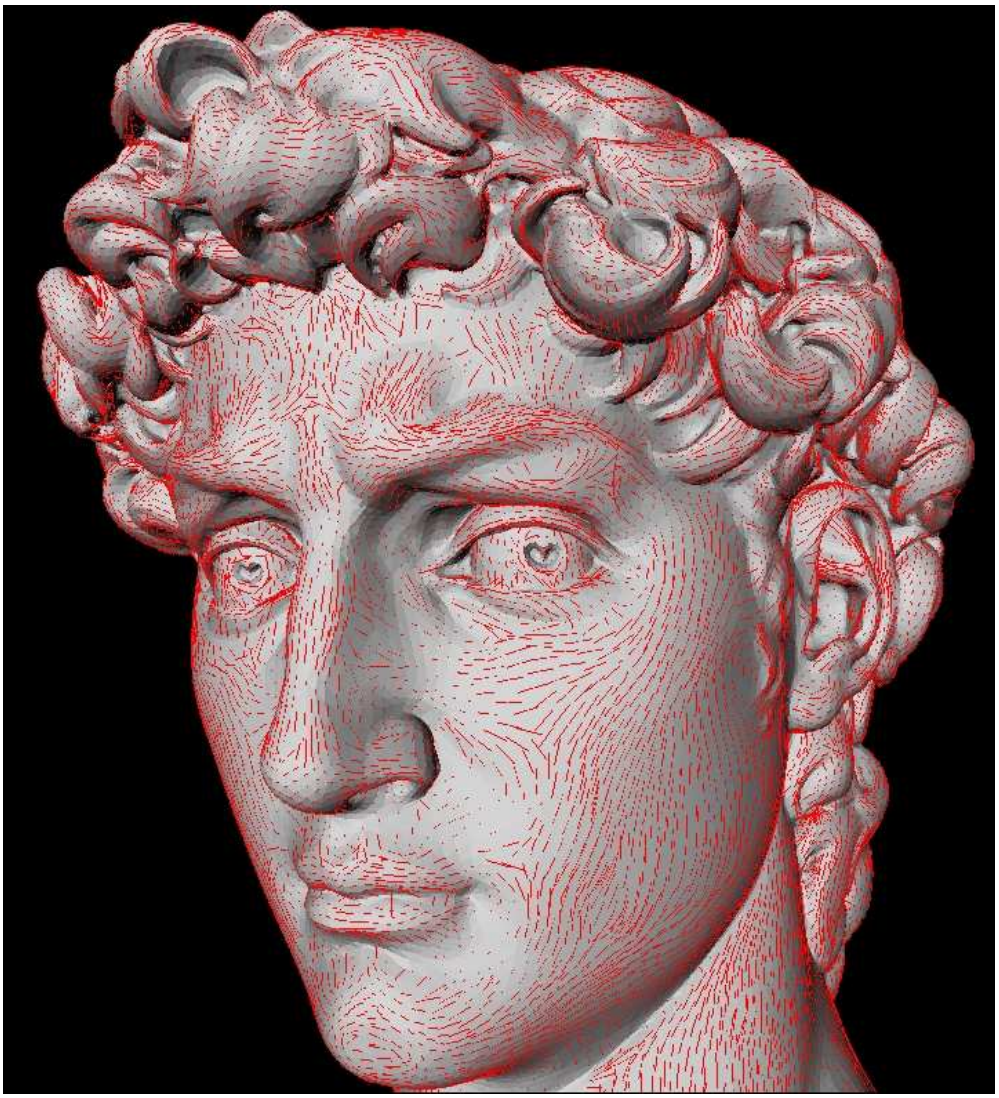
# Estimating Differential Quantities using Polynomial fitting of Osculating Jets

Frederic.Cazals@inria.fr
Marc.Pouget@inria.fr



# Smooth surfaces, point clouds, meshes, Differential quantities

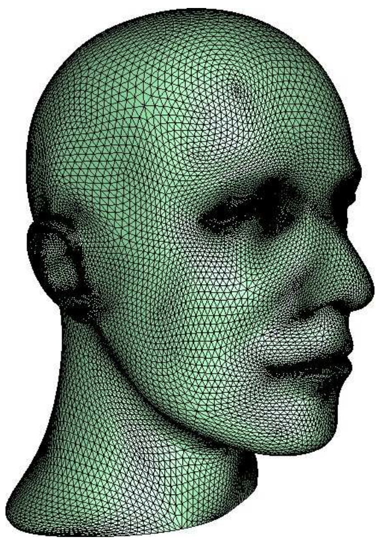
## Smooth surfaces & Differential quantities:

