Multi-restricted numbers
and
Multi-restrained Stirling numbers

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Partitions of a 4-set with k nonempty subsets

$k=1$	$k=2$	$k=3$	$k=4$
1234	1|234	1|2|34	1|2|3|4
	2|134	1|3|24	
	3|124	1|4|23	
	4|123	2|3|14	
	12|34	2|4|13	
	13|24	3|4|12	
	14|23		

Stirling numbers of the second kind

$S_2(n,k)$ = the number of partitions of an n-set with k nonempty subsets

Example>

$S_2(4,1)=1 \quad S_2(4,2)=7 \quad S_2(4,3)=6 \quad S_2(4,4)=1$

Permutations of a 4-set with k disjoint cycles

$k=1$	$k=2$	$k=3$	$k=4$
(1234)	(1)(234), (1)(243)	(1)(2)(34)	(1)(2)(3)(4)
(1243)	(2)(134), (2)(143)	(1)(3)(24)	
(1324)	(3)(124), (3)(142)	(1)(4)(23)	
(1342)	(4)(123), (4)(132)	(2)(3)(14)	
(1423)	(12)(34), (13)(24)	(2)(4)(13)	
(1432)	(14)(23)	(3)(4)(12)	

$C(n,k)$ = the number of permutations of an n-set with k disjoint cycles

Stirling numbers of the first kind

$$S_1(n,k) = (-1)^{n-k} C(n,k)$$

Example>

$S_1(4,1) = -6$ $S_1(4,2) = 11$ $S_1(4,3) = -6$ $S_1(4,4) = 1$

Inverse Relations

$$[x]_n = \sum_{k=0}^{n} S_1(n,k) x^k \;:\; x^n = \sum_{k=0}^{n} S_2(n,k)[x]_k$$

$$S_i(n,k) = \begin{cases} 1 & \text{if } n = k = 0 \\ 0 & \text{if } n \cdot k = 0 \neq n^2 + k^2 \end{cases} \quad \text{for } i = 1,\ 2$$

$$[x]_n = (x)(x-1)\cdots(x-n+1)$$

1						
	1					
	-1	1				
	2	-3	1			
	-6	11	-6	1		
	24	-50	35	-10	1	
	-120	274	-225	85	-15	1

$S_1(n,k)$

1						
	1					
	1	1				
	1	3	1			
	1	7	6	1		
	1	25	15	10	1	
	1	31	90	65	15	1

$S_2(n,k)$

Partitions of a 4-set with k nonempty subsets

$k=1$	$k=2$	$k=3$	$k=4$
~~1234~~	~~1|234~~	1|2|34	1|2|3|4
	~~2|134~~	1|3|24	
	~~3|124~~	1|4|23	
	~~4|123~~	2|3|14	
	12|34	2|4|13	
	13|24	3|4|12	
	14|23		

Multi-restricted numbers of the second kind

$M_2^m(n,k)$ = the number of partitions of an n-set with k nonempty subsets of size $\le m$

Example>

$$M_2^2(4,1)=0 \quad M_2^2(4,2)=3 \quad M_2^2(4,3)=6 \quad M_2^2(4,4)=1$$

With $M_2^m(n,k)=\begin{cases}1 & \text{if } n=k=0\\ 0 & \text{if } n\cdot k=0\neq n^2+k^2\end{cases}$

$$\begin{array}{lllllllllll}
1 \\
 & 1 \\
 & 1 & 1 \\
 & 1 & S_2 & 1 \\
 & \vdots & \vdots & \vdots & \ddots \\
 & 1 & S_2 & S_2 & \cdots & 1 \\
 & & S_2 & S_2 & \cdots & S_2 & 1 \\
 & & M_2^m & S_2 & \cdots & S_2 & S_2 & 1 \\
 & & M_2^m & M_2^m & \ddots & S_2 & S_2 & S_2 & 1 \\
 & & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots \\
 & & M_2^m & M_2^m & \cdots & M_2^m & S_2 & S_2 & S_2 & \cdots & 1 \\
 & & & M_2^m & \cdots & M_2^m & M_2^m & S_2 & S_2 & \cdots & S_2 & 1 \\
 & & & M_2^m & \cdots & M_2^m & M_2^m & M_2^m & S_2 & \cdots & S_2 & S_2 & 1
\end{array}$$

