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Editorial

The mathematical psychology of Peter Fishburn

Peter Fishburn was amazingly prolific in a remarkable variety of fields, including many of the topics covered by the *Journal of Mathematical Psychology (JMP)*: utility theory and expected utility, choice probabilities, preference theory, interval orders and semiorders, lotteries and gambles, risk, conjoint measurement, and more generally the theory of measurement. His-widespread influence and impact is reflected in the nine books he authored, the over 500 journal articles he wrote, and his more than 80 collaborators. Outside of his work published in the *JMP*, one paradigm in his work, reflected in several papers in this special issue, stands out for its direct societal impact: social choice theory, notably Peter's work with Steve Brams on Approval Voting (e.g., Brams & Fishburn, 1983). Mention should also be made of his extensive work on discrete mathematics, covering such topics as graph theory, ordered sets, and geometry. The topics and motivations overlap with his work in mathematical psychology and specifically those in papers in this special issue on interval orders, representability, and social choice. His 20 papers in the journal *Discrete Applied Mathematics* are summarized in Brams, Gehrlein and Roberts (2022). Peter's work has been recognized through numerous awards and prizes, for example the Frank P. Ramsey Medal for his work in decision analysis and the John von Neumann Prize for his research in operations research and management science.

Remarkably, we count 31 Peter Fishburn articles in the *JMP*. These publications, to which we will restrict our discussion here, represent the wide swath of his research interests.

1. Semiorders, interval orders, and their generalizations

A considerable literature addresses the question of whether judgments of indifference between alternatives are transitive. Nontransitivity of indifference, e.g., in psychophysics, led Duncan Luce to explore the idea that alternative a will be preferred to alternative b if and only if in some sense it is sufficiently better on some scale. He fixed a threshold δ (corresponding to just noticeable difference) and asked for a function f so that if $\succ$ is a binary preference relation, then

$$a \succ b \Leftrightarrow f(a) > f(b) + \delta. \tag{1}$$

If $\succ$ represents preference, then f is a utility function. Luce (1956) introduced *semiorders,* the binary relations $\succ$ for which such a function f into the set of real numbers exists, and Scott and Suppes (1958) proved a representation theorem showing that three axioms for semiorders are necessary and sufficient for the existence of a function f satisfying Condition (1):

$\sim [a \succ a]$,
$a \succ b \ \& \ c \succ d \Rightarrow a \succ d$ or $c \succ b$, and
$a \succ b \ \& \ b \succ c \Rightarrow a \succ d$ or $d \succ c$.

Semiorders have been widely studied for preference judgments, judgments like "louder than," "heavier than," etc. In the representation (1), let $J(a)$ be the closed interval of numbers $[f(a) - \delta/2, f(a) + \delta/2]$. Another way to look at (1) is to say that

$$a \succ b \Leftrightarrow J(a) > J(b), \tag{2}$$

where if J and J' are two intervals, then $J > J'$ means that every number in J is strictly bigger than every number in J'. If we allow for intervals J in (2) to have different lengths, then the binary relations $\succ$ for which we can find intervals of real numbers satisfying Condition (2) are called ***interval orders***. Interval orders and axioms defining them were introduced by Peter Fishburn in a key *JMP* article (Fishburn, 1970a). Specifically, an interval order is defined by the first two semiorder axioms.

Fishburn (1981) considered the class P_n of binary relations satisfying (2) for which there is a representation using intervals of no more than n different lengths. P_1 is the class of semiorders and the union of all P_n is the class of interval orders. He showed that while P_1 is axiomatizable by a universal sentence in first-order logic, no P_n for $n \geq 2$ is axiomatizable in the same sense.

The semiorder representation uses closed real intervals. If we allow intervals to be closed, open or closed on one side and open on the other, then the size of the set of alternatives becomes a factor and there is also interest in the relation $>>$ on intervals, where $J >> J'$ if and only if $\inf(J) > \sup(J')$. Fishburn (1973b) explored the representation

$$a \succ b \Leftrightarrow J(a) >> J(b). \tag{3}$$

He observed that interval orders are exactly the binary relations satisfying Condition (2) for an underlying set of alternatives that is either finite or countable and semiorders are exactly the binary relations satisfying Condition (1) or Condition (3) with all intervals having the same length if the set of alternatives is finite. However, he pointed out that semiorders may not satisfy Condition (3) with all intervals having the same length if there is a countable set of alternatives. He explored conditions for when a semiorder can satisfy this condition on a countable set of alternatives.

Fishburn (1968) explored preferences among risky choices, i.e., where the alternatives are probability distributions on a set of consequences. He provided a set of axioms on a binary preference relation $\succ$ of this kind that imply that indifference is transitive. Two of those axioms are two of the three standard semiorder axioms of Luce (1956) and Scott and Suppes (1958). The third axiom is a version of a sure-thing axiom for risky choices due to Savage (1954). To present this axiom

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we use the notation λP+(1-λ)Q for the direct linear combination of P and Q. Then the axiom states that for all P, Q, and R:

$$P \succ Q \Rightarrow \lambda P + (1-\lambda)R \succ \lambda Q + (1-\lambda)R.$$

The paper noted that if P, Q, and S are three choices, and P($100) = 1, Q($101) = 1 and S($40) = S($200) = 0.5, then surely $Q \succ P$ but it is entirely possible that there is no clear preference between P and S or between Q and S, so indifference is not transitive. The paper also explored ways to weaken the axioms given or replace them by others that preserve the possibility of nontransitive indifference in risky choices.

Many classes of partial orders that accommodate thresholds of discriminability have been studied over the years. These include not only semiorders and interval orders, but bisemiorders, split semiorders, unit tolerance orders, unit bitolerance orders, tolerance orders, bitolerance orders, semitransitive orders, and subsemiorders. A review paper (Fishburn, 1997) described a variety of such generalizations of semiorders and their interrelationships (equivalences, proper inclusions, etc.)

In the current special issue, Rébillé (2024) "*A representation of interval orders through a bi-utility function*" builds upon this line of work by offering a characterization of interval orders using a condition related to precedence and succession relations. The author determines the conditions under which interval orders admit a bi-utility representation. This work nicely connects, and extends, classic approaches to preference modeling via interval orders.