- surface area
- tangent plane, principal curvatures and directions
- special points [MORE Difficult!] —umbilics, parabolic lines, curvature lines, ridges, geodesics, medial axis, skeleton

## Sampled surfaces & Applications:

- surface reconstruction, segmentation
- smoothing, re-meshing
- parameterization

# Smooth surface. . . or not?



Phenomenological ambiguity:
mesh or smooth surface?

Ill-defined notions:
smooth mesh, sharp edge, normal,. . .

Questions raised:
differential operators
convergence & robustness issues

# Differential Geometries

Classical (smooth) diff. geom.

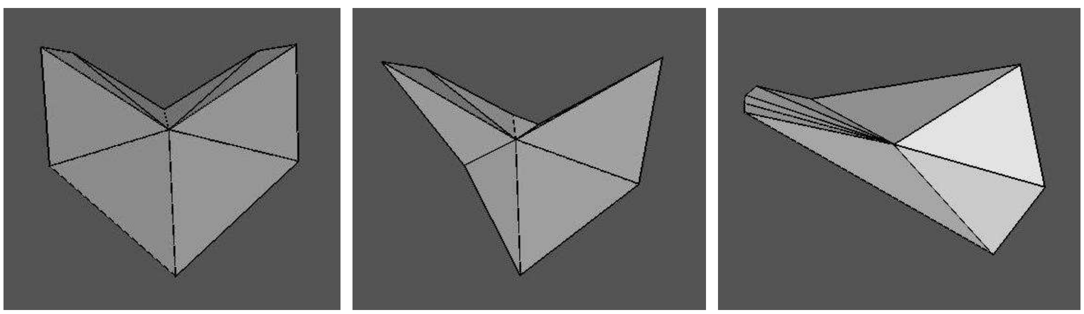
Diff. geom. for non-smooth objects

- normal cycle theory
- Clarke's theory
- Filipov's theory

# Smooth Diff Geom
# & Convergence issues

The angular defect exple

$$2\pi - \sum_i \gamma_i \sim \eta^2 (Ak_G + Bk_m^2 + Ck_M^2)$$



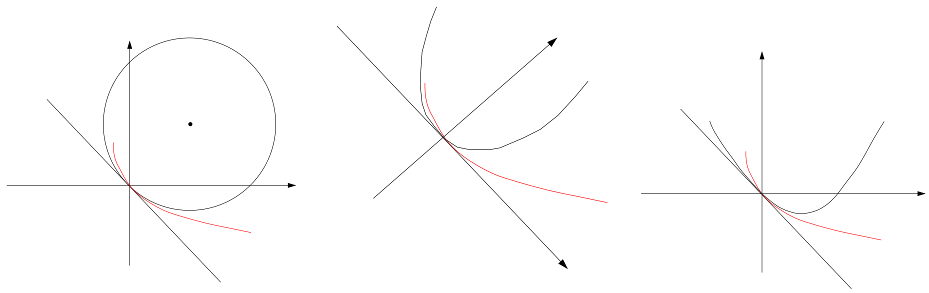
Triangulations of $z = (2x^2 - y^2)/2$

Convergence wishes: pointwise, global, various topologies

- local cv, usual topology: this paper
- "global" cv, topology of currents: Cohen-Steiner & Morvan, ACM SoCG'03

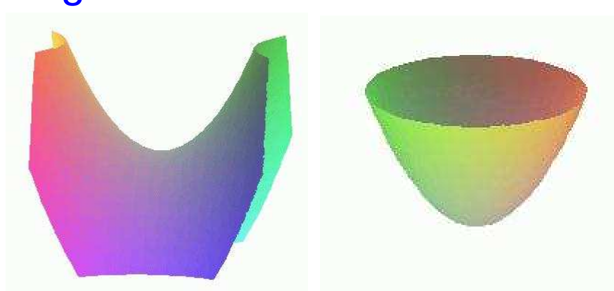
# Estimating Differential Properties using Polynomial fitting