1) $M_2^m(n,k)=0$ if $n<k$ or $n>km$

2) $M_2^m(n,n)=1$

3) $M_2^m(n,k)\le S_2(n,k)$

4) $M_2^m(n,k)=S_2(n,k)$ if $m\ge n$ or $m>n-k$

5) $\lim_{m\to\infty} M_2^m(n,k)=S_2(n,k)$

m-restricted numbers of the 2nd kind, $M_2^m(n,k)$

Explicit Formula

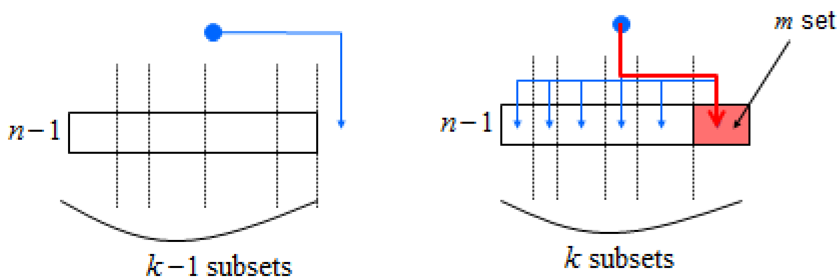
$$S_2(n,k) = \sum_{\substack{k_1+k_2+\cdots+k_n=k \\ k_1+2k_2+\cdots+nk_n=n}} \frac{n!}{(1!)^{k_1}(2!)^{k_2}\cdots(n!)^{k_n}\, k_1!k_2!\cdots k_n!}$$

$$M_2^m(n,k) = \sum_{\substack{k_1+k_2+\cdots+k_m=k \\ k_1+2k_2+\cdots+mk_m=n}} \frac{n!}{(1!)^{k_1}(2!)^{k_2}\cdots(m!)^{k_m}\, k_1!k_2!\cdots k_m!}$$

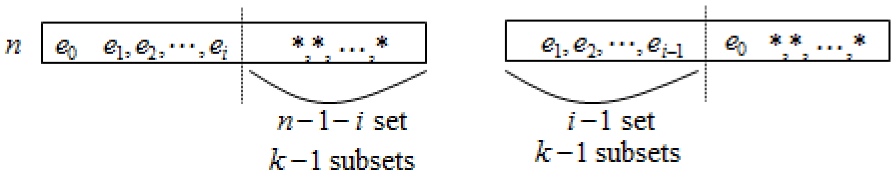
Recurrence Relation

$$S_2(n,k) = S_2(n-1,k-1) + k\cdot S_2(n-1,k)$$

$$M_2^m(n,k) = M_2^m(n-1,k-1) + k\cdot M_2^m(n-1,k) - \binom{n-1}{m}\cdot M_2^m(n-m-1,k-1)$$



Recurrence Relation



$$S_2(n,k) = \sum_{i=0}^{n-1} \binom{n-1}{i} S_2(n-1-i, k-1) \quad = \sum_{i=k}^{n-1} \binom{n-1}{i-1} S_2(i-1, k-1)$$

$$M_2^m(n,k) = \sum_{i=0}^{m-1} \binom{n-1}{i} M_2^m(n-1-i, k-1) = \sum_{i=n-m+1}^{n-1} \binom{n-1}{i-1} M_2^m(i-1, k-1)$$

