Faithful Representations of Graphs by Islands in the Extended Grid

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Grid

- The set of points in plane with integral coordinates.

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- Two distinct points $A = (X_A, Y_A)$ and $B = (X_B, Y_B)$ are *adjacent*, if $X_A = X_B|$ and $|Y_A - Y_B| \leq 1$ or $Y_A = Y_B$ and $|X_A - X_B| \leq 1$.

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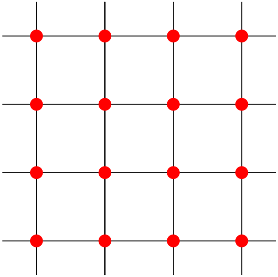

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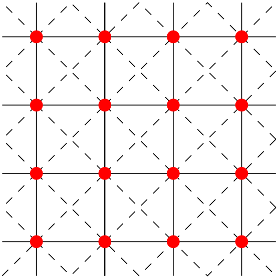

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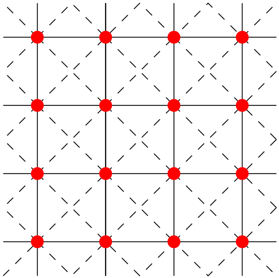

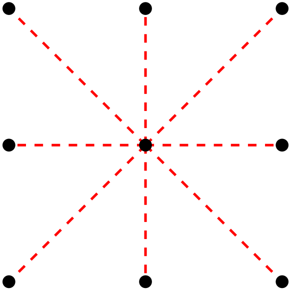

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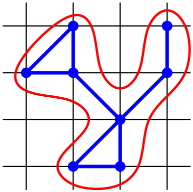
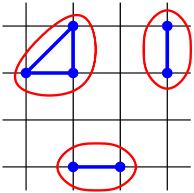


connected, forms island



not connected

Island adjacency

Two islands i, j are adjacent if there is a pair of points $P \in i, Q \in j$, that are adjacent in the extended grid.

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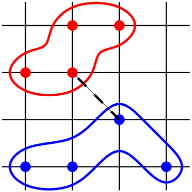

Faithful representation

- A *faithful representation* of a graph G by islands is a set $\mathcal{I}$ of vertex disjoint islands in the extended grid, such that the adjacency graph of islands of $\mathcal{I}$ is isomorphic to G.

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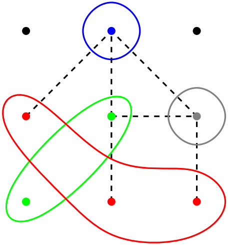

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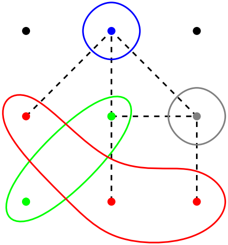

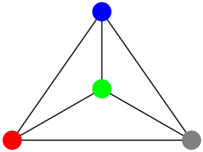

Motivation

- Motivation for this structure came from adiabatic quantum computation (AQC) where vertices of islands are qubits and edges are couplers.

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Adiabatic Quantum Computation is Equivalent to Standard Quantum Computation

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Hebrew University, Israel

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MIT, Cambridge, MA

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Université de Paris-Sud,
Orsay, France

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MIT, Cambridge, MA

Oded Regev
Computer Science Department,
Tel-Aviv University, Israel

Abstract

The model of adiabatic quantum computation has recently attracted attention in the physics and computer science communities, but its exact computational power has been unknown. We settle this question and describe an efficient adiabatic simulation of any given quantum algorithm. This implies that the adiabatic computation model and the standard quantum circuit model are polynomially equivalent. We also describe an extension of this result with implications to physical implementations of adiabatic computation. We believe that our result highlights the potential importance of the adiabatic computation model in the design of quantum algorithms and in their experimental realization.

1. Introduction

The study of adiabatic quantum computation was initiated several years ago by Farhi, Goldstone, Gutmann and Sipser [9], who suggested a novel quantum algorithm for solving classical optimization problems such as SATISFIABILITY (SAT). Their algorithm is based on a celebrated theorem in quantum mechanics known as 'the adiabatic theorem' [14, 19]. Although its exact worst-case behavior is not known, several simulations (e.g., [8]) on random instances of up to 20 quantum bits led to various optimistic speculations. The bad news is that there is now mounting evidence [4, 5, 23] that the algorithm of [9] takes exponential time in the worst-case for NP-complete problems. Nevertheless, adiabatic computations were since shown to be promising in other less ambitious directions: they exhibit inherent robustness against certain types of quantum errors [3], and they possess several interesting algorithmic capabilities, as we will soon review.