## Osculating quadric: not unique



**Thm .** There are 9 Euclidean conics and 17 Euclidean quadrics.

## Manifolds and Height functions



$$f(x,y) = ax + by + \frac{1}{2}(k_1 x^2 + k_2 y^2) + hot$$

## Polynomial Fitting & Variants

- two (or more) stages methods
- interpolation - approximation

# Height functions and jets

Height funtion = jet + h.o.t:

$$f(x,y) = J_{B,n}(x,y) + O(||(x,y)||^{n+1}),$$

with

$$J_{B,n}(x,y) = \sum_{k=1}^{n} H_{B,k}(x,y),\ H_{B,k}(x,y) = \sum_{j=0}^{k} B_{k-j,j} x^{k-j} y^{j}.$$

$\Rightarrow N_n = (d+1)(d+2)/2$ coefficients

# Differential Quantities

Tangent plane

$$n_S = (-B_{10}, -B_{01}, 1)^t / \sqrt{1 + B_{10}^2 + B_{01}^2}.$$

Second order info using the Weingarten map . . .

$$\{B_{10}, B_{01}, B_{20}, B_{11}, B_{02}\}$$

Higher order info Monge form of the surface

# Sample points, Interpolation, Approximation

Input: $N$ points $p_i(x_i, y_i, z_i = f(x_i, y_i))$

Interpolation: find a $n$-jet $J_{A,n}$:

$$f(x_i, y_i) = J_{B,n}(x_i, y_i) + O(||(x_i, y_i)||^{n+1}) = J_{A,n}(x_i, y_i), \quad i = 1 \ldots N.$$

Least-Square Approximation: find a $n$-jet $J_{A,n}$ minimizing:

$$\sum_{i=1}^{N} (J_{A,n}(x_i, y_i) - f(x_i, y_i))^2.$$

# Convergence issues

Sequence of converging points

$$p_i(x_i = a_i h, y_i = b_i h, z_i = f(x_i, y_i))$$

$a_i$ and $b_i$ arbitrary, $h \to 0$ —uniform convergence

Wish: $A_{ij} = B_{ij} + O(r(h))$

**Thm.**

$$A_{k-j,j} = B_{k-j,j} + O(h^{n-k+1}) \quad \forall k = 0, \ldots, n \;\; \forall j = 0, \ldots, k.$$

# Matrix set-up of the problem

$J_{B,n}$ jet of the height function sought
$J_{A,n}$ answer of the interpo./approx. problem

$N_n$-Vector of unknowns

$$A = (A_{0,0}, A_{1,0}, A_{0,1}, \ldots, A_{0,n})^t.$$

$N$-vector of ordinates, i.e. with $z_i = f(x_i, y_i)$:

$$B = (z_1, z_2, \ldots, z_N)^t = (J_{B,n}(x_i, y_i) + O(||(x_i, y_i)||^{n+1}))_{i=1,\ldots,N}.$$

Vandermonde $N \times N_n$ matrix

$$M = (1, x_i,\ y_i,\ x_i^2, \ldots,\ x_i y_i^{n-1},\ y_i^n)_{i=1,\ldots,N}.$$

Interpolation $N = N_n$, linear square system; solve $MA = B$

Approximation $N > N_n$, rectangular system; solve $\min ||MA - B||_2$

# Poised Bivariate

# Lagrange Interpolation

$\Pi_n$: space of bivar. polyn. of degree $\leq n$; $\dim(\Pi_n) = N_n = \binom{n+2}{n}$

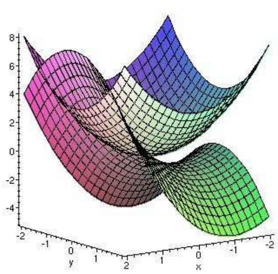
nodes $X = \{x_1, \ldots, x_N\}$

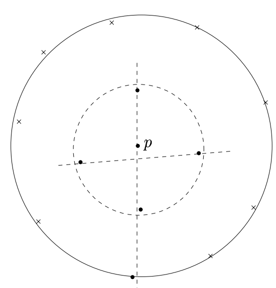
$f : \mathbb{R}^2 \to \mathbb{R}$

Interpolation problem poised for $X$:

for any $f$ $\exists$ unique $P \in \Pi_n \mid P(x_i) = f(x_i),\ i = 1, \ldots, N.$