Generating functions

$$f(t) = e^{tx} = \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)^x \Rightarrow f^{(n)}(0) = \sum_{k=1}^{n} S_2(n,k)[x]_n$$

$$f(t) = \left(1 + t + \frac{t^2}{2!} + \cdots + \frac{t^m}{m!}\right)^x \qquad \Rightarrow f^{(n)}(0) = \sum_{k=1}^{n} M_2^m(n,k)[x]_n$$

$$[x]_n = x(x-1)(x-2)\cdots(x-n+1)$$

Example $(m=2)$

$$f(t)=\left(1+t+\tfrac{t^2}{2!}\right)^x$$

$$f'(t)=x\left(1+t+\tfrac{t^2}{2!}\right)^{x-1}(1+t)$$

$$f''(t)=x(x-1)(1+t+\tfrac{t^2}{2!})^{x-2}(1+t)^2 + x(1+t+\tfrac{t^2}{2!})^{x-1}$$

$$f^{(3)}(t)=x(x-1)(x-2)(1+t+\tfrac{t^2}{2!})^{x-3}(1+t)^3 + 3x(x-1)(1+t+\tfrac{t^2}{2!})^{x-2}(1+t)$$

$$f^{(4)}(t)=x(x-1)(x-2)(x-3)(1+t+\tfrac{t^2}{2!})^{x-4}(1+t)^4 + 6x(x-1)(x-2)(1+t+\tfrac{t^2}{2!})^{x-3}(1+t)^2 + 3x(x-1)(1+t+\tfrac{t^2}{2!})^{x-2}$$

$$f(0)=1$$

$$f'(0)=[x]_1$$

$$f''(0)=[x]_1+[x]_2$$

$$f^{(3)}(0)=3[x]_2+[x]_3$$

$$f^{(4)}(0)=3[x]_2+6[x]_3+[x]_4$$

1						
	1					
	1	1				
		3	1			
		3	6	1		
			15	10	1	
			15	45	15	1

$M_2^2(n,k)$

Bessel Polynomials…

The polynomials y_n of degree n with a constant term equal to 1 satisfying

$$x^2 y_n'' + (2x+2)y_n' = n(n+1)y_n$$

$$y_0 = 1$$

$$y_1 = x+1$$

$$y_2 = 3x^2+3x+1$$

$$y_3 = 15x^3+15x^2+6x+1$$

$$\vdots$$

$$y_n = \sum_{k=0}^{n} \frac{(n+k)!}{2^k (k)!(n-k)!} x^k$$

$$||$$

$$M_2^2(n+k,n)$$

1						
	1					
	1	1				
		3	1			
		3	6	1		
			15	10	1	
			15	45	15	1

$M_2^2(n,k)$

Inverse Relationships

$$[S_2(n,k)] : [S_1(n,k)] = [M_2^m(n,k)] : [\qquad]$$

Multi-restricted numbers of the first kind

$M_1^m(n,k) =$ the (n,k)-th entry of $[M_2^m(n,k)]^{-1}$

1	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0
0	-1	1	0	0	0	0	0
0	2	-3	1	0	0	0	0
0	-5	11	-6	1	0	0	0
0	-35	-45	35	-10	1	0	0
0	10	175	-210	85	-15	1	0
0	-910	-315	1225	-700	175	21	1

3-restricted numbers of the 1st kind, $M_1^3(n,k)$

$S_2(n,k) =$ the number of partitions of an n-set with k nonempty subsets

$M_2^m(n,k) =$ the number of partitions of an n-set with k nonempty subsets of size $\le m$

$|S_1(n,k)| =$ the number of permutations of an n-set with k disjoint cycles

$|M_1^m(n,k)| \ne$ the number of permutations of an n-set with k disjoint cycles of length $\le m$

Permutations of a 4-set with k disjoint cycles

$k=1$	$k=2$	$k=3$	$k=4$
~~(1234)~~	~~(1)(234), (1)(243)~~	(1)(2)(34)	(1)(2)(3)(4)
~~(1243)~~	~~(2)(134), (2)(143)~~	(1)(3)(24)	
~~(1324)~~	~~(3)(124), (3)(142)~~	(1)(4)(23)	
~~(1342)~~	~~(4)(123), (4)(132)~~	(2)(3)(14)	
~~(1423)~~	(12)(34), (13)(24)	(2)(4)(13)	
~~(1432)~~	(14)(23)	(3)(4)(12)	

${}^{m}C(n,k)=$ the number of permutations of an n-set with k disjoint cycles of length $\le m$

