Cutting Planes and Elementary Closures for Non-linear Integer Programming

Juan Pablo Vielma

University of Pittsburgh

joint work with

D. Dadush and S. S. Dey

Georgia Institute of Technology

S. Modaresi and M. Kılınç

University of Pittsburgh

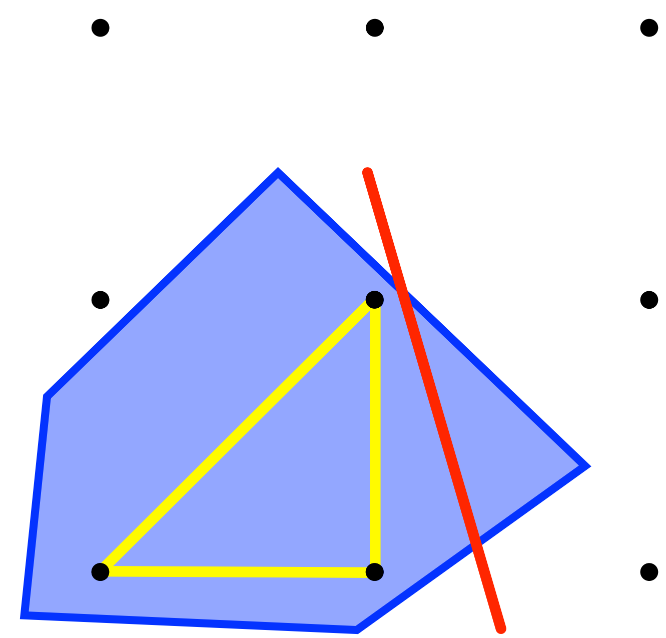
NSF CMMI-1030662 and ONR N000141110724

DIMACS, November, 2011 – Rutgers University, New Jersey

Cutting Planes for Integer Programming

- Valid Inequalities for the convex hull of integer feasible solutions.
- 50+ Years of development for Linear Integer Programming.
- Used to get tighter Linear Programming Relaxations.
- Crucial for state of the art solvers.

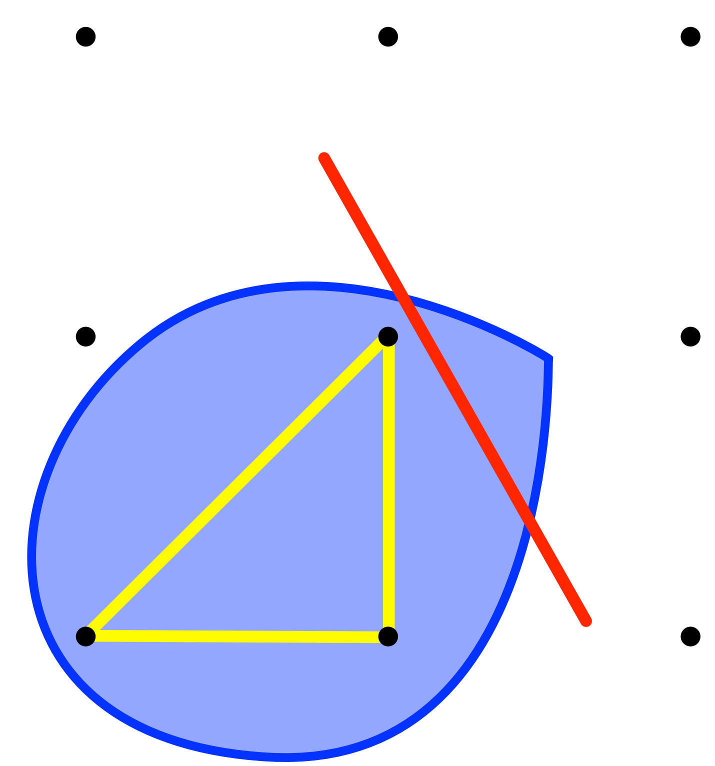
$$\max \quad \langle c, x \rangle$$
$$s.t.$$
$$Ax \leq b$$
$$x \in \mathbb{Z}^n$$



Convex Non-Linear Integer Programming

- Problems with convex continuous relaxation.
- Many applications, results and algorithms available.
- Cutting planes significantly less developed.
- Need new tools: linear results strongly rely on rationals.

$$
\begin{aligned}
\max \quad & \langle c, x \rangle \\
s.t. \quad & \\
& g_i(x) \leq 0,\ i \in I \\
& x \in \mathbb{Z}^n
\end{aligned}
$$



Two Classic Cutting Planes

- Chvátal-Gomory Cuts (Gomory 68, Chvátal 73):
  - AKA Gomory Fractional Cut
  - Simple, but yield pure cutting plane algorithm, Blossom's for Matching and Comb's for TSP.
- Split Cuts (Cook, Kannan and Shrijver 1990):
  - AKA MIG (Gomory 1960) and MIR (Nemhauser and Wolsey 1988)
  - Yield Flow Cover Cuts and modern IP solvers.