2. Subjective probability, comparative probability, qualitative probability

Sometimes we have some information about probabilities but have no way to calculate them explicitly. We depend on subjective probability estimates. Sometimes these are made directly, but an alternative approach is to use a binary relation of comparative probability $\succ$ on a set of events interpreted to mean that if A and B are two events, then $A \succ B$ means that A is judged more probable than B. Then we seek numbers P(E) for all events E so that the function P defines a (finitely) additive probability function and so that

$$A \succ B \Leftrightarrow P(A) > P(B). \qquad (4)$$

de Finetti (1931) asked whether certain axioms on the comparative probability relation $\succ$ that are necessary conditions for Representation (4) are also sufficient. Kraft, Pratt and Seidenberg (1959) showed that this was not true and gave an example where the events were all subsets of a 5-element set. Fishburn (1996a) extended this analysis to the case where $\succ$ is a linear order and where the set of events is all subsets of an n-element set. He characterized all non-representable relations $\succ$ when $n = 5$ and showed that for every $n \geq 5$, there is a non-representable relation $\succ$ that violates the simplest extension of de Finetti's additivity axiom yet has an order-preserving measure whenever the family of events is the set of all subsets of an (n-1)-element set.

Fishburn (1986) studied interval representations for comparative probability that are motivated by the semiorder and interval order representations. These representations are based on monotonic functions P_* and P^* on a set of events that are subsets of a set S, with $P^* \geq P_*$, $P_*(\phi) = P^*(\phi)$, and $P_*(S) = P^*(S) = 1$. The basic interval model has

$$A \succ B \Leftrightarrow P_*(A) > P^*(B).$$

Thus, A is regarded as more probable than B if and only if the interval $[P_*(A), P^*(A)]$ in [0, I] assigned to event A completely follows the interval $[P_*(B), P^*(B)]$ assigned to event B. Stronger models require a probability measure P such that $P_* \leq P \leq P^*$ and this paper studied additional restrictions on the functions such as superadditivity of P_* and subadditivity of P^*. Conditions on $\succ$ that are necessary and sufficient for most of the interval models were identified.

Fishburn and Roberts (1989) studied the comparative probability representation in Condition (4) when there is a unique solution P. Whereas extensive earlier work on uniqueness of such a representation had been for infinite algebras of events, there had been little work on the problem when the algebra of events is finite, with the exception of work by Luce (1967). The Fishburn-Roberts paper gave necessary and sufficient conditions on a binary relation $\succ$ on the Boolean algebra of all subsets of $S_n = \{1, 2, \ldots, n\}$ for there to be a unique probability measure satisfying Representation (4). The paper also investigated conditions on probability measures P on S_n such that there is some relation $\succ$ so that P is the uniquely agreeing probability representation satisfying Representation (4). Let P_n be the set of all such measures P on S_n. An example of a measure in P_4 is the measure that assigns the four probabilities 1/10, 2/10, 3/10, 4/10 to the four elements of S_n. If indifference $A \sim B$ means that neither $A \succ B$ nor $B \succ A$, then the following indifferences define an example of a preference relation that has this measure as its uniquely agreeing probability measure: $\{3\} \sim \{1,2\}$, $\{4\} \sim \{1,3\}$, $\{2,3\} \sim \{1,4\}$. Fishburn and Roberts studied various subclasses of P_n arising from various axiomatizations of the Representation (4). These included, for example, the class R_n of regular measures, those where for every atom x such that $P(x) > P(y)$ for some other atom y, P(x) is the sum of $P(x_i)$ for all atoms with smaller probability than x. The example we have just given is not regular, since for example 4/10 is not the sum of 1/10, 2/10, and 3/10. The paper concludes with the study of structural and combinatorial properties of the sets of unique solutions in different classes. For example, let $a_1a_2a_3a_4$ correspond to the probability measure that assigns probabilities $a_i/\sum_j a_j$ to the elements of S_n. Then the example above corresponds to the sequence 1234. It turns out that R_4 consists of exactly the sequences 1111, 1112, 1113, 1122, 1123, 1124 and P_4 consists of R_4 plus the two sequences 1223 and 1234. The study of sequences arising from similar uniqueness questions for other forms of representational measurement is the subject of a series of papers by Fishburn and his colleagues.

3. Multi-attribute alternatives: conjoint measurement/additive utility

Comparison of utility or other measures on different alternatives can be of great interest when the alternatives are multi-attribute. Here, the set of alternatives has a Cartesian product structure A_1 x A_2 x … x A_n representing n different attributes or criteria. A given alternative $(a_1, a_2, \ldots, a_n)$ can be viewed as giving the level or value of an alternative on each attribute. When the set of alternatives has such a product structure and we seek the utility f of an alternative, it would in principle be easier to calculate utility on each attribute separately and then add, i.e., to seek functions $f_1, f_2, \ldots, f_n$ so that

$$f(a_1, a_2, \ldots, a_n) = f_1(a_1) + f_2(a_2) + \ldots + f_n(a_n).$$

In this case, the utility function is additive. A fundamental idea in the theory of measurement is to seek additive utility representations that preserve a binary preference relation $\succ$, i.e., so that there are functions $f_1, f_2, \ldots, f_n$ so that

$$\begin{aligned}(a_1, a_2, \ldots, a_n) &\succ (b_1, b_2, \ldots, b_n) \Leftrightarrow \\ &f_1(a_1) + f_2(a_2) + \ldots + f_n(a_n) \\ &> f_1(b_1) + f_2(b_2) + \ldots + f_n(b_n).\end{aligned} \qquad (5)$$

This representation is called ***(additive) conjoint measurement***. Conditions sufficient for additive conjoint measurement were presented by Luce and Tukey (1964) following an earlier result of Debreu (1960). Note that conjoint measurement has applications in areas other than preference theory, e.g., in studying response strength, binaural loudness, and discomfort under different conditions of temperature and humidity (Roberts, 1979).

