Theory and Applications of Random Partition Processes

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Random partitions

- population genetics
- ecology
- physical science
- clustering
- machine learning/statistics.

Fragmentation trees

- phylogenetics
- linguistics

Complex networks

- physics
- population biology
- epidemiology

$[n] := \{1, \ldots, n\}$ (set of labels)

A partition B of $[n]$ is

- a set of non-empty disjoint subsets (blocks) $b \subset [n]$ such that $\bigcup_{b \in B} b = [n]$, e.g. $B = 124|35|6 \equiv 35|6|124 \equiv \{\{1,2,4\},\{3,5\},\{6\}\}$;
- an equivalence relation $B : [n] \times [n] \to \{0,1\}$ with $B(i,j) = 1 \Leftrightarrow i \sim_B j$;
- a symmetric Boolean matrix $(B_{ij}) := (B(i,j))$, e.g.

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For $B \in \mathcal{P}$, $\#B$ is number of blocks of B, e.g. $\#B = 3$ above;
For $b \in B$, $\#b$ is the number of elements of $b \subset \mathbb{N}$. e.g. $\#\{1,2,4\} = 3$.

$\mathcal{P}_{[n]}$: set partitions of $[n]$

$\mathcal{P}_{[n]}$ denotes the set of partitions of $[n]$

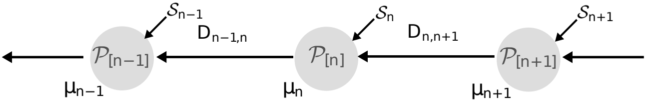
$$\begin{array}{ll} \mathcal{P}_{[1]}: & 1 \\ \mathcal{P}_{[2]}: & 12 \quad 1|2 \\ \mathcal{P}_{[3]}: & 123 \quad 1|23 \quad 12|3 \quad 13|2 \quad 1|2|3 \end{array}$$

Action by permutation: $\sigma = (12)(3)$, $\pi = 13|2 \Longrightarrow \pi^{\sigma} = 1|23$.

Restriction maps: $\mathbf{D}_{m,n} : \mathcal{P}_{[n]} \to \mathcal{P}_{[m]}$, $\mathbf{D}_{m,n}B := B_{|[m]}$ $(1 \leq m \leq n)$, e.g.

$$\mathbf{D}_{5,6}(1256|3|4) = 125|3|4.$$

$\mathcal{P}_{\infty}$ is the collection $(\mathcal{P}_{[n]}, n \geq 1)$ together with deletion $(D_{m,n}, m \leq n)$ and permutation maps, and all composite mappings, i.e. partitions of $\mathbb{N}$.



$\Pi := (\Pi_m, m \geq 0)$ is an exchangeable Feller chain on $\mathcal{P}_\infty$ if

- *exchangeable*: $\mathbf{D}_n \Pi =_{\mathcal{L}} (\mathbf{D}_n \Pi)^\sigma$ for all permutations $\sigma : [n] \to [n]$.
- *Feller*: $\mathbf{D}_n \Pi$ is a Markov chain for all $n \geq 1$;

For example,

$$\{1|2|34 \mapsto 134|2\} =_{\mathcal{L}} \{14|2|3 \mapsto 134|2\} =_{\mathcal{L}} \{14|2|3 \mapsto 124|3\}.$$

$$\left.\begin{array}{l} 1|2|34 \mapsto \left\{\begin{array}{lcr} 134|2 & & 1/9 \\ 13|24 & \text{w.p.} & 1/18 \\ 13|2|4 & & 1/18 \end{array}\right. \\ \\ 14|2|3 \mapsto \left\{\begin{array}{lcr} 134|2 & & 1/9 \\ 13|24 & \text{w.p.} & 1/18 \\ 13|2|4 & & 1/18 \end{array}\right. \end{array}\right\} \begin{array}{lll} 1|2|3 \mapsto 13|2 & \text{w.p.} & \frac{1}{9} + \frac{1}{18} + \frac{1}{18} = \frac{2}{9}. \\ 1|2|3 \mapsto 13|2 & \text{w.p.} & \frac{1}{9} + \frac{1}{18} + \frac{1}{18} = \frac{2}{9}. \end{array}$$

Motivation: Mitochondrial DNA (mtDNA) sequences

mtDNA sequences for 9 species (snake, iguana, lizard, crocodile, bird, whale, cow, human, monkey)

