# The Communication Complexity of Efficient Allocation Problems

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*Combinatorial auctions*: allocation of heterogeneous indivisible objects among bidders

- Objective: elicit information about bidders' preferences in order to achieve (or approximate) efficiency (or revenue maximization)
- Mechanism Design: incentives for honest reporting in Direct Revelation Mechanisms
  - E.g., with private values, the Vickrey-Groves-Clarke DRM implements efficiency
- Problem: Preference revelation requires enormous communication: e.g., with 30 objects, valuations over $2^{30} > 1$ billion subsets

- Consider simpler mechanisms:
  - Walrasian equilibria with per-object prices,
  - English auctions with per-object prices,
  - Ascending-bid combinatorial auctions?

Our paper: What is the *minimum* amount of communication required to achieve or approximate efficiency?

- Uses techniques developed in parallel in economics and computer science

Economics: *Dimensionality of Message Spaces* (Hurwicz 1960, Mount and Reiter 1974)

- Communication burden = the number of real numbers announced to verify that the right allocation is implemented ("information smuggling" is ruled out by regularity)
- Formalization of Hayek (1945): the price mechanism is "informationally efficient" in convex economies
- With nonconvexities (as in combinatorial allocation problems), a Walrasian linear-price equilibrium may not exist, may need more extensive communication (Calsamiglia 1977)

Computer science: *Communication Complexity* (Yao 1979, Kushilevitz and Nisan 1997)

- Considers *discrete* inputs (e.g., valuations given with a finite "precision")
- Measures communication with the number of bits sent (= $\log_2$ of the cardinality of the message space)

Our analysis:

- Applies to both the continuous and discrete measures
- Provides a lower bound on the amount communication needed to verify efficiency
- The bound is increasing with the class of agents' preferences
- With unrestricted or submodular preferences, the bound yields an exponential number of real numbers or bits
- *Approximation?* An exponential number of bits is needed to achieve 50% efficiency with unrestricted valuations
- *Approximation on expectation?* For some probability distributions over valuations, any subexponential mechanism loses a constant fraction of expected surplus
- In contrast, under, e.g., the gross substitute condition, polynomial communication achieves efficiency

Our lower bounds are derived from the "verification/nondeterministic" scenario:

- An omniscient oracle (e.g., "Walrasian auctioneer") must prove that an allocation is "correct": announces a message, each agent "accepts" or "rejects"
  - Message accepted ⇒ "correct" allocation
  - An acceptable message exists in all states
- "Deterministic" communication (without an oracle) is harder
- Deterministic communication is facilitated by having multiple rounds, where at each round only information previously revealed to be "relevant" is transmitted (e.g., "tatonnement," multi-round auctions suggested to the FCC)
- Multi-round mechanisms *do not help* with nondeterministic communication ⇒

Our lower bounds apply to both nondeterministic and deterministic, single- and multi-round mechanisms.

# 1 Setup

- $N =$ Finite set (number) of agents
- $K =$ Set of outcomes (allocations)
- $V^i \subset \mathbb{R}^K$ - set of agent $i$'s possible valuations. Agent $i$ observes only his own valuation $v^i \in V^i$. A *state* is a valuation profile $v = \left(v^1, ..., v^N\right) \in V^1 \times \ldots \times V^N$.
- Efficient (surplus-maximizing) choice correspondence:

$$K^*(v) = \arg\max_{k \in K} \sum_{i \in N} v_k^i$$

**Example 1** *Allocating objects from a set $L$: $K = N^L$.*

- $V^i \subset \mathbb{R}^K$ is *normalized* if $v^i \in V^i \Rightarrow v^i + (\lambda, \ldots, \lambda) \notin V^i$ for $\lambda \neq 0$ (e.g., require $v_1^i = 0$ for all $v^i \in V^i$).