We briefly describe the adiabatic computation model (a more precise description appears in Section 2.1). A computation in this model is specified by two Hamiltonians named H_{init} and H_{final} (a Hamiltonian is simply a Hermitian matrix). The ground state (eigenvector with smallest eigenvalue) of H_{init} is required to be an easy to prepare state, such as a tensor product state. The output of the adiabatic computation is the lowest eigenvector of the final Hamiltonian H_{final}. Hence, we choose an H_{final} whose ground state represents the solution to our problem. We require the Hamiltonians to be *local*, i.e., to only involve interactions between a constant number of particles (this is analogous to allowing gates operating on a constant number of qubits in the standard model). This, in particular, makes sure that the Hamiltonians have a short classical description (by simply listing the matrix entries of each local term). The running time of the adiabatic computation is determined by the minimal spectral gap[1] of all the Hamiltonians on the straight line connecting H_{init} and H_{final}: $H(s) = (1-s)H_{\text{init}} + sH_{\text{final}}$ for $s \in [0, 1]$. More precisely, the adiabatic computation is polynomial time if this minimal spectral gap is at least inverse polynomial.

The motivation for the above definition is physical. The Hamiltonian operator corresponds to the energy of the quantum system, and for it to be physically realis-

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1 The spectral gap is the difference between the lowest and second lowest eigenvalue.

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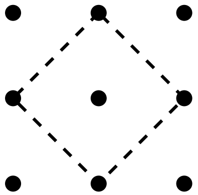

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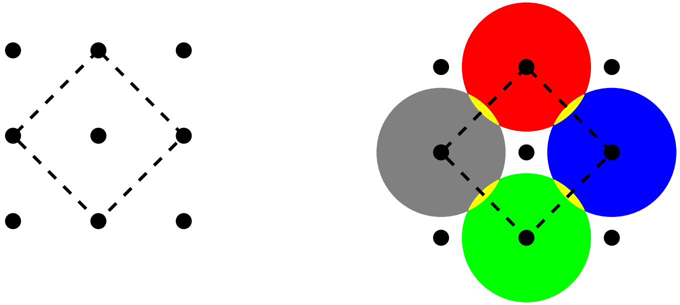

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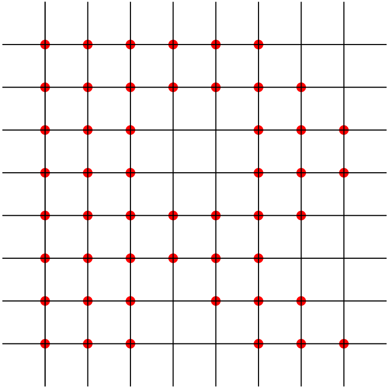

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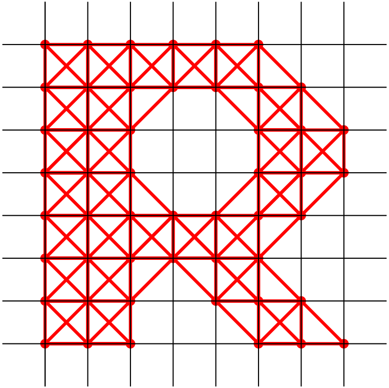

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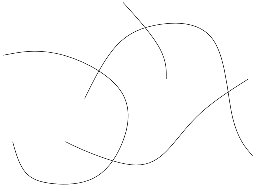

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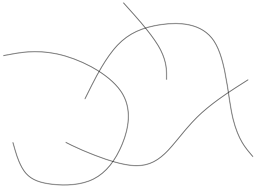

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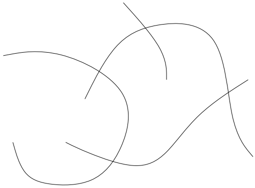

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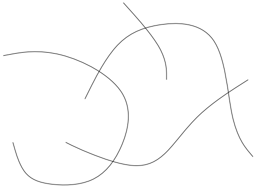

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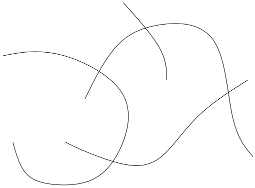

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- In that time it was not even known if this problem is decidable!!!