The representation in Condition (5) generalizes to the case where we seek a function F and functions $f_1, f_2, \ldots, f_n$ so that

$$(a_1, a_2, \ldots, a_n) \succ (b_1, b_2, \ldots, b_n) \Leftrightarrow F[f_1(a_1), f_2(a_2), \ldots, f_n(a_n)] > F[f_1(b_1), f_2(b_2), \ldots, f_n(b_n)],$$

and even to more complicated cases called ***nondecomposable conjoint measurement*** (Krantz et al., 1971) such as where we seek functions $F, f_1, f_2, \ldots, f_n$ and $g_1, g_2, \ldots, g_n$ so that

$$(a_1, a_2, \ldots, a_n) \succ (b_1, b_2, \ldots, b_n) \Leftrightarrow F[f_1(a_1), f_2(a_2), \ldots f_n(a_n), g_1(a_1), g_2(a_2), \ldots g_n a_n)] \succ F[f_1(b_1), f_2(b_2), \ldots f_n(b_n), g_1(b_1), g_2(b_2), \ldots g_n(b_n)]. \quad (6)$$

Fishburn (1975) discussed two examples of Representation (6), namely:

$$(a, x) \succ (b, y) \Leftrightarrow f_1(a) + g_1(a)g_2(x) > f_1(b) + g_1(b)g_2(y)$$
and
$$(a, x) \succ (b, y) \Leftrightarrow f_1(a) + f_2(x) + g_1(a)g_2(x) > f_1(b) + f_2(y) + g_1(b)g_2(y).$$

The paper presented independence axioms that are necessary and sufficient for these special representations within the context of so-called bisymmetric structures.

Fishburn's work extended the theory of additive conjoint measurement, and more generally the theory of additive measurement, in various ways. For example, an ***additive difference model*** arises from rewriting Condition (5) as

$$(a_1, a_2, \ldots, a_n) \succ (b_1, b_2, \ldots, b_n) \Leftrightarrow \sum [f_i(a_i) - f_i(b_i)] > 0,$$

or more generally where we use a function of the differences on the right-hand side, i.e.,

$$(a_1, a_2, \ldots, a_n) \succ (b_1, b_2, \ldots, b_n) \Leftrightarrow \sum F\{f_i(a_i) - f_i(b_i)\} > 0. \quad (7)$$

Fishburn (1992) provided two axiomatizations of Representation (7). He also studied specializations of this model for homogenous Cartesian product structures such as those arising when there are time streams or finite-state decisions under uncertainty.

Consider the case $n = 2$ where we consider alternatives in a set $A \times X$. Representation (5) is then

$$(a, x) \succ (b, y) \Leftrightarrow f_1(a) + f_2(x) > f_1(b) + f_2(y).$$

But what if attributes in A are the dominant factor? Then we might consider the ***lexicographic representation***:

$$(a, x) \succ (b, y) \Leftrightarrow f_1(a) > f_1(b) \text{ or } [f_1(a) = f_1(b) \;\&\; f_2(x) > f_2(y)]. \quad (8)$$

Fishburn (1980) studied variants of the Representation (8). This representation was generalized by Luce (1978) to a situation where A is the dominant factor if the difference between a and b is sufficiently large but if the difference is sufficiently small, the additive model applies. Here, Luce considered a binary relation $\succ_A$ defined by

$$a \succ_A b \Leftrightarrow (a, x) \succ (b, y) \text{ for all } x, y \text{ in } X.$$

For example, as Fishburn (1980) pointed out, you might wish to buy a car and A gives the expected annual operating cost and X the expected cruising range. Then $a \succ_A b$ would apply if the cost differential is sufficiently large so that no matter how much better the expected cruising range of y than that of x, you would still prefer a car with attributes (a,x) to one with attributes (b,y). Luce assumed that $\succ_A$ is a semiorder. He assumed that if neither $a \succ_A b$ nor $b \succ_A a$, then the additive model of Condition (5) would be used to make a choice. Thus, he sought functions f_A, f_X, and δ_A so that

$$a \succ_A b \Leftrightarrow f_A(a) > f_A(b) + \delta_A$$

and if neither $a \succ_A b$ nor $b \succ_A a$, then

$$(a, x) \succ (b, y) \Leftrightarrow f_A(a) + f_X(x) > f_A(b) + f_X(y). \quad (9)$$

Fishburn (1980) introduced a variant of the model of Luce, one that combines a lexicographic component but replaces the additive model of Condition (9) with an additive difference model. As discussed above, an additive difference model arises from rewriting (5) as (7). Fishburn replaced Luce's condition (Condition (9)) with the condition

$$(a, x) \succ (b, y) \Leftrightarrow F_A\{f_A(a) - f_A(b)\} > F_X\{f_X(y) - f_X(x)\}.$$

A fundamental idea in theorems such as that of Luce and Tukey that give sufficient conditions for additive conjoint measurement is the idea of a **cancellation condition**. Fishburn (2001a) studied such cancellation conditions for the special case where $n = 2$ and A_1 and A_2 have M and N items, respectively. A weak order on a finite set Cartesian product structure has an additive real-valued order-preserving representation if and only if it satisfies a denumerable scheme of cancellation conditions $C(2)$, $C(3)$, … . For example, suppose $(a_1,a_2,\ldots,a_n) \succsim (b_1,b_2,\ldots,b_n)$ means that it is not the case that $(b_1,b_2,\ldots,b_n) \succ (a_1,a_2,\ldots,a_n)$. Then $C(2)$ is the condition that for all x, y, z, w,

$$x \succsim y \Leftrightarrow z \succsim w \text{ if for each } i = 1, 2, \ldots, n,$$
$$\text{either } (x_i = y_i \text{ and } z_i = w_i) \text{ or } (x_i = z_i \text{ and } y_i = w_i).$$

Given M and N, there is a largest K, denoted by $f(M, N)$, such that some weak order satisfies $C(2)$ through $C(K\text{-}1)$ but violates $C(K)$. Fishburn (2001a) studied the numbers $f(M,N)$.

As noted, Representation (5) can be generalized as Condition (7). Fishburn (1991) studied an additive conjoint model that does not require that the relation $\succ$ be transitive, which he calls ***nontransitive additive conjoint measurement***. This is the representation where Condition (7) is replaced by using a skew symmetric functional ϕ_i so that

$$(a_1, a_2, \ldots, a_n) \succ (b_1, b_2, \ldots, b_n) \Leftrightarrow \sum \phi_i(a_i, b_i) > 0.$$

The functional ϕ_i is skew symmetric if

$$\phi_i(a, b) + \phi_i(b, a) = 0,$$

for all a, b. A special case of a skew-symmetric functional is of course $\phi_i(a,b) = f_i(a) - f_i(b)$. Fishburn's motivation for skew-symmetry was that the magnitude of "subjective difference" should not depend on the "direction" from which it is viewed. He provided axioms necessary and sufficient to imply that such a representation exists in the case where all attribute sets are finite and also in the case where $n = 2$ and there is no assumption of finiteness. A third axiom gave sufficient conditions, when

$n \geq 3$, that imply that the representing functionals are unique up to multiplication by a positive constant.