# Almost Degenerate cases



# Approximation

$\min ||MA - B||_2$ has a unique solution $\Leftrightarrow rank(M) = N_n$

Residual of the system $\rho = ||MA - B||_2$

# SVD & Condition Numbers

Thm: SVD decomposition of a $\mathbb{R}^{m\times n}$ matrix $A$: $\exists$ orthogonal matrices $U$ and $V$:

$$\begin{cases} U^t AV = diag(\sigma_p, \ldots, sigma_1), p = \min(m,n), \\ \sigma_p \geq \ dots \geq sigma_1 \geq 0. \end{cases}$$

# Cond. Numbers and Jet fitting, Relative Errors

Condition numbers = magnification factor

`Error on solution = Error on input × conditioning.`

$$\begin{cases} \text{of } M\text{; interpol}: & \kappa_2(M) = ||M||_2 ||M^{-1}||_2 = \sigma_n/\sigma_1, \\ \text{approx.}: & \kappa_2(M) + \kappa_2(M)^2 \rho \text{ with } \rho = ||MX - B||_2 \text{ the residual.} \end{cases}$$

Thm. $X$ and $\widetilde{X}$ solutions of:

$\Rightarrow$ Interpol.: $MX = B$ and $(M + \Delta M)\widetilde{X} = B + \Delta B$,
$\Rightarrow$ Approx.: $\min ||MX - B||_2$ and $\min ||(M + \Delta M)\widetilde{X} - (B + \Delta B)||_2$,

with $\varepsilon > 0$ such that

$$||\Delta M||_2 / ||M||_2 \leq \varepsilon, ||\Delta B||_2 / ||B||_2 \leq \varepsilon, \varepsilon \kappa_2(M) < 1.$$

Then: $||X - \widetilde{X}||_2 / ||X||_2 \leq \varepsilon$ conditioning.

# Pre-conditioning

# the Vandermonde system

Vandermonde matrix:

$$M = (1, x_i,\ y_i,\ x_i^2, \ldots,\ x_i y_i^{n-1},\ y_i^n)_{i=1,\ldots,N}.$$

Column-scaling. $x_i$s, $y_i$s being of order $h$, scale $x_i^k y_i^l$ by $h^{k+l}$

New system:

$$D = diag(1, h, h, h^2, \ldots, h^n, h^n),$$

$$MA = B \Leftrightarrow MDD^{-1}A = B \Leftrightarrow M^{'}Y = B, i.e. X = DY.$$

Alternatives: Newton polynomials

# Surfaces and curves: selected results

Hypothesis $N$ points $p_i(x_i, y_i, z_i)$, with $x_i = O(h), y_i = O(h)$

Thm.[Interpolation or Approximation] The coefficients of degree $k$ of the Taylor expansion of $f$ to accuracy $O(h^{n-k+1})$:

$$A_{k-j,j} = B_{k-j,j} + O(h^{n-k+1}) \quad \forall k = 0, \ldots, n \;\; \forall j = 0, \ldots, k.$$

If interpolation is used and the origin is one of the samples then $A_{0,0} = B_{0,0} = 0$.

Rmk. If lfs is bounded from above and $h = O(\varepsilon \text{lfs}) \ldots$

Corrolary

- normal coeffs estimated with accuracy $O(h^n)$,
- coeffs of I, II, shape operator: estimated with accuracy $O(h^{n-1})$

# Curves

Thm.[Interpolation, details omitted]:

$$|A_k - B_k| \leq \varepsilon^{(n-k+1)} c \left(\frac{n}{2d}\right)^{\frac{n(n-1)}{2}}.$$

# Algorithm

Collecting $N_n$ samples

- Mesh case: ith rings
- PC case: local mesh, Power Diag. in the tangent plane

Fitting problem, degenerate cases —almost singular matrices

- Interpolation: choose samples differently
- Approximation: decrease degree, increase # pts

Differential quantities

- Order two info.: Weingarten map of the height func.
- Higher order info: retrieve the Monge form of the surface

# Convergence results: experimental Illustration

**Thm.**

$$A_{k-j,j} = B_{k-j,j} + O(h^{n-k+1}) \quad \forall k = 0,\ldots,n \;\; \forall j = 0,\ldots,k.$$