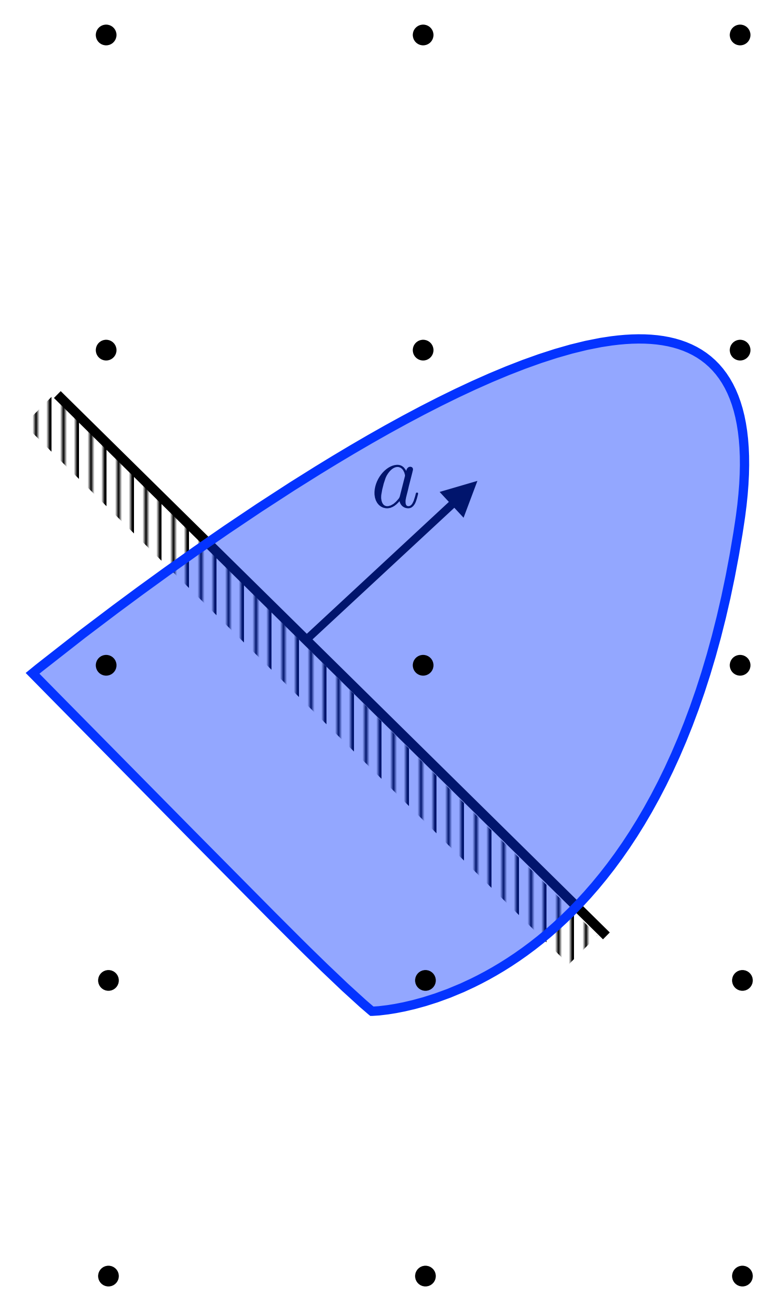
Outline

- Chvátal-Gomory Cuts for Non-Linear IP:
  - Polyhedrality of the Chvátal-Gomory Closure
- Split Cuts for Non-Linear IP:
  - Closed form Expressions.
  - Finite Generation v/s Polyhedrality of Split Closure
- Other Results and Open Questions

