Dynamic Pricing for Non-Perishable Products with Demand Learning

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Motivation

Dynamic Pricing with Demand Learning
Motivation

$N_0$

Product 1

Clearance

Regular Season

$\tau_1$

$\tau_1 + \tau_2$
Motivation

Dynamic Pricing with Demand Learning
Motivation

- For many retail operations, “capacity” is measured by store/shelf space.

- A key performance measure in the industry is 
  
  \textbf{Average Sales per Square Foot per Unit Time.}

- Trade-off between short-term benefits and the opportunity cost of assets.

  \textbf{Margin vs. Rotation.}

- As opposed to the airline or hospitality industries, selling horizons are flexible.

- In general, most profitable/unprofitable products are new items for which there is little demand information.
Outline

✓ Model Formulation.

✓ Perfect Demand Information.

✓ Incomplete Demand Information.
  - Inventory Clearance
  - Optimal Stopping ("outlet option")

✓ Conclusion.
Model Formulation

I) STOCHASTIC SETTING:
- A probability space \((\Omega, F, \mathbb{P})\).
- A standard Poisson process \(D(t)\) under \(\mathbb{P}\) and its filtration \(F_t = \sigma(D(s) : 0 \leq s \leq t)\).
- A collection \(\{\mathbb{P}_\alpha : \alpha > 0\}\) such that \(D(t)\) is a Poisson process with intensity \(\alpha\) under \(\mathbb{P}_\alpha\).
- For a process \(f_t\), we define \(I_{f(t)} := \int_0^t f_s \, ds\).

II) DEMAND PROCESS:
- Pricing strategy, a nonnegative (adapted) process \(p_t\).
- A price-sensitive unscaled demand intensity
  \[ \lambda_t := \lambda(p_t) \iff p_t = p(\lambda_t). \]
- A (possibly unknown) demand scale factor \(\theta > 0\).
- Cumulative demand process \(D(I_{\lambda(t)})\) under \(\mathbb{P}_\theta\).
- Select \(\lambda \in \mathcal{A}\) the set of admissible (adapted) policies
  \[ \lambda_t : \mathbb{R}_+ \to [0, \Lambda]. \]

\[ \text{Demand Intensity} \]

Exponential Demand Model
\[ \lambda(p) = \Lambda \exp(-\alpha p) \]
Increasing \(\theta\)

Dynamic Pricing with Demand Learning
Model Formulation

III) Revenues:

- Unscaled revenue rate $c(\lambda) := \lambda p(\lambda)$, \( \lambda^* := \arg\max_{\lambda \in [0, \Lambda]} \{c(\lambda)\} \), \( c^* := c(\lambda^*) \).

- Terminal value (opportunity cost): $R$  
  Discount factor: $r$

- Normalization: \( c^* = r R \).

IV) Selling Horizon:

- Inventory position: \( N_t = N_0 - D(I_\lambda(t)) \).

- \( \tau_0 = \inf\{t \geq 0 : N_t = 0\} \),  
  \( T := \{\mathcal{F}_t - \text{stopping times} \, \tau \, \text{such that} \, \tau \leq \tau_0\} \)

V) Retailer’s Problem:

\[
\max_{\lambda \in \mathcal{A}, \, \tau \in T} \mathbb{E}_\theta \left[ \int_0^\tau e^{-r t} p(\lambda_t) \, dD(I_\lambda(t)) + e^{-r \tau} R \right]
\]

subject to \( N_t = N_0 - D(I_\lambda(t)) \).
Suppose $\theta > 0$ is known at $t = 0$ and an inventory clearance strategy is used, i.e., $\tau = \tau_0$.

Define the value function

$$W(n; \theta) = \max_{\lambda \in A} \mathbb{E}_\theta \left[ \int_0^{\tau_0} e^{-rt} p(\lambda_t) dD(I_{\lambda}(t)) + e^{-r\tau} R \right]$$

subject to $N_t = n - D(I_{\lambda}(t))$ and $\tau_0 = \inf\{t \geq 0 : N_t = 0\}$.