Multi-restrained Stirling numbers of the first kind

$${}^{m}S_1(n,k)=(-1)^{n-k}\cdot {}^{m}C(n,k)$$

Example>

$${}^{2}C(4,1)=0 \qquad {}^{2}C(4,2)=3 \qquad {}^{2}C(4,3)=6 \qquad {}^{2}C(4,4)=1$$

$${}^{2}S_1(4,1)=0 \qquad {}^{2}S_1(4,2)=3 \qquad {}^{2}S_1(4,3)=-6 \qquad {}^{2}S_1(4,4)=1$$

With $${}^{m}S_1(n,k)=\begin{cases}1 & \text{if } n=k=0\\ 0 & \text{if } n\cdot k=0\neq n^2+k^2\end{cases}$$

$$\begin{array}{llllllllll}
1 \\
 & 1 \\
 & 1 & 1 \\
 & 1 & S_1 & 1 \\
 & \vdots & \vdots & \vdots & \ddots \\
 & 1 & S_1 & S_1 & \cdots & 1 \\
 & & S_1 & S_1 & \cdots & S_1 & 1 \\
 & & R_1^m & S_1 & \cdots & S_1 & S_1 & 1 \\
 & & R_1^m & R_1^m & \ddots & S_1 & S_1 & S_1 & 1 \\
 & & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots \\
 & & R_1^m & R_1^m & \cdots & R_1^m & S_1 & S_1 & S_1 & \cdots & 1 \\
 & & & R_1^m & \cdots & R_1^m & R_1^m & S_1 & S_1 & \cdots & S_1 & 1 \\
 & & & R_1^m & \cdots & R_1^m & R_1^m & R_1^m & S_1 & \cdots & S_1 & S_1 & 1
\end{array}$$

1) ${}^{m}S_1(n,k)=0$ if $n<k$ or $n>km$

2) ${}^{m}S_1(n,n)=1$

3) $|{}^{m}S_1(n,k)|\le|S_1(n,k)|$

4) ${}^{m}S_1(n,k)=S_1(n,k)$ if $m\ge n$ or $m>n-k$

5) $\lim_{m\to\infty}{}^{m}S_1(n,k)=S_1(n,k)$

m-restrained Stirling numbers of the 1st kind, ${}^{m}S_1(n,k)$

Inverse Relationships

$$[S_2(n,k)] : [S_1(n,k)] = [M_2^m(n,k)] : [M_1^m(n,k)]$$
$$= [\qquad\qquad] : [{}^m S_1(n,k)]$$

Multi-restrained Stirling numbers of the second kind

${}^m S_2(n,k) =$ the (n,k)-th entry of $[{}^m S_1(n,k)]^{-1}$

1	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0
0	1	1	0	0	0	0	0
0	1	3	1	0	0	0	0
0	−5	7	6	1	0	0	0
0	−65	−15	25	10	1	0	0
0	−455	−455	0	65	15	1	0
0	−1295	−4725	−1715	140	140	21	1

${}^3 S_2(n,k)$

Explicit Formula

$$S_1(n,k) = \sum_{\substack{k_1+k_2+\cdots+k_n=k \\ k_1+2k_2+\cdots+nk_n=n}} \frac{(-1)^{n-k} n!}{1^{k_1} 2^{k_2} \cdots m^{k_m} \cdots n^{k_n} (k_1!)(k_2!)\cdots(k_m!)\cdots(k_n!)}$$

$$ {}^m S_1(n,k) = \sum_{\substack{k_1+k_2+\cdots+k_m=k \\ k_1+2k_2+\cdots+mk_m=n}} \frac{(-1)^{n-k} n!}{1^{k_1} 2^{k_2} \cdots m^{k_m} (k_1!)(k_2!)\cdots(k_m!)}$$