Chvátal-Gomory Cuts

CG Cuts and Support Function

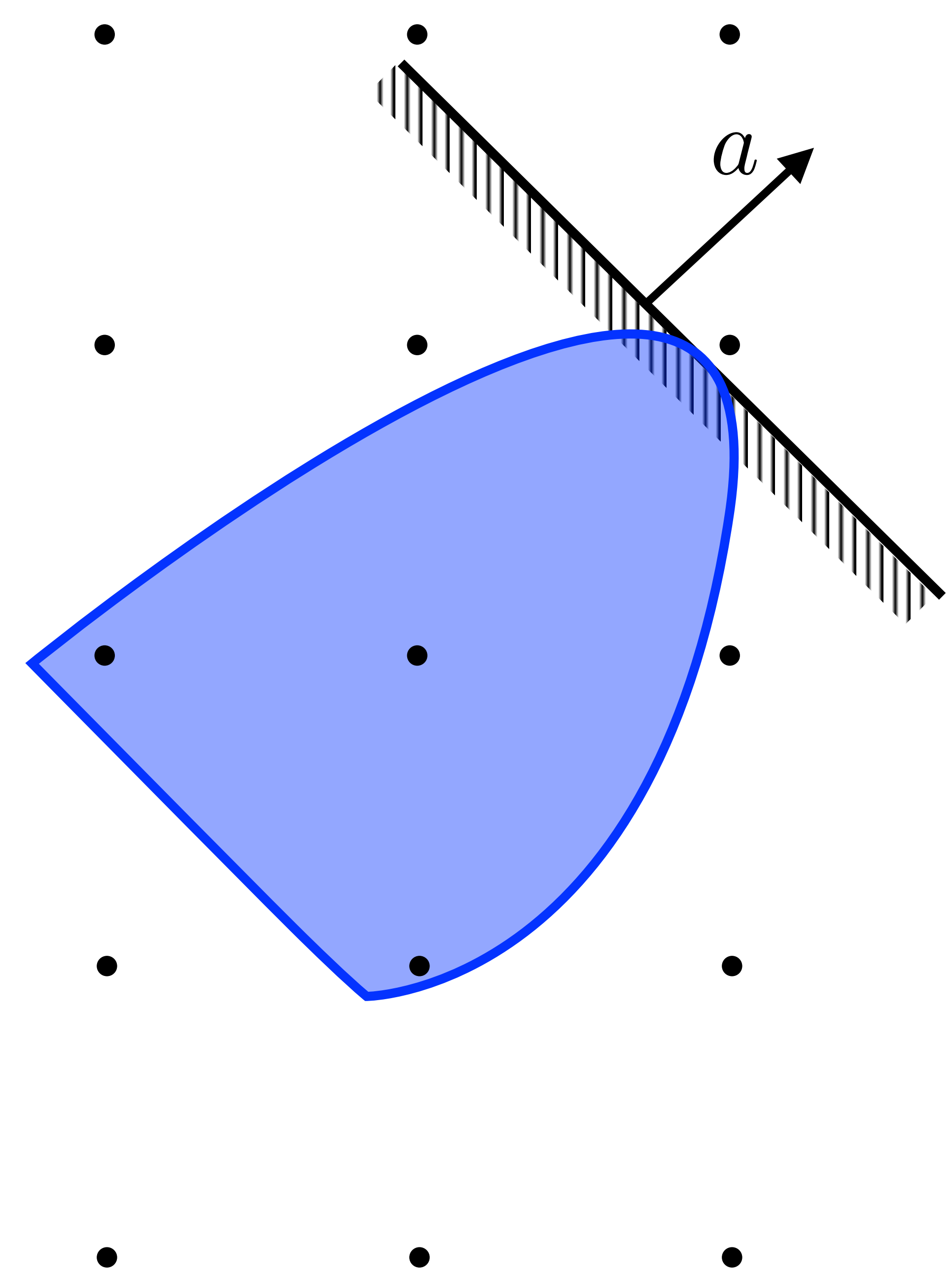
$$\sigma_C(a) := \sup\{\langle a, x\rangle \,:\, x \in C\}$$