The solution satisfies the recursion

$$\frac{r W(n; \theta)}{\theta} = \Psi(W(n-1; \theta) - W(n; \theta)) \quad \text{and} \quad W(0; \theta) = R,$$

where $\Psi(z) \triangleq \max_{0 \leq \lambda \leq \Lambda} \{\lambda z + c(\lambda)\}$.

**Proposition.** For every $\theta > 0$ and $R \geq 0$ there is a unique solution $\{W(n) : n \in \mathbb{N}\}$.

- If $\theta \geq 1$ then the value function $W$ is increasing and concave as a function of $n$.
- If $\theta \leq 1$ then the value function $W$ is decreasing and convex as a function of $n$.
- $\lim_{n \to \infty} W(n) = \theta R$. 
Value function for two values of $\theta$ and an exponential demand rate $\lambda(p) = \Lambda \exp(-\alpha p)$.

The data used is $\Lambda = 10$, $\alpha = 1$, $r = 1$, $\theta_1 = 1.2$, $\theta_2 = 0.8$, $R = \Lambda \exp(-1)/(\alpha r) \approx 3.68$. 

Dynamic Pricing with Demand Learning 9
**Corollary.** Suppose $c(\lambda)$ is strictly concave.

The optimal sales intensity satisfies:

$$\lambda^*(n; \theta) = \arg\max_{0 \leq \lambda \leq \Lambda} \{ \lambda (W(n-1; \theta) - W(n; \theta)) + c(\lambda) \}.$$

- If $\theta \geq 1$ then $\lambda^*(n; \theta) \uparrow n$.

- If $\theta \leq 1$ then $\lambda^*(n; \theta) \downarrow n$.

- $\lambda^*(n; \theta) \downarrow \theta$.

- $\lim_{n \to \infty} \lambda^*(n, \theta) = \lambda^*$.

What about inventory turns (rotation)?

**Proposition.** Let $s(n, \theta) \triangleq \theta \lambda^*(n, \theta)$ be the optimal sales rate for a given $\theta$ and $n$.

If $\frac{d}{d\lambda}(\lambda^p(\lambda)) \leq 0$, then $s(n, \theta) \uparrow \theta$. 

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Exponential Demand $\lambda(p) = \Lambda \exp(-\alpha p)$.

$\Lambda = 10$, $\alpha = r = 1$, $\theta_1 = 1.2$, $\theta_2 = 0.8$, $R = 3.68$. 

Dynamic Pricing with Demand Learning
Full Information

Summary:

- A tractable dynamic pricing formulation for the inventory clearance model.

- $W(n; \theta)$ satisfies a simple recursion based on the Fenchel-Legendre transform of $c(\lambda)$.

- With full information products are divided in two groups:
  - High Demand Products with $\theta \geq 1$: $W(n, \theta)$ and $\lambda^*(n)$ increase with $n$.
  - Low Demand Products with $\theta \leq 1$: $W(n, \theta)$ and $\lambda^*(n)$ decrease with $n$.

- High Demand products are sold at a higher price and have a higher selling rate.

- If the retailer can stop selling the product at any time at no cost then:
  - If $\theta < 1$ stop immediately ($\tau = 0$).
  - If $\theta > 1$ never stop ($\tau = \tau_0$).

- In practice, a retailer rarely knows the value of $\theta$ at $t = 0$!
Incomplete Information: Inventory Clearance

**Setting:**

- The retailer does not know $\theta$ at $t = 0$ but knows $\theta \in \{\theta_L, \theta_H\}$ with $\theta_L \leq 1 \leq \theta_H$.

- The retailer has a prior belief $q \in (0, 1)$ that $\theta = \theta_L$.

- We introduce the probability measure $\mathbb{P}_q = q \mathbb{P}_{\theta_L} + (1 - q) \mathbb{P}_{\theta_H}$.

- We assume an inventory clearance model, i.e., $\tau = \tau_0$.

**Retailer’s Beliefs:**

Define the belief process $q_t := \mathbb{P}_q[\theta | \mathcal{F}_t]$.