Fishburn (1972a) studied a notion of degree of interdependence of a preference relation $\succ$ on a finite subset A of a Cartesian product set A_1 x A_2 x … x A_n that is defined in terms of the highest order of preference interaction among the A_i that must be taken into account in a real-valued, interdependent additive representation for $\succ$. The degree is zero when indifference holds throughout A, and zero or one in the additive conjoint measurement case. A degree of n signifies complete preference interdependence among the A_i. For example, in an economic context, if the degree is 4 and A_3 and A_4 represent two complementary goods and A_1 and A_2 are relatively independent of each other and of A_3 and A_4, then we might be able to obtain the following generalization of additive conjoint measurement:

$$\begin{aligned}(a_1,a_2,a_3,a_4) \quad &\succ (b_1,b_2,b_3,b_4) \Leftrightarrow \\ &f_1(a_1)+f_2(a_2)+f_{34}(a_3,a_4) \\ &> f_1(b_1)+f_2(b_2)+f_{34}(b_3,b_4).\end{aligned}$$

This is different from nondecomposability; it involves functions f_1, f_2, f_{34} instead of functions f_1, f_2, f_3, f_4 but no interaction term involving two or more of the f_i. The paper provided an axiomatic characterization of degree of interdependence.

This line of work is strongly represented in the following four special issue papers. Nakamura (2024) "*Stochastic additive differences*" considers binary choice probabilities as a transformation of the additive difference evaluations of the chosen and unchosen outcomes. Building on Fishburn (1991, 1992), the author presents axiomatic characterizations of stochastic difference models.

Franco, Laros, and Wiberg (2024) "*Nondecomposable Item Response Theory models: Fundamental measurement in psychometrics*" develops a novel Item Response Theory (IRT) framework that is built upon nondecomposable conjoint measurement theory (Fishburn, 1974, 1975). The authors provide both classic and Bayesian instantiations of this framework. This provides a nice intellectual bridge between fundamental measurement, as described by the Fishburn theories of conjointment measurement, and modern Rasch modeling. This paper demonstrates how a nondecomposable IRT framework extends traditional Rasch modeling.

Dhurkari (2024) "*Multi-Attribute Gain Loss (MAGL) method to predict choices*" brings together several ideas from marketing, behavioral decision research, and multi-attribute decision theory, to develop a model of consumer choice and market share that accounts for context and available choice set. The paper posits a two-step preference elicitation process, by which the decision maker first assigns weights to attributes, then defines their preference, separately and independently for each attribute, over all available alternatives. A processing and aggregation step then helps the decision maker consolidate that information into overall scores for the choice options. The paper argues that the approach is useful for consumer product design and market share analysis.

Kurihara (2024) "*Sufficient conditions making lexicographic rules over the power set satisfy extensibility*" presents a series of results linking decisions over alternative and "null" alternatives. Building upon Fishburn (1992), null alternatives are negations of alternatives under consideration, e.g., one might prefer to keep John off the city council ("null" John) rather than elect Sally. These results link to lexicographic decision making and the usage of signed orders (Fishburn, 2001).

4. Expected utility, risk

Throughout his career, Peter Fishburn was interested in von Neumann-Morgenstern expected utility theory. Suppose an act or gamble or lottery can lead to one of E_1, E_2, …, E_n, mutually exclusive events, each occurring with a certain probability $P(E_i)$, so that $\sum P(E_i) = 1$. If event E_i occurs, there is a consequence or payment c_i. Von Neuman and Morgenstern (1947) provided axioms on a preference relation $\succ$ that allowed them to conclude that there is utility function f so that one gamble x is preferred to another gamble y under $\succ$ if and only if the first has a higher expected utility than the second, i.e.,

$$x \succ y \Leftrightarrow \sum P(E_i)f(c_i) > \sum P(E_i^{'})f(c_i^{'}),$$

where the first sum is over the events and consequences of gamble x and the second over the events and consequences of gamble y. Moreover, the utility function is unique up to change of scale and zero point, i.e., it defines an interval scale.

Fishburn (1967) presented a set of axioms for additive conjoint measurement in utility theory when the set of consequences, C, is a subset of the Cartesian product of two sets, and possibly a proper subset – which was a different assumption than typical. For example, he observed that if the first component of C is an action and the second is an outcome of such an action, not every possible outcome is a conceivable result of an action, so not every ordered pair (action, outcome) is of interest. Fishburn presented axioms for preference patterned after the von Neumann-Morgenstern axioms. These axioms imply the existence of a utility function on C and gambles formed from the elements in C but do not imply the additive form of conjoint measurement. However, Fishburn then showed how to strengthen one of the axioms so that additivity holds, i.e., the utility of each consequence equals the sum of the utilities of its two factors. He then explored the uniqueness of the two component functions.

According to Bell and Fishburn (2003), expected utility theory has become widely viewed as inadequate for describing individual decision making under risk. The various arguments supporting this observation had earlier led Fishburn to publish a variety of papers exploring generalizations and variants of von Neumann-Morgenstern utility theory, including ones in which some of the von Neumann-Morgenstern axioms are modified or omitted. For instance, Fishburn (1982b) axiomatized a generalization of the main theorem in the context of gambles/lotteries in which he didn't use the transitivity or independence axioms of von Neumann and Morgenstern. Instead, he gave axioms providing necessary and sufficient conditions for preferences to be representable using a bivariate rather than a univariate function. In this work, he studied convex combinations of gambles x and y. If $x(c)$ is the probability of outcome c in gamble x and $y(c)$ is the probability of outcome c in gamble y and λ is a number between 0 and 1, then in gamble $x\lambda y$, (which is sometimes denoted $\lambda x+(1-\lambda)y$), the probability of outcome c is $\lambda x(c)+(1-\lambda)y(c)$. Fishburn (1982b) then sought a skew-symmetric bilinear function ϕ so that

$$x \succ y \Leftrightarrow \phi(x,y) > 0. \tag{10}$$

Condition (10) is called the ***SSB model***. Such a skew-symmetric function ϕ has the two properties:

$$\phi(x,y) = -\phi(y,x), \qquad \text{(skew-symmetry)}$$
and
$$\phi(x\lambda y,z) = \lambda\phi(x,z) + (1-\lambda)\phi(y,z), \qquad \text{(bilinearity)}$$
and similarly for the second argument. In contrast, in linear utility,

$$f(x\lambda y) = \lambda f(x) + (1-\lambda)f(y). \tag{11}$$

Moreover, in contrast to the von Neumann Morgenstern uniqueness theorem, the new result provided a function defining a ratio scale, i.e., one unique up to multiplication by a positive constant. Fishburn's work here was motivated by an extensive literature, which he reviewed, that critiques different axioms of von Neumann and Morgenstern and also linear utility theory. For example, he pointed out that individuals tend to subjectively inflate small probabilities and discount large probabilities.