CG Cuts and Support Function

$$\sigma_C(a) := \sup\{\langle a, x\rangle \, : \, x \in C\}$$

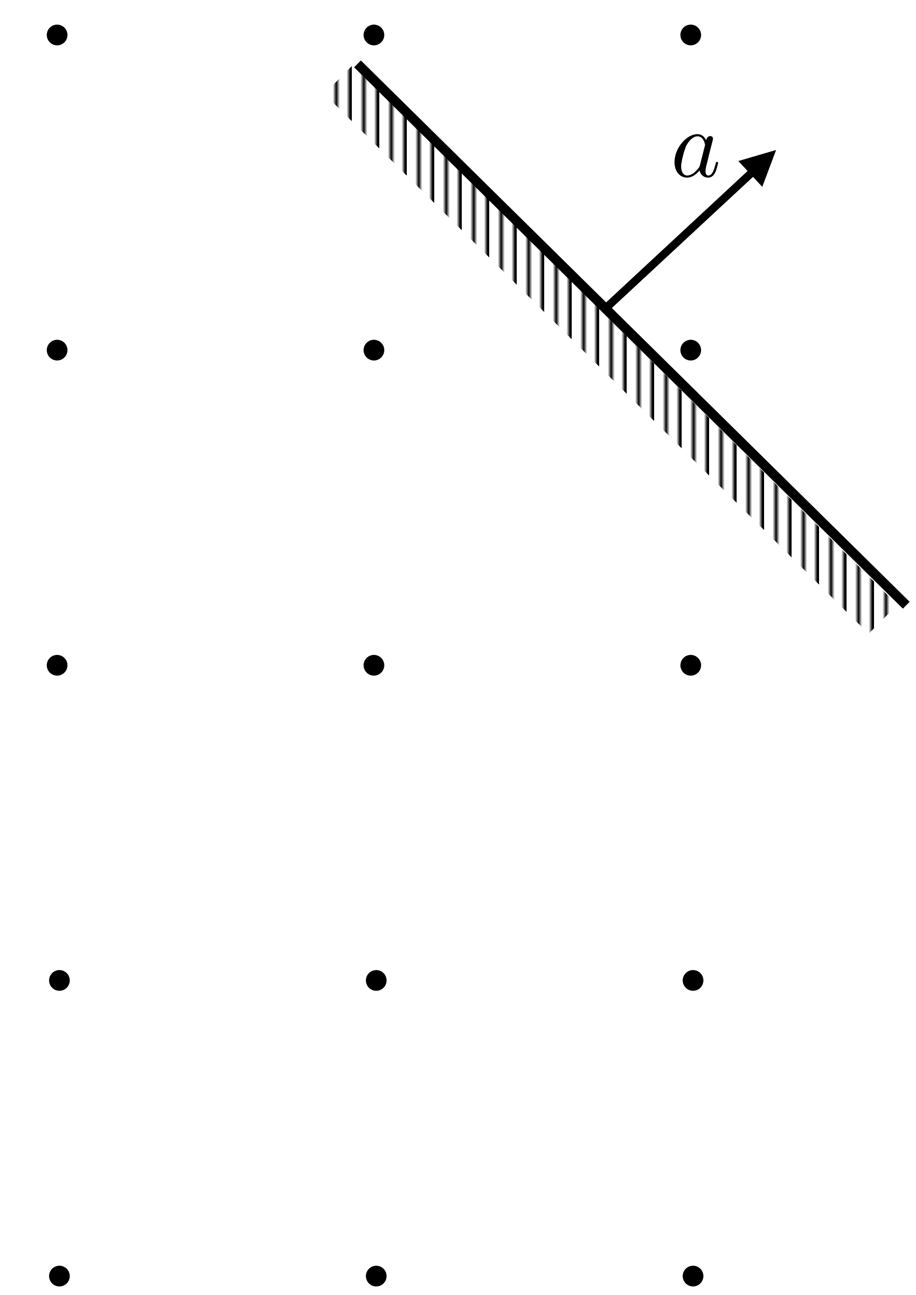
$$C \subseteq H := \{x \in \mathbb{R}^n \, : \, \langle a, x\rangle \leq \sigma_C(a)\}$$



CG Cuts and Support Function

$$\sigma_C(a) := \sup\{\langle a, x\rangle \,:\, x \in C\}$$

$$C \subseteq H := \{x \in \mathbb{R}^n \,:\, \langle a, x\rangle \leq \sigma_C(a)\}$$