**Proposition.** $q_t$ is a $\mathbb{P}_q$-martingale that satisfies the SDE

$$
\mathrm{d}q_t = -\eta(q_{t-}) [\mathrm{d}D_t - \lambda_t \bar{\theta}(q_{t-}) \mathrm{d}t],
$$

where

$$
\bar{\theta}(q) := \theta_L q + \theta_H (1 - q)
$$

and

$$
\eta(q) := \frac{q (1 - q) (\theta_H - \theta_L)}{\theta_L q + \theta_H (1 - q)}.
$$
Incomplete Information: Inventory Clearance

**Retailer’s Optimization:**

\[
V(N_0, q) = \sup_{\lambda \in A} \mathbb{E}_q \left[ \int_0^{\tau_0} e^{-r t} p(\lambda_t) \, dD(I_\lambda(s)) + e^{-r \tau_0} R \right]
\]

subject to

\[
N_t = N_0 - \int_0^t dD(I_{\lambda}(s)),
\]

\[
\dot{q}_t = -\eta(q_{t-}) [dD_t - \lambda_t \theta(q_{t-}) dt], \quad q_0 = q,
\]

\[
\tau_0 = \inf\{ t \geq 0 : N_t = 0 \}.
\]

**HJB Equation:**

\[
rV(n, q) = \max_{0 \leq \lambda \leq \Lambda} \left[ \lambda \theta(q) [V(n - 1, q - \eta(q)) - V(n, q) + \eta(q) V_q(n, q)] + \bar{\theta}(q) c(\lambda) \right],
\]

with boundary condition \( V(0, q) = R, V(n, 0) = W(n; \theta_H), \) and \( V(n, 1) = W(n; \theta_L). \)

**Recursive Solution:**

\[
V(0, q) = R, \quad V(n, q) + \Phi \left( \frac{r V(n, q)}{\theta(q)} \right) - \eta(q) V_q(n, q) = V(n - 1, q - \eta(q)).
\]
Incomplete Information: Inventory Clearance

Proposition.
- The value function $V(n, q)$ is
  a) monotonically decreasing and convex in $q$,
  b) bounded by
    \[ W(n; \theta_L) \leq V(n, q) \leq W(n; \theta_H), \quad \text{and} \]
    \[ V(n, q) \xrightarrow{n \to \infty} R \bar{\theta}(q), \quad \text{uniformly in} \ q. \]
  c) uniformly convergent as $n \uparrow \infty$,
  \[
  V(n, q) \xrightarrow{n \to \infty} R \bar{\theta}(q), \quad \text{uniformly in} \ q.
  \]
- The optimal demand intensity satisfies
  \[
  \lim_{n \to \infty} \lambda^*(n, q) = \lambda^*.
  \]
Conjecture:
The optimal sales rate $\bar{\theta}(q) \lambda^*(n, q) \downarrow q$. 
**Asymptotic Approximation:** Since

\[
\lim_{n \to \infty} V(n, q) = R \bar{\theta}(q) = \lim_{n \to \infty} \{q W(n, \theta_L) + (1 - q) W(n, \theta_H)\},
\]

we propose the following approximation for \( V(n, q) \)

\[
\tilde{V}(n, q) := q W(n, \theta_L) + (1 - q) W(n, \theta_H).
\]

**Some Properties of \( \tilde{V}(n, q) \):**

- Linear approximation easy to compute.
- Asymptotically optimal as \( n \to \infty \).
- Asymptotically optimal as \( q \to 0^+ \) or \( q \to 1^- \).
- \( \tilde{V}(n, q) = \mathbb{E}_q[W(n, \theta)] \neq W(n, \mathbb{E}_q[\theta]) =: V^{CE}(n, q) = \text{Certainty Equivalent.} \)
Relative Error (\%) := \frac{V^{\text{approx}}(n, q) - V(n, q)}{V(n, q)} \times 100\%.

Exponential Demand \( \lambda(p) = \Lambda \exp(-\alpha p) \):

Inventory = 5, \( \Lambda = 10 \), \( \alpha = r = 1 \), \( \theta_H = 5.0 \), \( \theta_L = 0.5 \).
Incomplete Information: Inventory Clearance

For any approximation \( V_{\text{approx}}(n, q) \), define the corresponding demand intensity using the HJB

\[
\lambda_{\text{approx}}(n, q) := \arg \max_{0 \leq \lambda \leq \Lambda} \left[ \lambda \bar{\theta}(q) [V_{\text{approx}}(n-1, q-\eta(q)) - V_{\text{approx}}(n, q)] + \lambda \kappa(q) V_{q_{\text{approx}}}(n, q) + \bar{\theta}(q) c(\lambda) \right].
\]

Relative Price Error (%) := \( \frac{p(\lambda_{\text{approx}}) - p(\lambda^*)}{p(\lambda^*)} \times 100\% \).