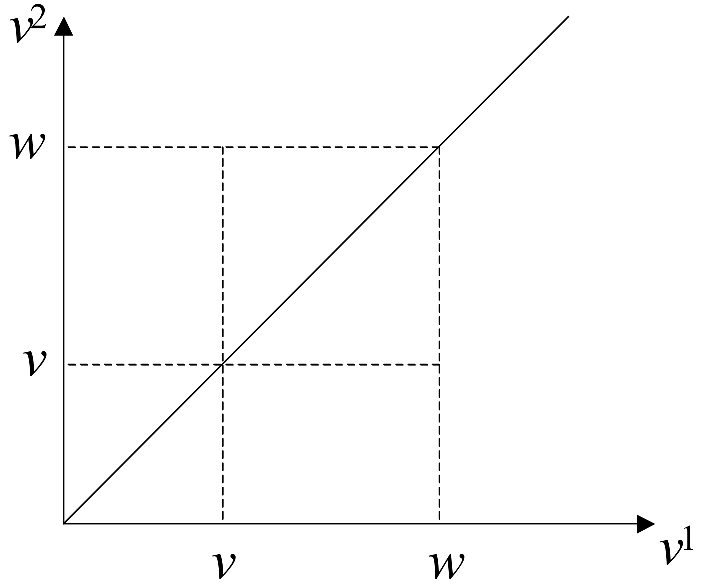
# 2 The Basic Technique

- Construction of a set called “fooling set” in computer science / “set with the uniqueness property” in economics

Example: a single indivisible object to be allocated between two agents

- Agent $i$’s valuation = $v^i \in [\underline{v}, \overline{v}]$.
- Efficiency: give the object to the agent with the higher valuation

State space"



- Suppose the same message $m$ occurs in two "diagonal" states $(v, v)$ and $(w, w)$:

$\Rightarrow$ Each agent $i$ "accepts" $m$ when $v^i \in \{v, w\}$

$\Rightarrow$ $m$ occurs also in states $(w, v)$ and $(v, w)$.

$\Rightarrow$ these states yield the same outcome.

("privacy-preservation" $\Rightarrow$ $m$ occurs in a "rectangle")

- But efficiency requires giving the object to agent 1 in state $(w, v)$ and to agent 2 in state $(v, w)$ - contradiction!

- Therefore, all distinct diagonal states yield distinct messages
- This argument generalizes to arbitrary many agents and outcomes:

**Lemma 1** ***(Indifference Fooling Lemma)*** *Suppose each $V^i$ is normalized, and a pair of states $v, w \in V^1 \times \ldots \times V^N$ with $v \neq w$ satisfies $K^*(v) = K^*(w) = K$. Then in any efficient mechanism, states $v$ and $w$ cannot give rise to the same message.*

- Thus, the set of "diagonal" states:
  $$\left\{v \in V^1 \times \ldots \times V^N : K^*(v) = K\right\}$$
  constitutes a "fooling set."

# 3 Application to Combinatorial Allocations

- Allocation of a set $L$ of objects: $K = N^L$, $k(l) \in N$ is the agent holding object $l$

  Maintained assumptions:

- No externalities: $v_k^i = u^i(k^{-1}(i))$, where $u^i \in U^i \subset \mathbb{R}^{2^L}$ is agent $i$'s valuation *over his own consumption*
- For each $u^i \in U^i$:
  - Monotonicity: $u^i(S)$ is nondecreasing in $S$.
  - Normalization: $u^i(\emptyset) = 0$.
- For the *discrete case*, in addition assume integer valuations: $u^i(S) \subset \{0...R\}$

  A lower bounding trick: take $N = 2$, define *dual valuation* $u^*(S) = u(L) - u(L \backslash S)$.

- All states $(u, u^*) \in U^1 \times U^2$ yield social indifference $\Rightarrow$ by the Indifference Fooling Lemma, give rise to different messages.

## 3.1 Unrestricted Valuations

Here $U^1 = U^2 = U =$ set of all valuations satisfying monotonicity and normalization.