Discrepancy $\delta$ on a $k$th order diff. quantity

$$\delta = \mid F_A(A_{\leq k}) - F_B(B_{\leq k}) \mid, \;\; \delta \approx c\, h^{n-k+1}$$

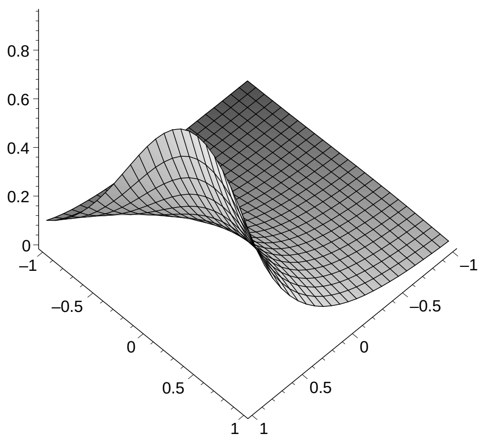
Conv. over a sequence of finer samples —$h \to 0$

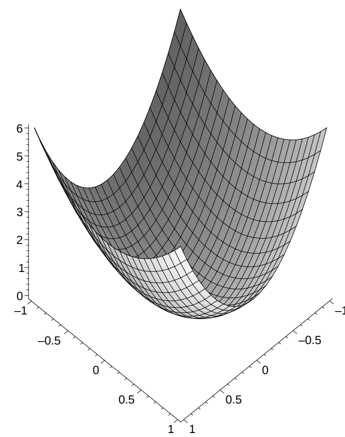
$$\log(1/\delta) \approx \log(1/c) + (n-k+1)\log(1/h)$$

Conv. when increasing the degree $n$

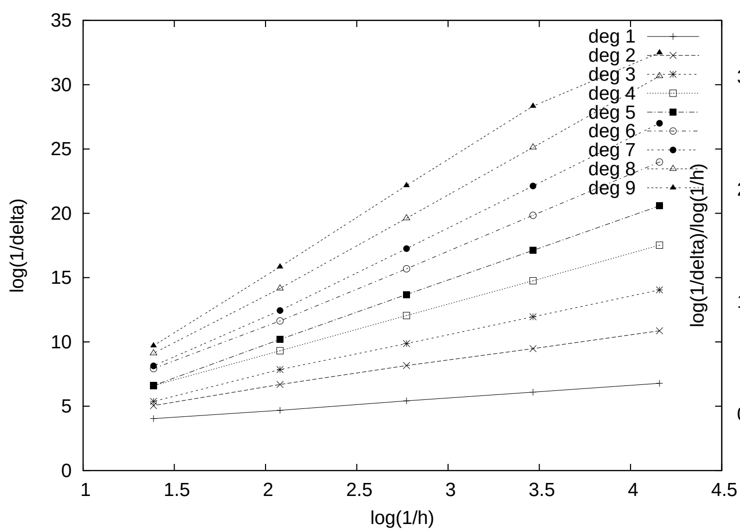
$$\frac{\log(1/\delta)}{\log(1/h)} \approx \frac{\log(1/c)}{\log(1/h)} + (n-k+1)$$