### Asymptotic Approximation (%)

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### Certainty Equivalent (%)

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Relative price error for the exponential demand model \( \lambda(p) = \Lambda \exp(-\alpha p) \), with \( \Lambda = 20 \) and \( \alpha = 1 \).
Incomplete Information: Inventory Clearance

When should the retailer engage in selling a given product?

When \( V(n, q) \geq R \).

Using the asymptotic approximation \( \tilde{V}(n, q) \), this is equivalent to

\[
q \leq \tilde{q}(n) := \frac{W(n; \theta_H) - R}{[W(n; \theta_H) - R] + [R - W(n; \theta_L)]}.
\]

\( \tilde{q}(n) \to \tilde{q}_\infty := \frac{\theta_H - 1}{\theta_H - \theta_L} \), as \( n \to \infty \).

Exponential demand rate \( \lambda(p) = \Lambda \exp(-\alpha p) \).

Data: \( \Lambda = 10, \alpha = 1, r = 1, \theta_H = 1.2, \theta_L = 0.8 \).
Incomplete Information: Inventory Clearance

**Summary:**

- Uncertainty in market size ($\theta$) is captured by a new state variable $q_t$ (a jump process).

- $V(n, q)$ can be computed using a recursive sequence of ODEs.

- Asymptotic approximation $\tilde{V}(n, q) := \mathbb{E}_q[W(n, \theta)]$ performs quite well.
  - Linear approximation easy to compute.
  - Value function: $V(n, q) \approx \tilde{V}(n, q)$.
  - Pricing strategy: $p^*(n, q) \approx \tilde{p}(n, q)$.

- Products are divided in two groups as a function of $(n, q)$:
  - Profitable Products with $q < \bar{q}(n)$ and
  - Non-profitable Products with $q > \bar{q}(n)$.

- The threshold $\bar{q}(n)$ increases with $n$, that is, the retailer is willing to take more risk for larger orders.
Incomplete Information: Optimal Stopping

**Setting:**
- Retailer does not know \( \theta \) at \( t = 0 \) but knows \( \theta \in \{ \theta_L, \theta_H \} \) with \( \theta_L \leq 1 \leq \theta_H \).
- Retailer has the option of removing the product at any time, “Outlet Option”.

**Retailer’s Optimization:**

\[
U(N_0, q) = \max_{\lambda \in \mathcal{A}, \tau \in \mathcal{T}} \mathbb{E}_q \left[ \int_0^\tau e^{-rt} p(\lambda_t) dD(I_\lambda(t)) + e^{-r\tau} R \right]
\]

subject to
\[
N_t = N_0 - D(I_\lambda(t)),
\]
\[
d_{q_t} = -\eta(q_{t-}) [dD(I_\lambda(t)) - \lambda_t \bar{\theta}(q_{t-}) dt], \quad q_0 = q.
\]

**Optimality Conditions:**

\[
\begin{cases}
U(n, q) + \Phi \left( \frac{rU(n,q)}{\theta(q)} \right) - \eta(q) U_q(n, q) = U(n - 1, q - \eta(q)) & \text{if } U \geq R \\
U(n, q) + \Phi \left( \frac{rU(n,q)}{\theta(q)} \right) - \eta(q) U_q(n, q) \leq U(n - 1, q - \eta(q)) & \text{if } U = R.
\end{cases}
\]
Incomplete Information: Optimal Stopping

Proposition.