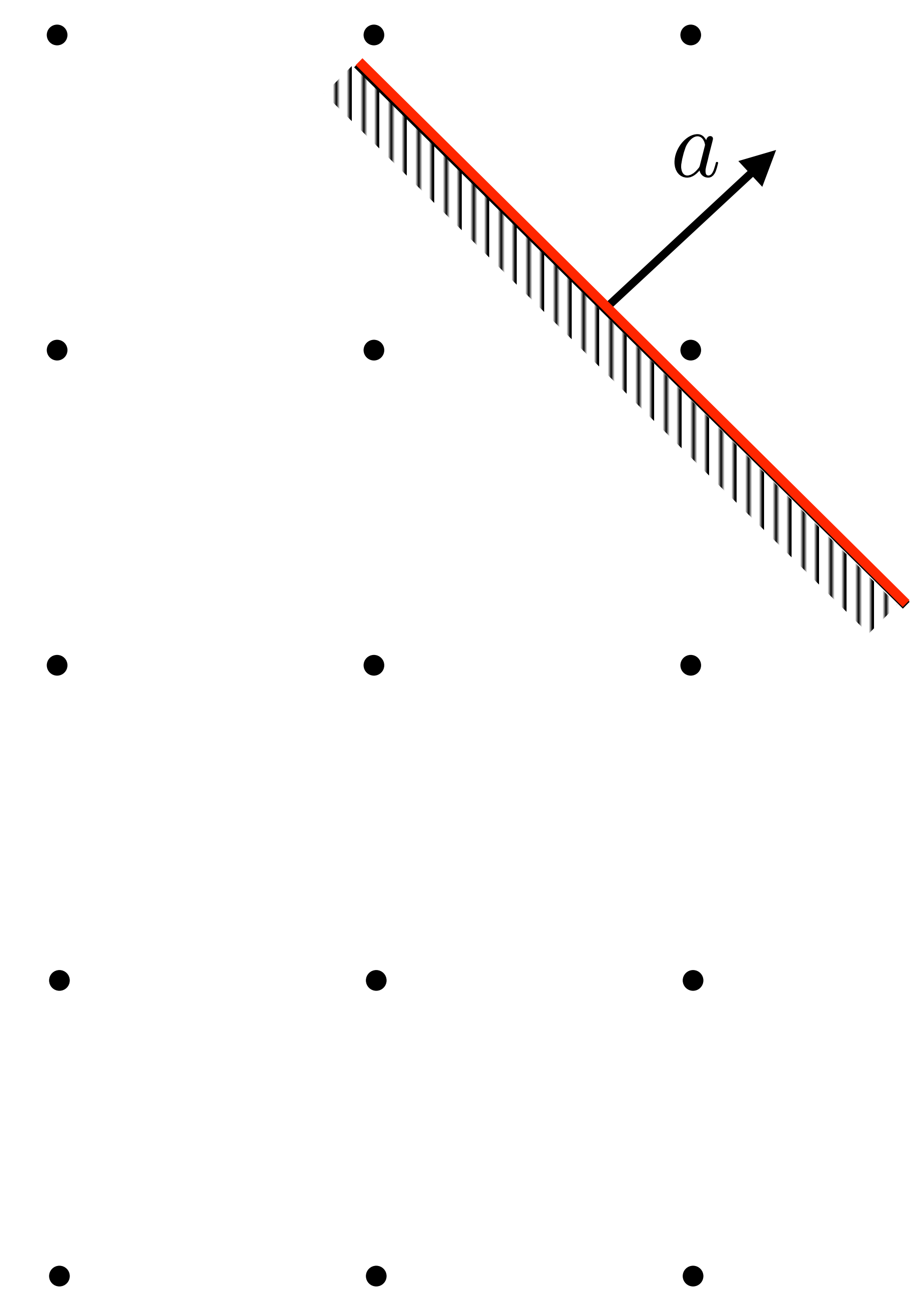


CG Cuts and Support Function

$$\sigma_C(a) := \sup\{\langle a, x\rangle \,:\, x \in C\}$$

$$C \subseteq H := \{x \in \mathbb{R}^n \,:\, \underbrace{\langle a, x\rangle}_{\in \mathbb{Z}} \leq \sigma_C(a)\}$$