1	snake	T	A	G	G	A	T	T	G	A	T	A	C	C	C
2	iguana	T	A	G	G	A	T	T	G	A	T	A	C	C	C
3	lizard	T	A	G	G	A	T	T	G	A	T	A	C	C	C
4	crocodile	T	A	G	G	A	T	T	G	A	T	A	C	C	C
5	bird	T	G	G	G	A	T	T	G	A	T	A	C	C	C
6	whale	T	G	G	G	A	T	T	G	A	T	A	C	C	C
7	cow	A	A	G	C	A	T	C	T	A	C	A	C	C	C
8	human	A	A	C	C	C	C	C	C	C	C	A	T	C	C
9	monkey	T	G	G	G	A	T	T	G	A	T	A	C	C	C

$$1234569|78 \to 123478|569 \to 12345679|8 \to \cdots$$

How to model this sequence of partitions?

$\mathcal{P}_{[\infty]:k}$, k-colorings of $\mathbb{N}$ and partition matrices

$\mathcal{P}_{[\infty]:k}$: partitions with at most k blocks

$\mathcal{L}_{[n]:k}$: k-colorings of $[n]$ (labeled partitions)

- $x \in \mathcal{L}_{[n]:k}$: $x = x^1 x^2 \cdots x^n$, e.g. $x = 12112 \Rightarrow (134, 25)$.
- Write a k-coloring as a set-valued vector $L = (L_1, \ldots, L_k)$.
- Natural map $\mathcal{B}_n : \mathcal{L}_{[n]:k} \to \mathcal{P}_{[n]:k}$ by removing colors

$$(34, 1, 256) \longrightarrow_{\mathcal{B}_6} 1|256|34.$$

- DNA example: with A, C, G, T as 1, 2, 3, 4:
  $x = TTTTTTAAT \Rightarrow (78, \emptyset, \emptyset, 1234569) \longrightarrow_{\mathcal{B}_9} 1234659|78.$

$\mathcal{M}_{[n]:k}$: $k \times k$ partition matrices

$$\begin{pmatrix} 234 & 1456 & 2 \\ 15 & \emptyset & 146 \\ 6 & 23 & 35 \end{pmatrix} \begin{pmatrix} 34 \\ 1 \\ 256 \end{pmatrix} = \begin{pmatrix} 1234 \\ 6 \\ 5 \end{pmatrix}.$$

In general,

$$\begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1k} \\ M_{21} & M_{22} & \cdots & M_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ M_{k1} & M_{k2} & \cdots & M_{kk} \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \\ \vdots \\ L_k \end{pmatrix} = \begin{pmatrix} \bigcup_{j=1}^k (M_{1j} \cap L_j) \\ \bigcup_{j=1}^k (M_{2j} \cap L_j) \\ \vdots \\ \bigcup_{j=1}^k (M_{kj} \cap L_j) \end{pmatrix}.$$

Constructing Markov chains on $\mathcal{L}_{[\infty]:k}$

Let:

- Λ_0 be an exchangeable initial state
- χ be a probability measure on $\mathcal{M}_{[\infty]:k}$
- $M_1, M_2, \ldots$ be i.i.d. random partition matrices with distribution χ (independent of Λ_0).

For each $m \geq 1$, put

$$\Lambda_m := M_m \Lambda_{m-1}^T = M_m M_{m-1} \cdots M_1 \Lambda_0^T.$$

$\Lambda := (\Lambda_m, m \geq 0)$ is a Markov chain on k-colorings of $\mathbb{N}$.

Example, $\Lambda_0 = (1345, 26)$; $M_1 = \begin{pmatrix} 2345 & 256 \\ 16 & 134 \end{pmatrix}$; $M_2 = \begin{pmatrix} 1345 & 24 \\ 26 & 1356 \end{pmatrix}$.

$$\begin{aligned}
\Lambda_0 &= && && (1345, 26) \\
\Lambda_1 &= M_1 \Lambda_0^T &&= \begin{pmatrix} 2345 & 256 \\ 16 & 134 \end{pmatrix} \begin{pmatrix} 1345 \\ 26 \end{pmatrix} &&= (23456, 1) \\
\Lambda_2 &= M_2 \Lambda_1^T = M_2 M_1 \Lambda_0^T &&= \begin{pmatrix} 1345 & 24 \\ 26 & 1356 \end{pmatrix} \begin{pmatrix} 23456 \\ 1 \end{pmatrix} &&= (345, 126) \\
&\vdots \qquad \vdots && \qquad \vdots
\end{aligned}$$