In utility theory, we often compare choices among collections of items, e.g., packages of car options, potential committees chosen from a group of candidates, potential winning research proposals, etc. It is tempting, as Fishburn (1993) observed, to combine the values of the elements of a subset by an operation such as addition. However, as he pointed out, the interdependencies among items in a subset may lead to significant distortions if we simply add up the values. This led Fishburn to study independence axioms on subset structures, in particular in the linear utility theory of von Neuman and Morgenstern, but also in the skew symmetric bilinear utility theory discussed in Fishburn (1982b) and summarized above, as well as in weighted linear utility theory where we seek both a utility function *f* and a weighting function *w* so that

$$x \succ y \Leftrightarrow f(x)w(x) > f(y)w(y).$$

In generalizing the semiorder model of Eq. (1), Nakamura (1990) studied the representation

$$a \succ b \Leftrightarrow f(a) > f(b) + \delta(b).$$

He extended this to the representation

$$x \succ y \Leftrightarrow \phi(x, y) > w(x, y),$$

where ϕ is skew symmetric. This model incorporates imprecise discriminability into judgments of preference and allows preferences to be nontransitive. In Fishburn and Nakamura (1991), the case of $w \equiv 1$ is studied, i.e., the representation

$$x \succ y \Leftrightarrow \phi(x, y) > 1,$$

in contrast to the SSB model of Condition (10). This paper provided necessary and sufficient conditions for the existence of such a skew symmetric bilinear representation. All but one of the axioms are implied by the SSB model and also the linear utility model of von Neumann and Morgenstern.

A key axiom in von Neumann-Morgenstern expected utility theory is an Archimedean axiom. Fishburn and LaValle (1992) explored the non-Archimedean situation. They pointed out that Archimedean axioms do not have the same "normative status" as axioms like transitivity and linearity and are there to help produce real-valued representations. They also pointed out that if every lottery that offers positive probability of a wrong outcome is inferior to one that only produces non-wrong outcomes, even if undesirable, then one has a (plausible) non-Archimedean situation. They also explored a variety of multiattribute independence conditions. The paper was written in the context of lexicographic expected utility. Consider a set *E* of probability distributions on an outcome set *S*. Then a function *f* on probability distributions is ***linear*** if it satisfies Eq. (11).

$$f(x\lambda y) = \lambda f(x) + (1 - \lambda) f(y).$$

It is ***Archimedean*** if $x \succ y \succ z$ implies that there are α, β so that $x\alpha z \succ y$ and $y \succ x\beta z$.

We define the lexicographic relation $\succ_L$ on vectors $c = (c_1, c_2, \ldots, c_n)$ and $d = (d_1, d_2, \ldots, d_n)$ by

$$c \succ_L d \ \Leftrightarrow c \neq d \text{ and } \min j \text{ so that } c_j \neq d_j \text{ has } c_j > d_j.$$

In exploring ***lexicographic expected utility*** in the multi-attributed case, Fishburn and LaValle studied the representation where there is a linear utility function *F* into *n*-space such that

$$x \succ y \Leftrightarrow F(x) \succ_L F(y)$$

More specifically, they explored the representation where there are functions $f_1, f_2, \ldots, f_n$ so that each f_i is linear and so that

$$x \succ y \Leftrightarrow (f_1(x_1), f_2(x_2), \ldots, f_n(x_n)) \succ_L (f_1(y_1), f_2(y_2), \ldots, f_n(y_n)).$$

Fishburn (1972c) also explored weakening of the von Neuman-Morgenstern axioms. He showed that several important implications of the preference axioms follow from significantly weaker and psychologically more plausible axioms. The paper was partially motivated by the observation on which semiorders are based that indifference may not be transitive. It discussed four axioms that use convex linear combinations of probability distributions, two of which allow indifference to be nontransitive. The axioms are:

$$x \succ y \Rightarrow x\lambda z \succ y\lambda z,$$
$$x \succ y \text{ and } z \succ w \Rightarrow x\lambda z \succ y\lambda w,$$
$$x\lambda z \succ y\lambda z \Rightarrow x \succ y,$$

and

$$x \sim z \text{ and } y \sim z \Rightarrow x\lambda y \sim z.$$

Here $\sim$ means indifference, i.e., $x \sim y$ if and only if neither $x \succ y$ nor $y \succ x$.

Bell and Fishburn (2003) recalled the observation that individuals tend to subjectively inflate small probabilities and discount large probabilities, and noted that as a result, several theories propose different weighting functions for outcomes interpreted as gains and losses. In their paper, they considered a specific probability weighting function and explored the effects of a very weak version of the independence axiom of expected utility theory. According to the paper and the cited literature, of the EU axioms, the one most frequently violated in observed behavioral situations is the independence axiom. The axiom says that if *x* is a two-outcome gamble with certainty equivalent *c*, then the even-chance lottery that yields *x* or *c*, each with probability 1/2, will be considered indifferent to *c*.

Assuming that there is an underlying von Neumann-Morgenstern utility function, Fishburn (1976) explored two models for describing an individual's binary choice probabilities between gambles. The first model looks at the odds of choosing gamble *x* over gamble *y* as a ratio involving the incremental expected utility advantage *A*(*x,y*) of gamble *x* over gamble *y*. Specifically, the model is:

$$P(x, y)/P(y, x) = [A(x, y)/A(y, x)]^{\alpha}$$

The parameter α is called the individual's ***index of perspicacity***. Zero perspicacity corresponds to a completely random chooser with $P(x,y) = ½$ for all *x, y*. Large perspicacity corresponds to a chooser who almost always chooses the gamble with the larger expected utility. The second model considered looks at the same odds using the expected utility loss *L*(*x,y*) of *x* with respect to *y*:

$$P(x, y) \big/ P(y, x) = [L(x, y)/L(y, x)]^{\beta}$$

where β is a different index of perspicacity.