$$\text{if } a,\, x \in \mathbb{Z}^n$$



CG Cuts and Support Function

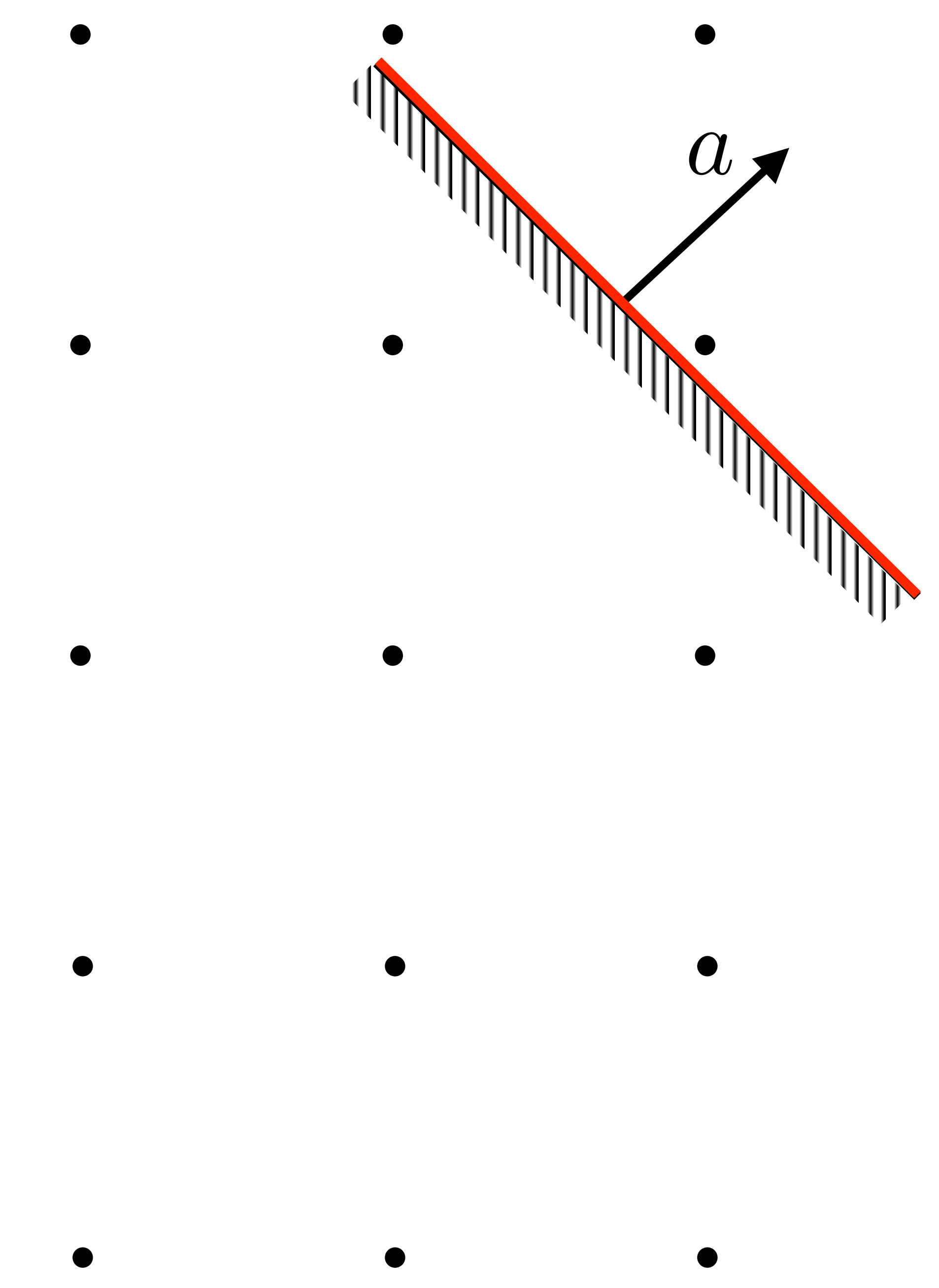
$$\sigma_C(a) := \sup\{\langle a, x\rangle \,:\, x \in C\}$$

$$C \subseteq H := \{x \in \mathbb{R}^n \,:\, \underbrace{\langle a, x\rangle}_{\in \mathbb{Z}} \leq \sigma_C(a)\}$$