# Convergence

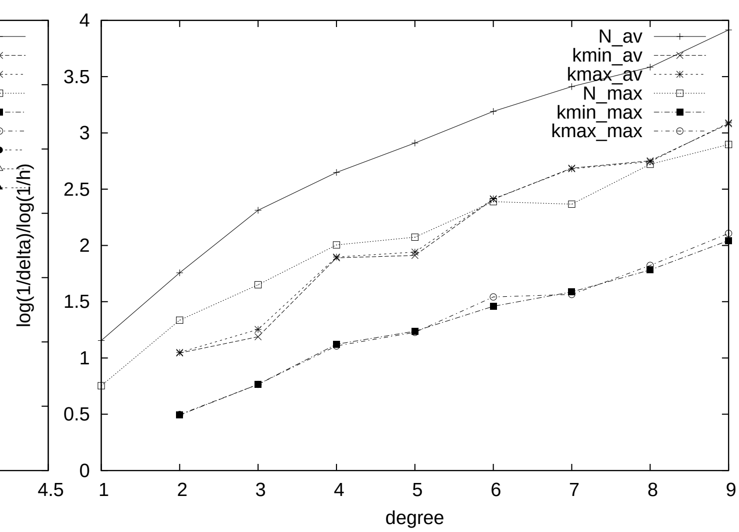




$f(u,v) = 0.1e^{2u+v-v^2}$ and $g(u,v) = 4u^2 + 2v^2$

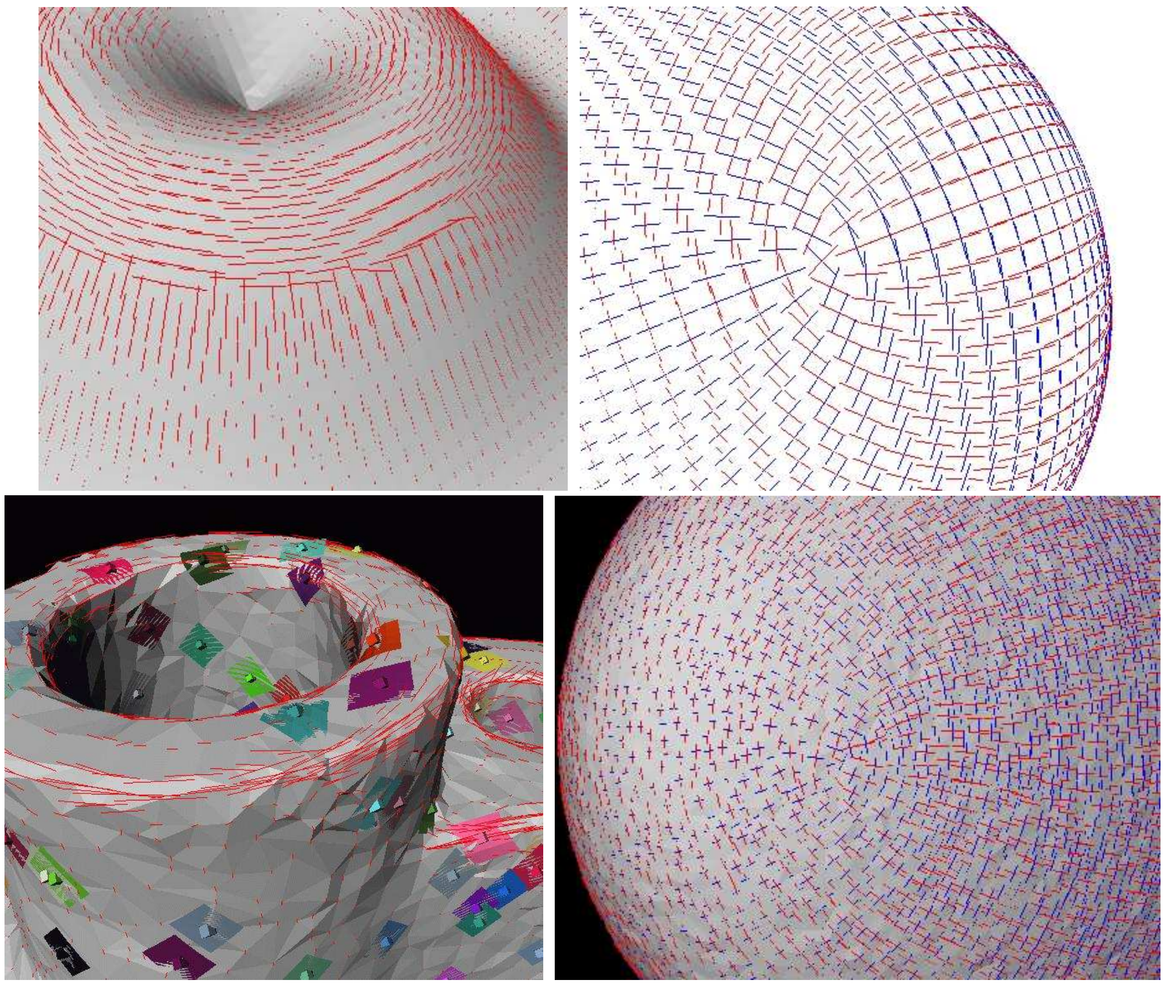



Exponential model: Convergence of the normal estimate wrt h, approximation fitting




Polynomial model: Convergence of normal and curvature wrt the degree of the approximation fitting

# Illustrations



# Illustrations