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- Schaefer,Sedgwick and Štefankovič (2003) proved that recognizing of STRING graphs is NP-complete.

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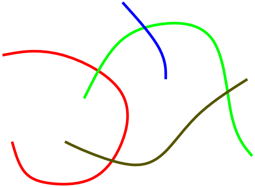

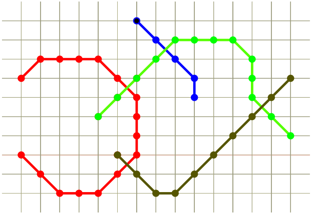

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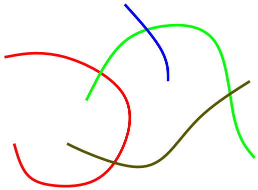

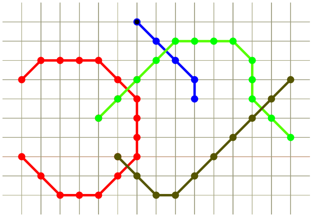

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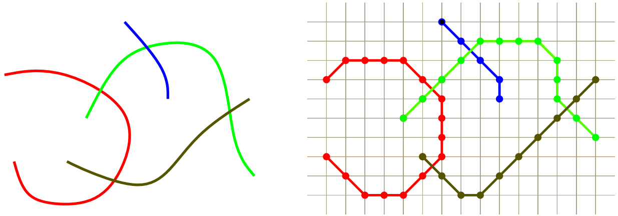

- There exist graphs which require exponential number of grid points.
- The recognition of ISLAND graphs is NP-complete.

Problem

- The complexity of recognition of k-ISLAND graph.

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- We show that k-ISLAND is NP-complete for $k < 3$ and $k > 5$.

Our approach

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- For cases $k = 1, 2$ we made reduction from NAE-3-SAT and use so called "logic engine".
- For cases $k > 5$ we use reduction from PLANAR-3-CONNECTED-(3,4)-SAT.

Logic engine

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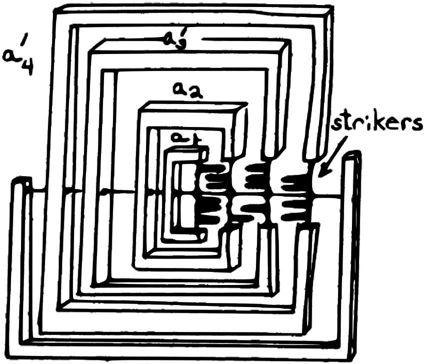
- Very intuitive model of a standard reduction from the problem NAE-SAT



Small islands

Theorem

The problem 1-ISLAND is NP-complete.

Small islands

Theorem

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Theorem

The problem 2-ISLAND is NP-complete.

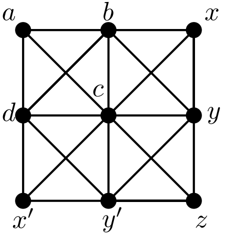
How to prove it

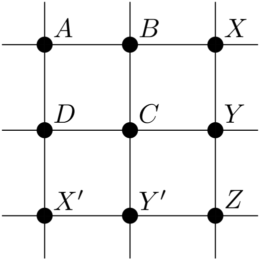
For "logic engine" construction are important to produce:

- rigid element (frames, shafts)
- flexible element (rod)

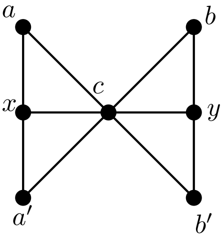
Case k = 1

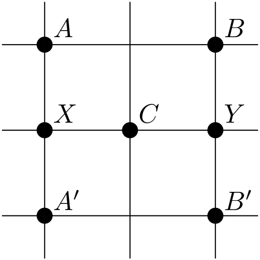
- rigid element





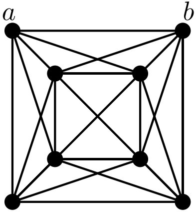
- flexible element

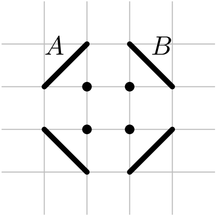




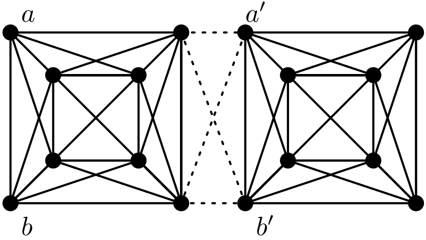
Case $k = 2$

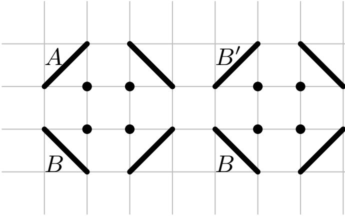
- rigid element





- flexible element





Example k = 1

$$(\neg X_1 \vee X_2 \vee \neg X_3) \wedge (\neg X_1 \vee X_2 \neg X_4) \vee (\neg X_2 \vee \neg X_3 \vee X_4)$$