The idea of stationary value or utility arises, among other ways, in the context of consumer demand behavior over successive time periods. Fishburn (1966) explored notions of stationary value mechanisms and stationary transition value mechanisms in time-dependent processes. The paper presented two axiomatizations of these concepts in the context of expected utility theory. Specifically, suppose that an individual starts in a state s_0 of a system that passes through a series of states $s_1, s_2, \ldots, s_n$. Let $\succ$ be a binary preference relation between *n*-step histories. We say that there is a ***stationary value mechanism*** if there is a function *v* so that

$$\begin{aligned}(s_1, s_2, \ldots, s_n) \quad &\succ (t_1, t_2, \ldots, t_n) \Leftrightarrow \\ &v(s_1) + v(s_2) + \ldots + v(s_n) \\ &> v(t_1) + v(t_2) + \ldots + v(t_n).\end{aligned}$$

There is a ***stationary transition value mechanism*** if there is a function *v* so that

$$\begin{aligned}&(s_1, s_2, \ldots, s_n) \succ (t_1, t_2, \ldots, t_n) \Leftrightarrow \\ &v(s_0, s_1) + v(s_1, s_2) + \ldots + v(s_{n-1}, s_n) > v(t_0, t_1) + v(t_1, t_2) + \ldots + v(t_{n-1}, t_n).\end{aligned}$$

A variety of independence axioms for multiattribute utility functions have been studied in the literature, in particular in the context of the study of expected utility. Fishburn and Keeney (1974) explored seven such independence concepts for a preference relation on a set of simple probability measures over multiattribute consequences. Some of these concepts involved gambles, so involved a risky situation, and others involved preference over *n*-tuples in the consequence set, i.e., they were in a riskless context. The paper explored conditions under which the concepts over gambles can be derived from those in the riskless context, and more generally, what is the relationship among the different independence axioms. For example, the weakest independence axiom is called "weak indifference independence." The paper showed that if the components A_i of the Cartesian product set are convex and the von Neumann-Morgenstern utility function is continuous, then under certain assumptions, weak indifference independence implies considerably stronger independence axioms.

Suppose f is a real-valued function on a subset A of a product structure A_1 x A_2 x ... x A_n. Motivated by von Neumann-Morgenstern expected utility theory, Fishburn (1971) sought conditions under which f is ***additive***, i.e., where there are functions f_i on A_i so that

$$f(a_1, a_2, \ldots, a_n) = f_1(a_1) + f_2(a_2) + \ldots + f_n(a_n). \tag{12}$$

Let m be a positive integer and $a^1, a^2, \ldots, a^m, b^1, b^2, \ldots, b^m$ be elements of A. We say that $(a^1,a^2,\ldots,a^m)\ I_A\ (b^1,b^2,\ldots,b^m)$ if for each i from 1 up to n, $a_i^1, a_i^2, \ldots, a_i^m$ is a permutation of $b_i^1, b_i^2, \ldots, b_i^m$. We say that f satisfies Condition 1(A) if

$$(a^1, a^2, \ldots, a^m) I_A (b^1, b^2, \ldots, b^m) \Rightarrow \sum_{j=1}^{m} f(a^j) = \sum_{j=1}^{m} f(b^j).$$

Then Fishburn (1971) showed that f is additive, i.e., there are functions f_i satisfying Eq. (12), if and only if f satisfies Condition 1(A).

There is a long tradition in mathematical psychology and related disciplines in economics to formalize theories of risk and to develop the foundations of risk measurement. Much of the early work in mathematical psychology dealing with risk was due to Clyde Coombs. In some sense, a great deal of Fishburn's work on expected utility can be viewed as dealing with risk, though Coombs and others did not consider EU theory as an adequate description of risk (Tversky, 1992). Peter Fishburn was concerned with developing foundations of risk measurement, and in particular with measurement of perceived risk. In a paper on this topic (Fishburn, 1982a), he expanded on an earlier Bell Labs unpublished draft that focused on risk as probable loss. In this paper, he studied measures of risk that include effects of gains on perceived risk and adopted the position that increased gains can reduce the risk of fixed probable losses without completely negating this risk. The paper axiomatized several numerical measures of risk and discussed effects of loss and gain probabilities. In this formalization, outcomes are divided into favorable ones and unfavorable ones. A given risky decision was defined by probabilities of gain and loss and the gain and loss distributions. Fishburn studied a binary preference relation on the set of such decisions and gave six axioms that lead to a risk measure that is continuous in gain and in loss probability. He also axiomatized expected risk for loss and discussed an approach to the case where the probabilities for loss and gain are separable and the axioms reflect traditional approaches to conjoint measurement.

Two papers in this special issue naturally connect with this body of work. Nakamura (2024) "*Subjective expected utility with signed threshold*" provides a new axiomatization of subjective expected utility with signed thresholds. The signed threshold perspective is useful for modeling alternatives with both positive and negative features. This paper offers an interesting new perspective on this classic utility framework by considering alternative axiom systems that require neither transitivity of preference nor certain other common axioms.

Bardakhchyan and Allahverdyan (2024) "*Regret theory, Allais' paradox and Savage's omelet*" offers a new choice property based on Regret theory that nicely interfaces with several well-known decision problems and theories. Specifically, the authors demonstrate how their new regret criterion defined over binary lotteries can resolve the classic Allais Paradox and Savage's Omelet. This property and related analyses provide a sharper understanding of the various classic decision problems and help lay foundations for improving choice theory.