- $u \in U \Rightarrow u^* \in U$, $(u, u^*) \in U \times U$
- By the Lemma, all such states yield different messages $\Rightarrow \dim U$ [$\log_2 |U|$] is a lower bound in the continuous [discrete] case (tight)
- In the continuous case, $\dim U = 2^L - 1$
- In the discrete case, $|U| \geq 2^{\binom{L}{L/2}}$ (by considering functions with $u(S) = 0$ for $|S| < L/2$ and $u(S) = 1$ for $|S| > L/2$)

**Corollary 1** *With continuous unrestricted valuations, the dimensionality of the message space* $\geq 2^L - 1$. *With discrete unrestricted valuations, the number of bits* $\geq \binom{L}{L/2}$.

- Thus the number of real numbers/bits transmitted must be exponential in $L$
- Open problem: how does the required communication grow with $N$?

## 3.2 Submodular Valuations

$U$ contains submodular valuations (equivalently, decreasing marginal utility $u(S \cup l) - u(S)$)

- Problem: $(u, u^*) \in U \times U \Rightarrow u(S) \equiv 0$
- Will apply the Indifference Fooling Lemma on a restricted set $\tilde{K}$ of outcomes that give each agent exactly $L/2$ -objects.
- Consider the set $\widetilde{U} \subset U$ of valuations defined as follows:

$$\begin{array}{ll} u(S) = R & \text{for } |S| > L/2, \\ u(S) = 2d|S| & \text{for } |S| < L/2, \text{ where } d = R/L. \\ u(S) \in [R-d, R] & \text{for } |S| = L/2 \\ u(\{1, .., L/2\}) = R & \end{array}$$

- Such valuations are indeed submodular.
- For each $u \in \widetilde{U}$, define

$$\begin{array}{ll} \widehat{u}(S) = 2R - d - u(L \backslash S) & \text{for } |S| = L/2, \\ \widehat{u}(S) = R & \text{for } |S| > L/2. \\ \widehat{u}(S) = 2d\,|S| & \text{for } |S| < L/2 \end{array}$$

- $u \in \widetilde{U} \Rightarrow \widehat{u} \in U$

- Note that
  $$\begin{aligned} u(S)+\widehat{u}(L\backslash S) &= 2R-d \quad \text{for } |S| = L/2, \\ u(S)+\widehat{u}(L\backslash S) &\le 2R-2d \ \text{ for } |S| \neq L/2. \end{aligned}$$
  $\Rightarrow$ In each state $(u, \widehat{u})$ with $u \in \widetilde{U}$, the set of optimal allocations is exactly $\tilde{K}$.
- By the Lemma, all such states yield different messages $\Rightarrow \dim \widetilde{U}$ [$\log_2 \left|\widetilde{U}\right|$] is a lower bound in the continuous [discrete] case
- In the continuous case, $\dim \widetilde{U} = \binom{L}{L/2} - 1$
- In the discrete case, $\left|\widetilde{U}\right| = (d+1)^{\binom{L}{L/2}-1}$.

**Corollary 2** *With continuous submodular valuations, the dimensionality of the message space* $\geq \binom{L}{L/2} - 1$. *With discrete submodular valuations, the number of bits* $\geq \left(\binom{L}{L/2} - 1\right)(\log_2 (R/L) + 1)$.

## 3.3 Homogeneous Valuations

Here the objects are identical: $U^1 = U^2 = U$ is restricted to include only $u(S) = \phi(|S|)$

- $u \in U \Rightarrow u^* \in U$, $(u, u^*) \in U \times U$
- By the Lemma, all such states yield different messages $\Rightarrow \dim U$ $[\log_2 |U|]$ is a lower bound in the continuous [discrete] case (tight)
- In the continuous case, $\dim U = L$
- In the discrete case, $|U| = \binom{R+L}{R}$ (number of monotone functions $\{1..L\} \rightarrow \{0..R\}$)

**Corollary 3** *With continuous homogeneous valuations, the dimensionality of the message space* $\geq L$. *With discrete homogeneous valuations, the number of bits* $\geq \log_2 \binom{R+L}{R}$.