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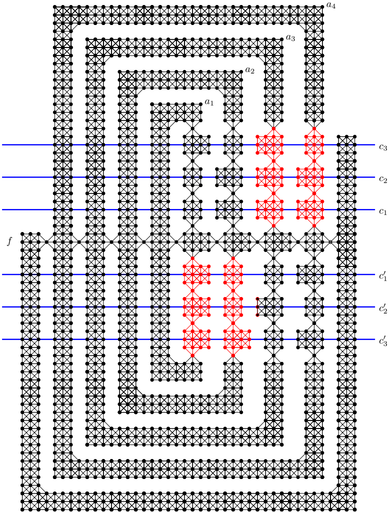

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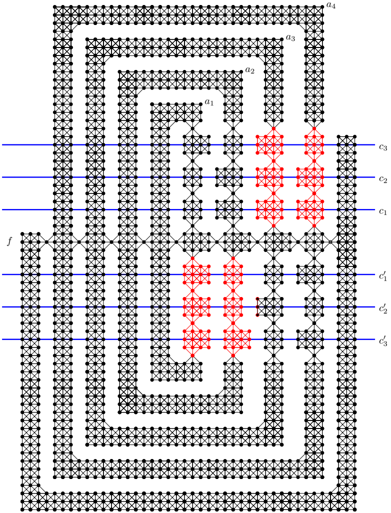

X_1 false
X_2 false
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Example k = 2

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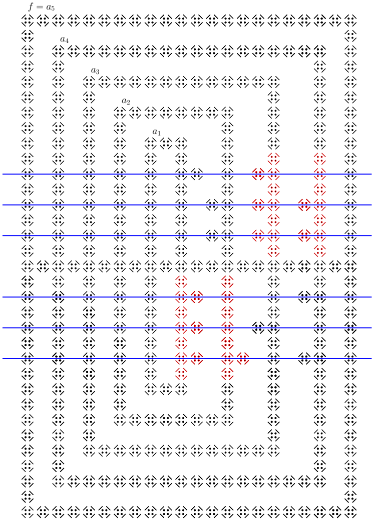

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- Input formula has exactly 3 distinct literals in each clause.
- Each variable occurs in at most 4 clauses
- Incidence graph of formula is vertex-3-connected and planar.

Reduction for large islands

- PLANAR-3-CONNECTED-(3,4)-SAT was introduced by by Kratochvíl. It can be used to show that STRING graphs are NP-hard.

Reduction for large islands

- PLANAR-3-CONNECTED-(3,4)-SAT was introduced by by Kratochvíl. It can be used to show that STRING graphs are NP-hard.
- We will use the same reduction for recognition of large islands.

Idea of the proof

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  - If ϕ is satisfiable then $H_\phi \in 6-$ISLAND.
  - If ϕ is not satisfiable then $H_\phi \notin$ STRING $=$ ISLAND.

The first step of construction

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- Edges piece-wise linear and following the grid lines.