5. Choice functions, choice probabilities

Choosing an alternative from a set of possibilities has been of great interest to psychologists for many years. Peter Fishburn wrote a number of papers exploring this concept. Suppose $A = \{a_1, a_2, \ldots, a_n\}$ is a finite set of alternatives from which we want to choose a best choice, e.g., A could be a set of restaurants, a set of job candidates, a set of new cars. Not all of the alternatives are always available (e.g., some restaurants are closed on Mondays). We also may not always make the same choice, so there is a probability that a given element will be chosen. If S is a subset of A, we can denote by $P(i;S)$ the probability that a_i will be chosen if S is the set of available alternatives. If S just has the two elements a_i and a_j, we write $P(i;S)$ as $P(i,j)$. Let $P(S;A)$ denote the probability that if all alternatives in A are available, then the chosen alternative will belong to S. A pioneering paper of Thurstone (1927) presented alternatives as having utilities that vary randomly (from a normal distribution) and so the probability of choosing a_i from S depends on its current utility value. Luce (1959) introduced the ***choice axiom***, which says that if $P(i,j) \neq 0$ for all i,j, then

$$P(i;A) = P(i;S)P(S;A).$$

If it is possible that $P(i,j) = 0$ for some i,j, then Luce adds the condition that whenever $P(i,j) = 0$, then

$$P(S;A) = P(S \setminus \{a_i\}; A \setminus \{a_i\}).$$

The choice axiom and this extra condition imply that there is a function v assigning a real number to each alternative a_i so that

$$P(i;S) = \frac{v(a_i)}{\sum_{a_j \in S} v(a_j)}$$

Fishburn (1994) studied the interplay between choice probabilities and probabilities of rankings, noting that most earlier work had concentrated on deriving the latter from the former. Instead, he studied two axioms for a probability distribution on rankings that are necessary and sufficient to induce choice probabilities satisfying Luce's choice axiom. This topic is of particular interest now given the increasing use of ranked ballots in elections, another topic to which Fishburn made major contributions. Suppose S and T are disjoint subsets of A whose union is A. Suppose that $s = s_1s_2\ldots s_k$ is a ranking of elements in S and $t = t_1t_2\ldots t_l$ is a ranking of elements of T. Then st denotes the ranking where t follows s, i.e., $st = s_1s_2\ldots s_k\, t_1t_2\ldots t_l$. The first axiom says that if s and s' are rankings of S and t and t' are rankings of T, then if P denotes probability,

$$P(st)P(s't') = P(st')P(s't).$$

To explain the second axiom, suppose that S is a subset of A and r is a ranking of elements outside of S. We use the notation $[S]$ to mean the set of all rankings of the set S and $r[S]$ to be the set of rankings where ranking r comes first followed by some ranking of elements of S. Then if a is chosen from S, the second axiom says that there exists a ranking s of A/S so that

$$P(sa[S/\{a\}]) = P(a[S/\{a\}])P(s[S]).$$

That is, there is a ranking s so that the unconditional probability that a is ranked first in S equals the probability that a is ranked first in S given that the objects outside S have the ranking s and are ranked ahead of all objects in S. Fishburn (1994) proved that Axioms 1 and 2 imply Luce's choice axiom.

In contrast to the choice probabilities we have been discussing, there are other ways to approach individual choice behavior, namely of course preference relations but also choice functions. A ***choice function*** C defined on subsets S of the set A gives the set of alternatives of A chosen if the set S is presented as the set of alternatives. S could be viewed as the set of those alternatives that are acceptable. For examples of choice functions defined from probabilities, consider

$C_{min}(S) = \{a \in S : P(a; S) > 0\}$ and
$C_{max}(S) = \{a \in S : P(a; S) = \max\{P(u; S) : u \in S\}\}$.

Fishburn (1978) pointed out that there had been sizable literatures on the connections between choice probabilities and preference relations, and between preference relations and choice functions, but little had been done to connect choice probabilities and choice functions, which is the topic of this paper. Specifically, the paper studied a specific family of choice functions depending on a threshold parameter and defined from choice probabilities. Three traditional rationality conditions for choice functions were presented and the paper described conditions for the choice probability function that lead to these rationality conditions being satisfied for the family of choice functions being studied. He also showed what has to be true about the choice functions in the family so that the choice probability function satisfies a version of Luce's axiom for individual choice probabilities. The choice functions studied include a in $C(S)$ if and only if $P(a;S)$ is at least as large as a specified fraction of the largest $P(u;S)$ for u in S. Specifically, for a given parameter λ, the function C_λ is defined as follows:

$$C_\lambda(S) = \{a \in S : P(a; S) \geq \lambda \max_{u \in S} P(u; S)\}. \quad (13)$$

So, for example, if $\lambda = 0.1$, then if a is not in $C_\lambda(S)$, some element in $C_\lambda(S)$ is more than 10 times as likely as a to be chosen from S. A special case of Equality (13) arises when $\max_{u \in X} P(u; S) = 1$ in which case Equality (13) becomes

$$C_\lambda(A) = \{a \in S : P(a; S) \geq \lambda\}. \quad (14)$$

Fishburn showed that if C_λ satisfies Equality (14), then the three axioms on C_λ are necessary and sufficient for Luce's choice axiom.

Fishburn (1973a) studied the binary choice probabilities $P(i,j)$. If $a_i = a$ and $a_j = b$, we will denote $P(i,j)$ by $P(a,b)$. Motivated by interest in loudness judgments, we can ask whether a is judged louder than b a sufficiently large percentage of the time. A similar idea is useful for preference. Then we are interested in the binary relation R_λ where

$$aR_\lambda b \Leftrightarrow P(a,b) > \lambda.$$

This binary relation was introduced by Luce (1959) and was studied extensively in Roberts (1971), which was the paper motivating the approach in Fishburn (1973a). As Fishburn pointed out, the parameter λ can be viewed as an indicator of decisiveness. The relation is relevant to an increasingly stronger set of conditions called stochastic transitivity conditions, including:

Weak Stochastic Transitivity WST

$$P(a,b) \geq 1/2 \ \& \ P(b,c) \geq 1/2 \Rightarrow P(a,c) \geq 1/2.$$

Moderate Stochastic Transitivity MST

$$P(a,b) \geq 1/2 \ \& \ P(b,c) \geq 1/2 \Rightarrow P(a,c) \geq \min\{P(a,b), P(b,c)\}.$$

Strong Stochastic Transitivity SST

$$P(a,b) \geq 1/2 \ \& \ P(b,c) \geq 1/2 \Rightarrow P(a,c) \geq \max\{P(a,b), P(b,c)\}.$$

In his paper, Fishburn (1973a) defined additional transitivity conditions for the family of orders R_λ and studied which imply which. In particular, he introduced additional concepts for this family of orders motivated by the axioms arising in the study of semiorders and interval orders. Specifically:

Interval Stochastic Transitivity IST

$$\max\{P(a,b), P(c,d)\} \geq \min\{P(a,d), P(c,b)\}$$

JST:

$$\max\{P(a,b), P(b,c)\} \geq \min\{P(a,d), P(d,c)\}$$

The paper showed that every R_λ is an interval order if and only if IST holds. Also, every R_λ is a semiorder if and only if IST and JST both hold.