Recurrence Relations

$$S_1(n,k)=\sum_{i=1}^{n}(-1)^{i-1}(n-1)(n-2)\cdots(n-i+1)S_1(n-i,k-1)$$

$$^{m}S_1(n,k)=\sum_{i=1}^{m}(-1)^{i-1}(n-1)(n-2)\cdots(n-i+1)\cdot{}^{m}S_1(n-i,k-1)$$

$$S_1(n,k)=S_1(n-1,k-1)-(n-1)S_1(n-1,k)$$

$$^{m}S_1(n,k)={}^{m}S_1(n-1,k-1)-(n-1)\cdot{}^{m}S_1(n-1,k)$$
$$-(-1)^{m}(n-1)(n-2)\cdots(n-m)\cdot{}^{m}S_1(n-m-1,k-1)$$

$$C(n,k)=C(n-1,k-1)+(n-1)C(n-1,k)$$

$$^{m}C(n,k)={}^{m}C(n-1,k-1)+(n-1)\cdot{}^{m}C(n-1,k)$$
$$-(n-1)(n-2)\cdots(n-m)\cdot{}^{m}C(n-m-1,k-1)$$

To find a generating function…

[Faá di Bruno's formula] $f(t)=\left(g(t)\right)^{x}$

$$f^{(n)}(t)=\sum_{k=1}^{n}\sum_{\substack{k_1+k_2+\cdots+k_m=k\\ k_1+2k_2+\cdots+mk_m=n}}\frac{n!}{(k_1!)(k_2!)\cdots(k_m!)}\prod_{i=1}^{n}\left(\frac{g^{(i)}(t)}{i!}\right)^{k_i}(g(t))^{x-k}[x]_k$$

$$\sum_{\substack{k_1+k_2+\cdots+k_m=k\\ k_1+2k_2+\cdots+mk_m=n}}\frac{(-1)^{n-k}\,n!}{1^{k_1}2^{k_2}\cdots m^{k_m}(k_1!)(k_2!)\cdots(k_m!)}={}^{m}S_1(n,k)$$

$$\Rightarrow\begin{cases}g(0)=1\\ g^{(i)}(0)=\begin{cases}(-1)^{i-1}(i-1)! & \text{if } i\le m\\ 0 & \text{if else}\end{cases}\end{cases}$$

$$\Rightarrow g(t)=1+t-\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}+\cdots+(-1)^{m-1}\frac{t^m}{m}\quad\Rightarrow f(t)=\left(g(t)\right)^{x}$$

Extension of **3-restrained** Stirling **numbers**

$$ {}^{3}S_1(n,k)=\sum_{i=1}^{3}(-1)^{i-1}[n-1]_{i-1}\ {}^{3}S_1(n-i,k-1) $$

1					
15	1				
65	10	1			
0	25	6	1		
-455	-15	7	3	1	
-455	-65	-5	1	1	1

1				
1				
-1	1			
2	-3	1		
11	-6	1		
-20	35	-10	1	
40	-135	85	-15	1

Generating functions

$$ f(t)=\left(1+t-\frac{t^2}{2}+\frac{t^3}{3}-\cdots+(-1)^{m-1}\frac{t^m}{m}\right)^x \quad \Rightarrow f^{(n)}(0)=\sum_{k=1}^{n}{}^{m}S_1(n,k)[x]_n $$

$$ f(t)=\left(1+t-\frac{t^2}{2}+\frac{t^3}{3}-\cdots+(-1)^{m-1}\frac{t^m}{m}+\cdots\right)^x \Rightarrow f^{(n)}(0)=\sum_{k=1}^{n}S_1(n,k)[x]_n $$

$$ =(1+\ln(1+t))^x $$

$$ f(t)=\left(1+t+\frac{t^2}{2}+\frac{t^3}{3}+\cdots+\frac{t^m}{m}\right)^x \quad \Rightarrow f^{(n)}(0)=\sum_{k=1}^{n}{}^{m}C(n,k)[x]_n $$

$$ f(t)=\left(1+t+\frac{t^2}{2}+\frac{t^3}{3}+\cdots+\frac{t^m}{m}+\cdots\right)^x \quad \Rightarrow f^{(n)}(0)=\sum_{k=1}^{n}C(n,k)[x]_n $$

$$ =(1-\ln(1-t))^x $$

Reciprocity Law

$[a(n,k)]_{n,k\geq 0}$: an infinite invertible matrix with
$a(0,0)=\pm 1$
only finitely many nonzero elements in each row

Suppose $a(n+1,k+1)=\sum_{i=0}^{r-1} f_i(n)a(n-i,k)$ for all integers n, k and $n\geq 0$

Set $a(n,k)=0$ if only one of n and k is negative.