a) There is a unique continuously differentiable solution $U(n, \cdot)$ defined on $[0, 1]$ so that $U(n, q) > R$ on $[0, q_n^*)$ and $U(n, q) = R$ on $[q_n^*, 1]$, where $q_n^*$ is the unique solution of

$$R + \Phi \left( \frac{r R}{\theta(q)} \right) = U(n - 1, q - \eta(q)).$$

b) $q_n^*$ is increasing in $n$ and satisfies

$$\frac{\theta_H - 1}{\theta_H - \theta_L} \leq q_n^* \xrightarrow{n \to \infty} q_\infty \leq \text{Root} \left\{ \Phi \left( \frac{r R}{\theta(q)} \right) = \frac{\eta(q)}{q} (\theta_H - 1) R \right\} < 1.$$

c) The value function $U(n, q)$

- Is decreasing and convex in $q$ on $[0, 1]$
- Increases in $n$ for all $q \in [0, 1]$ and satisfies

$$\max\{R, V(n, q)\} \leq U(n, q) \leq \max\{R, m(q)\} \quad \text{for all } q \in [0, 1],$$

where

$$m(q) := W(n, \theta_H) - \frac{(W(n, \theta_H) - R)}{q_n^*} q.$$

- Converges uniformly (in $q$) to a continuously differentiable function, $U_\infty(q)$. 
Exponential demand rate $\lambda(p) = \Lambda \exp(-\alpha p)$. Data: $\Lambda = 10$, $\alpha = 1$, $r = 1$, $\theta_H = 1.2$, $\theta_L = 0.8$. 

Dynamic Pricing with Demand Learning
**Incomplete Information: Optimal Stopping**

**Approximation:**

\[
\tilde{U}(n, q) := \max\{R, W(n, \theta_H) - \frac{(W(n, \theta_H) - R)}{\tilde{q}_n} q\}
\]

where \(\tilde{q}_n\) is the unique solution of

\[
R + \Phi \left( \frac{r R}{\tilde{\theta}(q)} \right) = \tilde{U}(n - 1, q - \eta(q)).
\]

Exponential demand rate \(\lambda(p) = \Lambda \exp(-\alpha p)\).

Data: \(\Lambda = 10\), \(\alpha = 1\), \(r = 1\), \(\theta_H = 1.2\), \(\theta_L = 0.8\).
**Incomplete Information: Optimal Stopping**

**Summary:**

- $U(n, q)$ can be computed using a recursive sequence of ODEs with free-boundary conditions.
- For every $n$ there is a critical belief $q_n^*$ above which it is optimal to stop.
- Again, the sequence $q_n^*$ is increasing with $n$, that is, the retailer is willing to take more risk for larger orders.
- The sequence $q_n^*$ is bounded by
  \[
  \frac{\theta_H - 1}{\theta_H - \theta_L} \leq q_n^* \leq \hat{q} := \text{Root} \left\{ \Phi \left( \frac{r R}{\theta(q)} \right) = \frac{\eta(q)}{q} (\theta_H - 1) R \right\}
  \]
- The “outlet option” increases significantly the expected profits and the range of products $(n, q)$ that are profitable.
  \[
  0 \leq U(n, q) - V(n, q) \leq (1 - \theta_L)^+ R.
  \]
- A simple piece-wise linear approximation works well.
  \[
  \tilde{U}(n, q) := \max \{ R, W(n, \theta_H) - \frac{(W(n, \theta_H) - R)}{\tilde{q}_n} q \}.
  \]
Concluding Remarks

- A simple dynamic pricing model for a retailer selling non-perishable products.

- Captures two common sources of uncertainty:
  - Market size measured by $\theta \in \{\theta_H, \theta_L\}$.
  - Stochastic arrival process of price sensitive customers.

- Analysis gets simpler using the Fenchel-Legendre transform of $c(\lambda)$ and its properties.

- We propose a simple approximation (linear and piecewise linear) for the value function and corresponding pricing policy.

- Some properties of the optimal solution are:
  - Value functions $V(n, q)$ and $U(n, q)$ are decreasing and convex in $q$.
  - The retailer is willing to take more risk ($\uparrow q$) for higher orders ($\uparrow n$).
  - The optimal demand intensity $\lambda^*(n, q) \uparrow q$ and the optimal sales rate $\bar{\theta}(q) \lambda^*(n, q) \downarrow q$.

- Extension: $R(n) = R + \nu n - K \mathbb{1}(n > 0)$. 