As noted earlier, one problem with estimating preferences between different subsets of a given set of alternatives, as well as associated choice probabilities, is that there might be interdependencies among the items in a subset, including for example substitutability or the desire for diversity. Fishburn (2001b) studied **signed orders** that were introduced earlier to address this difficulty. These are orders that make a copy of each item in a set A, creating a set A^*, and consider the copy of a^* to be the "anti-item" of a. If a is preferred to b, then the anti-item b^* of b is preferred to the anti-item a^* of a. In other words, anti-items allow you to consider what you don't want. The paper studies the relationship between binary choice probabilities $P(a,b)$ for distinct a, b in $A \bigcup A^*$ and probability distributions on the set of all linear signed orders on $A \bigcup A^*$. Of particular interest are conditions on the binary choice probability function P that are necessary and sufficient for the existence of a probability distribution on the set L of linear signed orders such that $P(a,b)$ is the sum of the probabilities of all linear orders R in L so that aRb. Fishburn studied this problem through the use of the linear signed order polytope, which is the convex hull of the set of incident vectors of the linear signed orders on $A \bigcup A^*$ and where each coordinate is associated with an ordered pair of distinct elements in $A \bigcup A^*$.

In the special issue, three papers examine choice functions. Pekeč (2024) "*A characterization of the existence of succinct linear representation of subset-valuations*" considers how to determine a valuation function for a bundle of choice options out of a master set of options. This result is useful as determining subset valuation using direct approaches can result in combinatorial explosion, greatly complicating computability. These results identify the situation when a simple linear representation exists and provide guidance on how subset valuation can be carried out in a parsimonious manner.

Barokas (2024) "*Majority-approval social choice*" axiomatizes a lexicographic social choice rule that reduces to Condorcet's majority rule when there is no top-cycle, i.e., when one or more Condorcet winners exist. Otherwise, it becomes a type of approval voting among the candidates in the top cycle. The paper also discusses the procedure's desirable properties and how it relates to other proposals in the social choice literature.

Marchant and Sen (2024) "*Stochastic choice with bounded processing capacity*" contributes to the literatures on bounded rationality, on probabilistic choice, and on consideration sets. The decision maker has a preference ranking $\succ$ among options. But, because of limited capacity to consider only up to k many items at a time, they sometimes need to go through two steps. In choice sets with fewer than k options, they pick their most preferred option. From larger choice sets, they first probabilistically consider a subset of k many options and then pick the best option in that consideration set according to $\succ$. The paper provides a comprehensive axiomatic characterization and discusses its properties.

6. Theory of measurement

A great deal of the work described above falls in the field that is called the ***representational theory of measurement*** and that was reflected in the foundational books of Falmagne (1985), Krantz et al. (1971), Luce et al. (1990), Narens (1985), Pfanzagl (1959), Roberts (1979), and Suppes et al. (1989). A surprising amount of work relevant to modern concepts of measurement theory was done by Norbert Wiener in the early 20th century. Fishburn and Monjardet (1992) went back to this work and reformulated it in modern language for readers of the JMP. Fishburn's mark on the theory of measurement is reflected in the many papers discussed above, almost all of which deal with representations, scales of measurement reflecting certain observed relations, and axioms under which such scales may be obtained. His mark on measurement theory is also reflected in this same representational approach that underlies many of his books, specifically Fishburn (1964, 1970b, 1972b, 1982c, 1985, 1988).

Three contributions in the special issue address the topic of measurement, broadly construed. Candeal (2024) "*A characterization of two-agent Pareto representable orderings*" axiomatically studies special types of partial orders on a nonempty set X that allow for a two-agent Pareto representation: For any element x of X, all other elements that are non-comparable must satisfy a specific decomposition property. The paper shows that, for any |X| up to five, any partial order is two-agent Pareto representable. It also discusses other scenarios with no cardinality constraints on X, including connections to the dimension of a partially ordered set.

Karpov (2024) "*Structure of single-peaked preferences*" studies various domain restriction conditions, such as generalizations of unfolding and of single-peaked preferences. It offers ways to compute the number of single-peaked preference profiles recursively for several concepts of single-peakedness and at various levels of generality. The results include important connections to work on acyclic sets of linear orders that Peter Fishburn published in *Social Choice and Welfare* in 1996, 2002, 2005 (Fishburn, 1996b, 2002, 2005).

Carpentiere, Giarlotta and Watson (2024) "*Modal preference structures*" develops a preference system which acts as an aggregate of a collection of individual binary preferences. This aggregate provides a way to ensure transitive and complete, i.e., rational, group-level preferences. The authors provide four properties that characterize such modal preference structures.

7. Closing comment

The variety of topics covered in Peter's *JMP* papers illustrates the dramatic impact his work has had on numerous areas of mathematical psychology, and this is just a hint at the many contributions that Peter made to utility theory, measurement theory, and other subjects. He was a constant source of ideas that have influenced the careers of the authors of this article and those of his many other colleagues, and will continue to be fundamental concepts in mathematical psychology for generations to come.

Two of us had personal interactions with Peter. In 2000, Mike Regenwetter organized a conference titled "Random Utility 2000" at which Peter was one of the invited speakers. During the talk of another speaker, Mike was sitting next to Peter and noticed that Peter was writing very neat mathematical content, and some text, on sheets of paper, in a small font, with a sharp pencil. When he asked Peter what he was doing, Mike received a stunning response. Peter explained that his routine was to work on a given problem for three weeks, then either abandon it or write up a manuscript by hand. Since he did not interact with computers (his secretary handled email for him) he gave the handwritten manuscripts to his secretary to enter into the computer for him and then submit to journals. As it turned out, Peter was not taking notes. He was using the time to write up his latest paper.

Fred Roberts observed this mode of research first hand. He and Peter wrote a dozen papers together and Peter would invariably produce various handwritten drafts of his ideas for Fred to modify and add to. This was especially true during Fred's two sabbaticals at Bell Labs, chosen so he could work with Peter. Peter's three-week principle was violated during this period, but only because Fred wasn't as fast as Peter was. Their collaboration traces back to the late 1960s where it started via mail, since they were on opposite coasts. That too slowed things down! It continued during memorable discussions while hiking in Banff National Park or playing tennis at Duke, while on breaks from conferences. He was a good friend as well as a professional colleague, and his ideas profoundly influenced Fred's career.

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