$\Rightarrow$

There is a unique extension satisfying the given recurrence relation:

$$a(-n,-k)=(-1)^{n-k}b(k,n)$$

where $[b(n,k)]_{n,k\geq 0}$ is the inverse of $[a(n,k)]_{n,k\geq 0}$

$$b(n-1,k-1)=\sum_{i=0}^{r-1} f_i(k+i-1)b(n,k+i) \quad \text{for all natural } n, k$$

Extensions

$[S_2(n,k)]=[S_1(n,k)]^{-1} \qquad \Rightarrow \qquad S_2(-n,-k)=(-1)^{n-k}\cdot S_1(k,n)$

$[M_2^m(n,k)]=[M_1^m(n,k)]^{-1} \qquad \Rightarrow \qquad M_2^2(-n,-k)=(-1)^{n-k}\cdot M_1^m(k,n)$

$[{}^mS_2(n,k)]=[{}^mS_1(n,k)]^{-1} \qquad \Rightarrow \qquad {}^mS_1(-n,-k)=(-1)^{n-k}\cdot {}^mS_2(k,n)$

Horizontal recurrence relations

$$M_1^m(n-1,k-1)=\sum_{i=0}^{m-1}\binom{k+i-1}{i}M_1^m(n,k+i)$$

$$S_1(n-1,k-1)=\sum_{i=0}^{n-1}\binom{k+i-1}{i}S_1(n,k+i)$$

$$S_1(n,k)=\sum_{i=k}^{n}\binom{i}{k}S_1(n+1,i+1)$$

$$^mS_2(n-1,k-1)=\sum_{i=0}^{m-1}(-1)^i k(k+1)(k+2)\cdots(k+i-1)\cdot{}^mS_2(n,k+i)$$

$$S_2(n-1,k-1)=\sum_{i=0}^{n-1}(-1)^i k(k+1)(k+2)\cdots(k+i-1)\cdot S_2(n,k+i)$$

$$S_2(n,k)=\sum_{i=k}^{n}(-1)^{i-k}(i)_{i-k}S_2(n+1,i+1)$$

A Markov Chain Example

Cows are drawn one after the other from a farm initially containing r whites, according to the following scheme:
If white is drawn, place a black cow in the farm;
If black is drawn, put it back.
What is the probability draw k whites in n trials, $P(k;n)$?
Blacks cannot be drawn more than m times.

$$P(k;n)=\frac{r-k+1}{r}P(k-1;n-1)+\frac{k}{r}P(k;n-1)-\binom{n-1}{m}\frac{r-k+1}{r^{m+1}}P(k-1;n-1-m)$$

$$P(1;1)=1=\frac{[r]_1}{r^1}\qquad P(n;n)=\frac{[r]_n}{r^n}\qquad P(k;n)=a_{n,k}\frac{[r]_k}{r^n}$$

$$a_{n,k}\frac{[r]_k}{r^n}=\frac{r-k+1}{r}a_{n-1,k-1}\frac{[r]_{k-1}}{r^{n-1}}+\frac{k}{r}a_{n-1,k-1}\frac{[r]_k}{r^{n-1}}-\binom{n-1}{m}\frac{r-k+1}{r^{m+1}}a_{n-1-m,k-1}\frac{[r]_{k-1}}{r^{n-1-m}}$$

$$a_{n,k}=a_{n-1,k-1}+ka_{n-1,k}-\binom{n-1}{m}a_{n-1-m,k-1}\qquad\Rightarrow\qquad P(k;n)=M_2^m(n,k)\frac{[r]_k}{r^n}$$

References

[CS] *On the Unimodality and Combinatorics of the Bessel Numbers*, Discrete Math., 264 (2003), 45-53.

[CS] On the combinatorics of Multi-restricted numbers, Ars. Com., 75(2005), p.44-63

[CLNS] Reciprocity for Stirling and multi-restricted numbers, J. Comb, Th., Ser. A, 113 (2006), 1050-1060

[C] Multi-restrained Stirling numbers, Ars. Com. to be appeared

Thank you very much!

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