if $a,\ x \in \mathbb{Z}^n$

$$\langle a, x\rangle \leq \lfloor \sigma_C(a) \rfloor$$

Valid for $H \cap \mathbb{Z}^n$



CG Cuts and Support Function

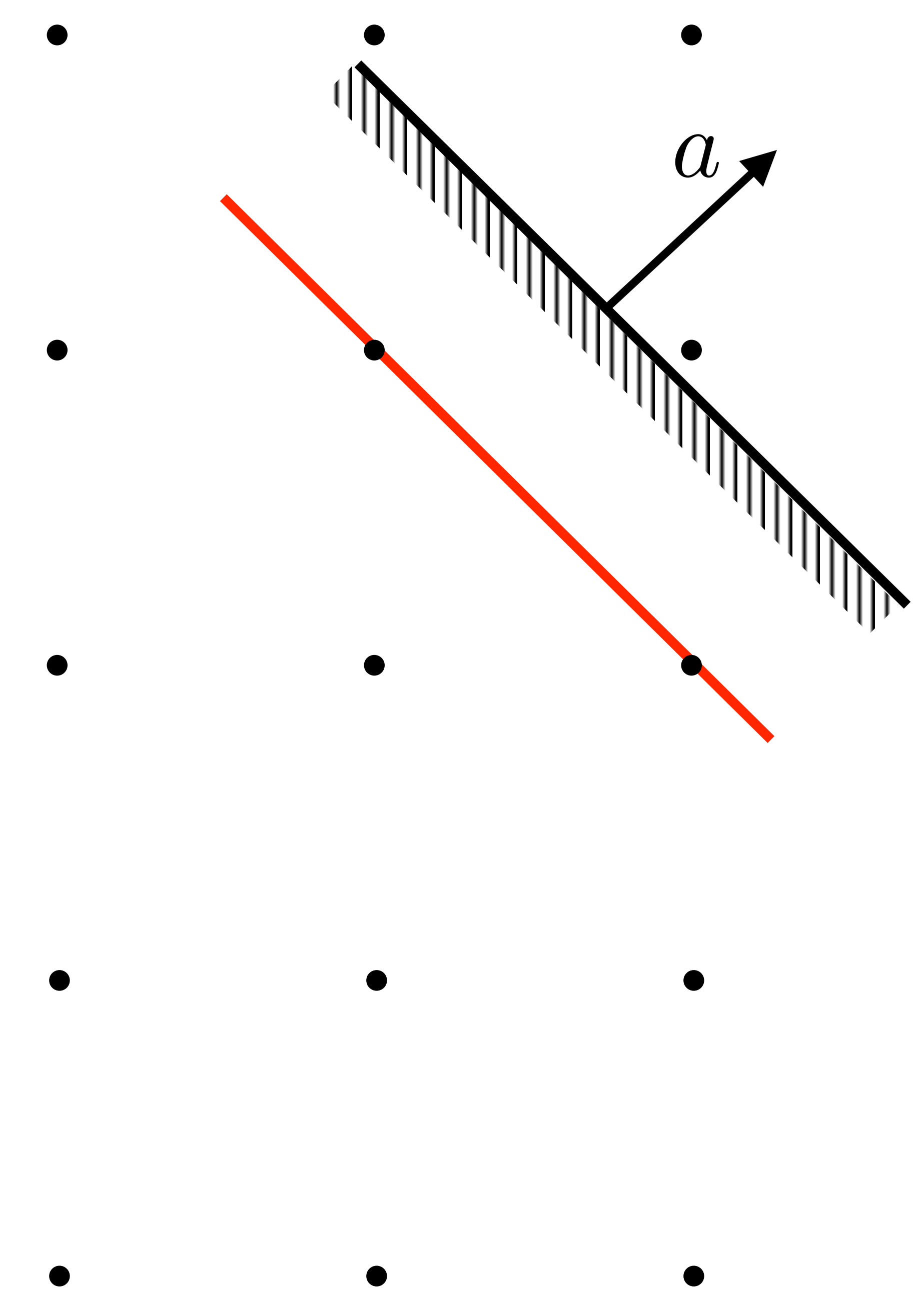
$$\sigma_C(a) := \sup\{\langle a, x\rangle \ : \ x \in C\}$$

$$C \subseteq H := \{x \in \mathbb{R}^n \ : \ \underbrace{\langle a, x\rangle}_{\in \mathbb{Z}} \leq \sigma_C(a)\}$$