- Let instead the good be divisible: $L = [0, 1]$. With an appropriate topology on $U$, $\dim U = \infty \Rightarrow$ infinite-dimensional communication needed.
- In contrast, if valuations are concave, a single-price Walrasian equilibrium exists

# 4 Approximation

- Uniform approximation: relax the efficient correspondence to

$$K_c^*(v) = \{k \in K | \sum_{i \in N} v_k^i \geq \frac{1}{c} \max_{j \in K} \sum_{i \in N} v_j^i\}, c > 1.$$

- Approximation factor $c$ close to 1 means "good approximation: means, high $c$ means "bad approximation"
- Finite communication achieves arbitrary approximation even in the continuous case
- Approximation of the discrete case offers a lower bound on the continuous case
- In the discrete case with input precision $R$, approximation to $c < \frac{NR}{NR-1}$ is equivalent to exact efficiency ⇒ can use previous results

## 4.1 Unrestricted Valuations

- Take Corollary 1 with $N = 2$, $R = 1$: approximation to $c < \frac{NR}{NR-1} = 2$ requires transmitting at least $\binom{L}{L/2}$ bits
- Generalization to $N > 2$: Approximation to $c > N$ requires exponential communication in $L$ as long as $N \leq L^{1/3-\varepsilon}$, $\varepsilon > 0$ (proof in Nisan (2001))
- Factor $N$- approximation is trivially achieved by auctioning all objects together as a bundle

## 4.2 Submodular Valuations

- Use Corollary 2 with $N = 2$, $R = 2L$: approximation to $c < \frac{NR}{NR-1} = \frac{4L}{4L-1}$ requires transmitting at least $\binom{L}{L/2} - 1$ bits.
- This is weak, but rules out "FPAS": $1 + \varepsilon$ -approximation with communication that is polynomial in $L, \log R, \varepsilon^{-1}$
  (E.g., achieving $\varepsilon(L) = 1/(5L)$ requires exponential communication in $L$)

- The only known upper bound is the 2-approximation of Lehmann, Lehmann, and Nisan (2001): allocate the objects one by one to the agent with the highest marginal valuation. This communicates only $NL \log R$ bits.

## 4.3 Homogeneous Valuations

An upper bound:

- Each agent $i$ sends $u_i(L)$ and his valuation rounded to $\varepsilon u_i(L)$, implement an efficient allocation given rounded valuations
- A monotonic homogeneous valuation with $\widehat{R} = \varepsilon^{-1}$ possible values can be described with $\widehat{R}$ "jump points," using at most $\log_2 (L+1)^{\widehat{R}} = \varepsilon^{-1} \log_2 (L+1)$ bits
- This achieves FPAS – approximation to within a factor of $1+\varepsilon$ using polynomial communication in $\log R, \log L, N, \varepsilon^{-1}$ (in contrast to the submodular case)
- Note: here can approximate arbitrarily closely using $O(\log L)$ bits, while exact efficiency in the continuous case requires $L$ real numbers!

## 4.4 Average case approximation

- Suppose only *expected* efficiency is required to be close to optimal, given a probability distribution over states.

**Theorem 1** *There exists a distribution $D$ over valuations and a constant $c > 1$, such that when all agents' valuations are chosen independently from $D$, every mechanism that achieves expected efficiency within a factor of c from optimal requires the communication of an exponential number of bits in $L$.*

- Proof by reduction to the "disjointness problem" and appeal to the results of Babai, Frankl, and Simon (1986)

# 5 Conclusion

- The communication “bottleneck” is distinct from, and may be harder in practice than the computational bottleneck: kicks in with dozens of objects, rather than hundreds/thousands
- Any practical mechanism can only achieve/approximate efficiency with some restriction on valuations (or on probability distributions over them)
- Example: LP relaxation finds an optimal allocation (e.g., under “gross substitutes”) $\Rightarrow$ a Walrasian equilibrium with per-object prices exists

For such cases, we suggest deterministic polynomial communication based on a primal-dual LP algorithm. Two improvements over Ausubel (2000):

- It does not require “gross substitutes”
- It achieves $1+\varepsilon$ -approximation with polynomial communication in $\log \varepsilon$ (rather than in $\varepsilon^{-1}$)