Homogeneous Cut-and-Paste chains

Theorem (C. 2012)

Every exchangeable Feller chain Λ on $\mathcal{L}_{[\infty]:k}$ can be constructed from an i.i.d. sequence $M_1, M_2, \ldots$ so that

$$\Lambda_m = M_m M_{m-1} \cdots M_1 \Lambda_0, \quad m \geq 1.$$

Corollary (C. 2012)

Every exchangeable Feller chain Π on $\mathcal{P}_{[\infty]:k}$ can be obtained as the projection $\mathcal{B}_\infty(\Lambda)$*, where Λ is an exchangeable Feller chain on $\mathcal{L}_{[\infty]:k}$.*

Matrix permanents

Recall: we can regard a partition B as a symmetric Boolean matrix $(B_{ij}) := (B(i,j))$, e.g.

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = 124|35|6.$$

For an $n \times n$ matrix X, the α-permanent of X is given by

$$\mathrm{per}_\alpha X := \sum_{\sigma \in \mathrm{Sym}_n} \alpha^{\#\sigma} \prod_{j=1}^{n} X_{j\sigma(j)}.$$

Hard to compute, but for a partition B, we have

$$\mathrm{per}_\alpha B = \prod_{b \in B} \alpha^{\uparrow \# b}.$$

Moreover, there is the identity

$$\mathrm{per}_\alpha X = \sum_{B \in \mathcal{P}_{[n]:k}} \frac{k!}{(k - \#B)!} \mathrm{per}_{\alpha/k}(X \cdot B),$$

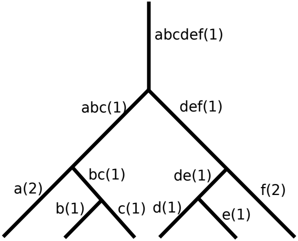
$X \cdot B$ is the Hadamard product.

Permanental partition process (C. 2012)

For X a non-negative $n \times n$ matrix with positive diagonal entries and $\alpha > 0$, we have a general class of partition-valued Markovian transition probabilities on $\mathcal{P}_{[n]:k}$:

$$P_n(B, B') = \frac{k!}{(k - \#B')!} \frac{\mathrm{per}_{\alpha/k}(X \cdot B \cdot B')}{\mathrm{per}_{\alpha}(X \cdot B)}, \quad B, B' \in \mathcal{P}_{[n]:k}.$$

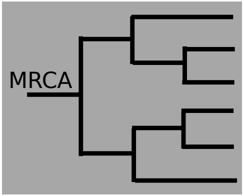
- Gives a parametric statistical model for dependent sequences of partitions.
- In cases of interest, X is a *discrete* parameter $\Longrightarrow$ hard to estimate.



$$= \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 4 & 2 & 2 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 1 \\ 2 & 3 & 4 & 1 & 1 & 1 \\ 1 & 1 & 1 & 4 & 3 & 2 \\ 1 & 1 & 1 & 3 & 4 & 2 \\ 1 & 1 & 1 & 2 & 2 & 4 \end{pmatrix} \end{array}$$

Example: Phylogenetic inference

unknown tree



	$t=1$	2	3	4	5	6	7	...
a	A	G	C	C	T	A	G	...
b	A	T	G	G	C	A	G	...
c	C	T	G	C	T	T	G	...
d	C	G	C	C	C	T	G	...
e	C	G	C	G	C	T	G	...
f	C	G	G	G	T	A	G	...

Use permanental partition transition probabilities with X as a rooted tree matrix in likelihood-based inference of the unknown tree.

Given sequence $B=(B_1, B_2, \ldots, B_m)$, obtain a likelihood

$$\mathcal{L}(X,\alpha;B) = \frac{k^{\downarrow \#B}\,\mathrm{per}_{\alpha}(X\cdot B)}{\mathrm{per}_{k\alpha} X} \prod_{j=1}^{m-1} \frac{k^{\downarrow \#B_{j+1}}\,\mathrm{per}_{\alpha/k}(X\cdot B_j\cdot B_{j+1})}{\mathrm{per}_{\alpha}(X\cdot B_j)}$$

How to (approximately) optimize with respect to X (restricted to the space of rooted trees)?

References

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