$ of $v$
Adaptive affine controls

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$\rightarrow$ Parameterized by integer $r > 0$.

At round $r$,

- Let $\kappa = \text{maximum overload}$ of any line within radius $r$ of $v$
- If $\kappa > c_r^v$, demand at $v$ reduced (scaled) by a factor
  \[ \max \left\{ 1, s_r^v (c_r^v - \kappa) + b_r^v \right\}. \]
Adaptive affine controls

For each demand $v$, and round $r$, control $c_r^v, b_r^v, s_r^v$ to be computed

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- If $\kappa > c_r^v$, demand at $v$ reduced (scaled) by a factor

$$\max \{1, \ s_r^v (c_r^v - \kappa) + b_r^v\}.$$  

The goal: pick control to maximize demand being served at the end of round $R$. 

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This talk: $r = n$ (number of nodes)
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This talk: $r = n$ (number of nodes)

**Special case: (optimal scaling problem)**

Insist that for each $r$, $(c_v^r, b_v^r, s_v^r) = (c^r, b^r, s^r)$ for every $v$
For each demand $v$, and round $r$, control $c^r_v$, $b^r_v$, $s^r_v$

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$$\min \{ 1, \left[ s^r_v (c^r_v - \kappa) + b^r_v \right]^+ \}.$$ 

This talk: $r = n$ (number of nodes)

**Special case: (optimal scaling problem)**

Insist that for each $r$, $(c^r_v, b^r_v, s^r_v) = (c^r, b^r, s^r)$ for every $v$

Then, equivalent problem:

- In round $r$, let $\alpha^r(K) \leq 1$ be chosen for each component of the network in round $r$
- If node $v \in$ component $K$, then its demand is scaled by $\alpha^r(K)$
Notation:

- $\hat{\beta}$ = supply/demand vector at time 0
- $\hat{f}$ = corresponding power flows at time 0
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Robust/stochastic version?
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- Robust/stochastic version?
General case: simulation-based optimization

Given a control vector $\tilde{u} = (c^r_v, b^r_v, s^r_v)$ (over all $v$ and $r$),

$\Theta(\tilde{u})$ = throughput (total demand) satisfied at end of cascade

- Maximization of $\Theta(\tilde{u})$ should be (very?) fast
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- From a strict perspective, $\Theta(\tilde{u})$ is not even continuous
General case: simulation-based optimization

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- From a strict perspective, \( \Theta(\tilde{u}) \) is not even continuous

\( \Theta(\tilde{u}) \) is obtained through a simulation
Derivative-free optimization

Conn, Scheinberg, Vicente, others

Rough description:

- Sample a number of control vectors $\tilde{u}$
- Use the sample points to construct a convex approximation to $\tilde{\Theta}$
- Optimize this approximation; this yields a new sample point

Scalability to large dimensionality?
“First order” method

Given a control vector $\tilde{u}$

1. Estimate the “gradient” $g = \nabla \tilde{\Theta}(\tilde{u})$ through finite differences.
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→ Easily parallelizable
Line searches

ls3.1

ls3.2

ls4.2

Daniel Bienstock (Columbia University)

Power grid vulnerability analysis

Dimacs 2010
Current parallel implementation: boss-nerd

- Boss carries out search algorithm
- Nerds simulate cascades with given control
- Communication using Unix sockets
Scaling

Example: 10000 nodes, 19309 lines
5 gradient steps
8-core i7 CPUs (3 machines total)

<table>
<thead>
<tr>
<th>cores</th>
<th>wall-clock sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
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<tr>
<td>16</td>
<td>14618</td>
</tr>
<tr>
<td>24</td>
<td>9918</td>
</tr>
</tbody>
</table>
Initial experiments with Eastern Interconnect

- 15023 nodes, 23769 lines.
- 2122 generator nodes, 6261 demand nodes
- “Equivalent” DC flow version
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- Methodology for experiments
  1. Generate an interdiction of the grid ("initial event")
  2. Compute control and simulate
Initial experiments with Eastern Interconnect

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- 2122 generator nodes, 6261 demand nodes
- “Equivalent” DC flow version

Methodology for experiments

1. Generate an interdiction of the grid (“initial event”)
2. Compute control and simulate
3. At least three rounds of cascade after initial event
Computing a control

(1) Solve scaling problem – let \((c^*, b^*, s^*)\) be optimal

\[ \Sigma_1, \ldots, \Sigma_k. \]

Example = demand quantiles.

Perform segmented gradient search starting from \((c^*, b^*, s^*)\).

Look for a control with \((c_{rv}, b_{rv}, s_{rv}) = \text{constant}\) for each given \(r\) and all \(v\) in a common \(\Sigma_i\).

Perform full gradient search starting from the output in (2).
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(1) Solve scaling problem – let \((c^*, b^*, s^*)\) be optimal

(2) Partition demand nodes into “small” number of segments \(\Sigma_1, \ldots, \Sigma_k\).
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Experiments

- $K$ random lines taken out
- Highly loaded lines more likely to be taken out; connectivity preserved
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- Highly loaded lines more likely to be taken out; connectivity preserved

<table>
<thead>
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<th>yield, (%)</th>
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<th>wallclock (sec)</th>
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Conjectures

- It is best to stop the cascade in the first round

- It is best to apply control in the first round only, and ride out the cascade
Conjectures

- It is best to stop the cascade in the first round
- It is best to apply control in the first round only, and ride out the cascade

(Answer: both wrong)
Details: cascade with 50 (highly loaded) random lines taken out

- No control $\Rightarrow$ yield = 0%
- Optimal round 1 only constant control $\Rightarrow$ yield = 38%
- Optimal scaling control $\Rightarrow$ yield = 45%
- Plus segmented gradient seach $\Rightarrow$ yield = 50%
Load distribution at time zero

\[(\text{load of arc } j = \frac{|f_j|}{u_j})\]

<table>
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Optimal round 1 scale = 0.51,
Load distribution at time zero

(load of arc \( j = \frac{|f_j|}{u_j} \))

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**Optimal** round 1 scale = 0.51, so 44 faults
Out-of-sample testing: use stochastic faults

at round \( r \),

- **\( e \) fails** if \( u_e < \tilde{f}_e^r \),

- **\( e \) does not fail** if \((1 - \gamma)u_e > \tilde{f}_e^r\), \((\gamma = \text{tolerance})\)

if \((1 - \gamma)u_e \leq \tilde{f}_e^r \leq u_e\) then **\( e \) fails with probability 1/2**
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$e$ fails if $u_e < \tilde{f}_e^r$,

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if $(1 - \gamma)u_e \leq \tilde{f}_e^r \leq u_e$ then $e$ fails with probability 1/2

What is the impact of $\gamma$?
$\gamma = 0.03, 0.10, 0.20,$

10000 runs