The first step of construction continued

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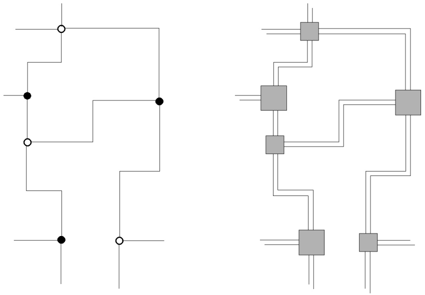
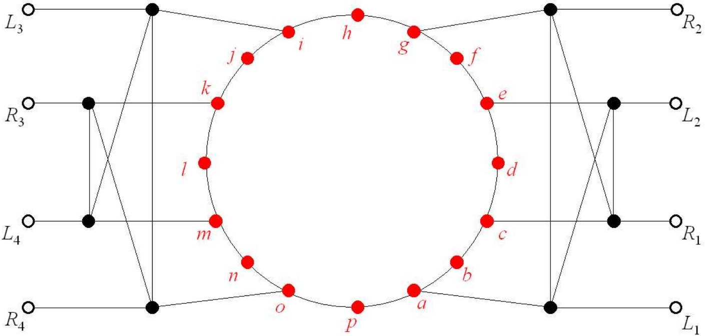
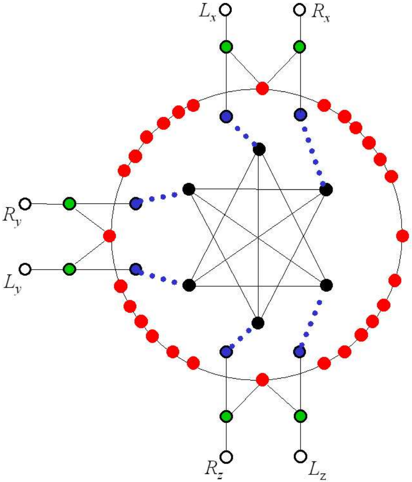


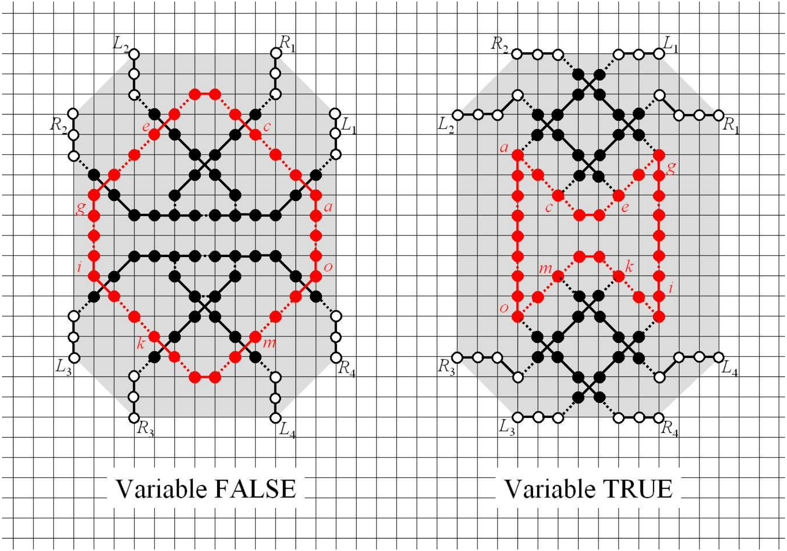
Variable gadget



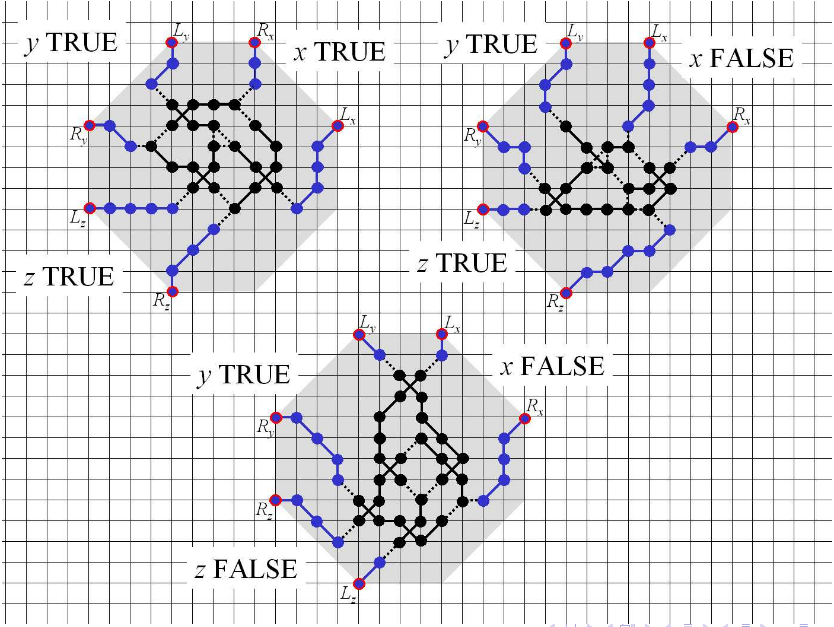
Clause gadget



Representations of the variable gadget



Representations of the clause gadget



The end

Thank you for your attention