if $a,\ x \in \mathbb{Z}^n$

$$\langle a, x\rangle \leq \lfloor \sigma_C(a) \rfloor$$

Valid for $H \cap \mathbb{Z}^n$



CG Cuts and Support Function

$$\sigma_C(a) := \sup\{\langle a, x\rangle \,:\, x \in C\}$$

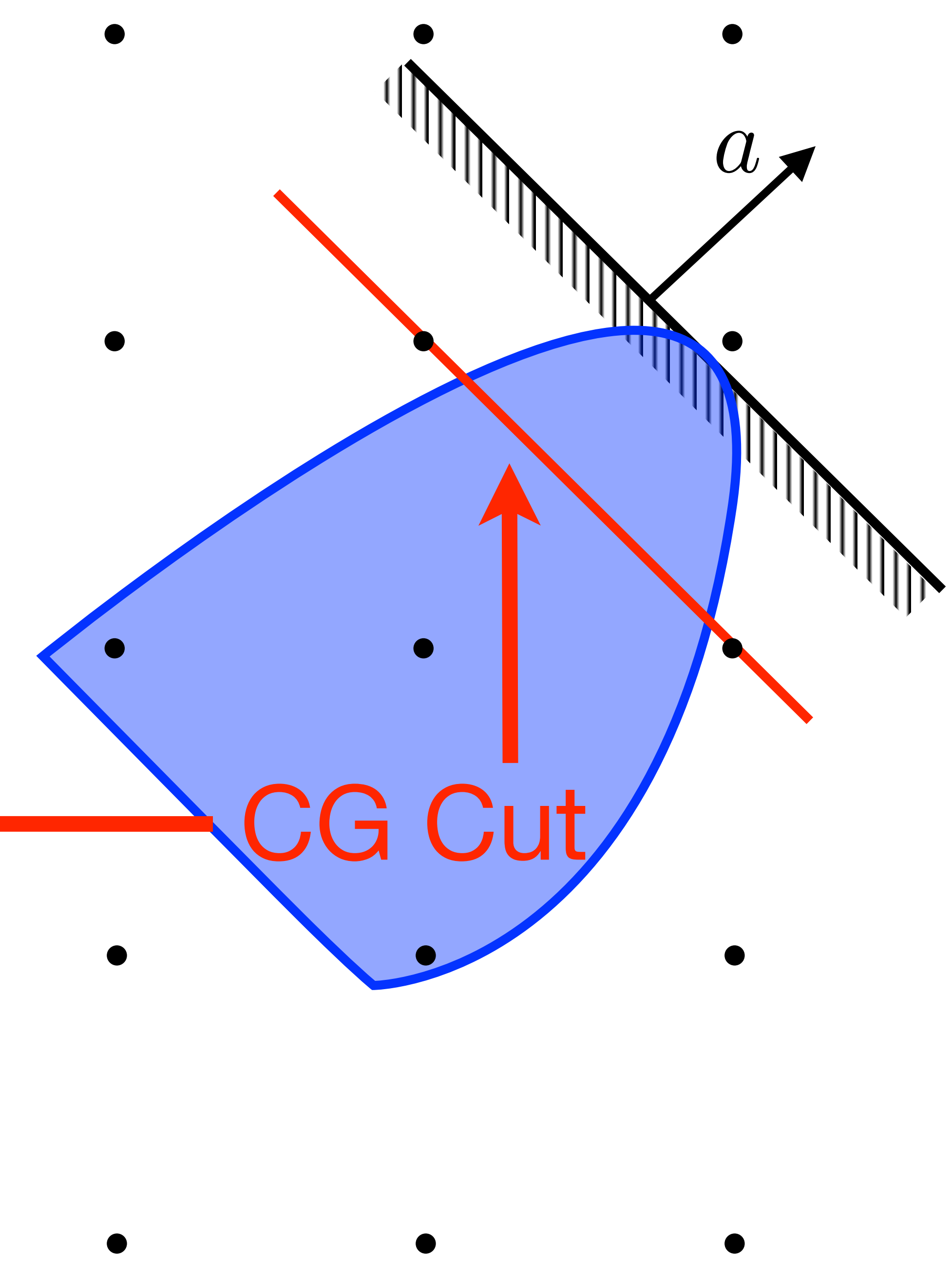
$$C \subseteq H := \{x \in \mathbb{R}^n \,:\, \underbrace{\langle a, x\rangle}_{\in \mathbb{Z}} \leq \sigma_C(a)\}$$

if $a,\, x \in \mathbb{Z}^n$

$$\langle a, x\rangle \leq \lfloor \sigma_C(a) \rfloor$$

Valid for $H \cap \mathbb{Z}^n$

Valid for $C \cap \mathbb{Z}^n$



CG Cuts and Support Function

$$\sigma_C(a) := \sup\{\langle a, x\rangle : x \in C\}$$

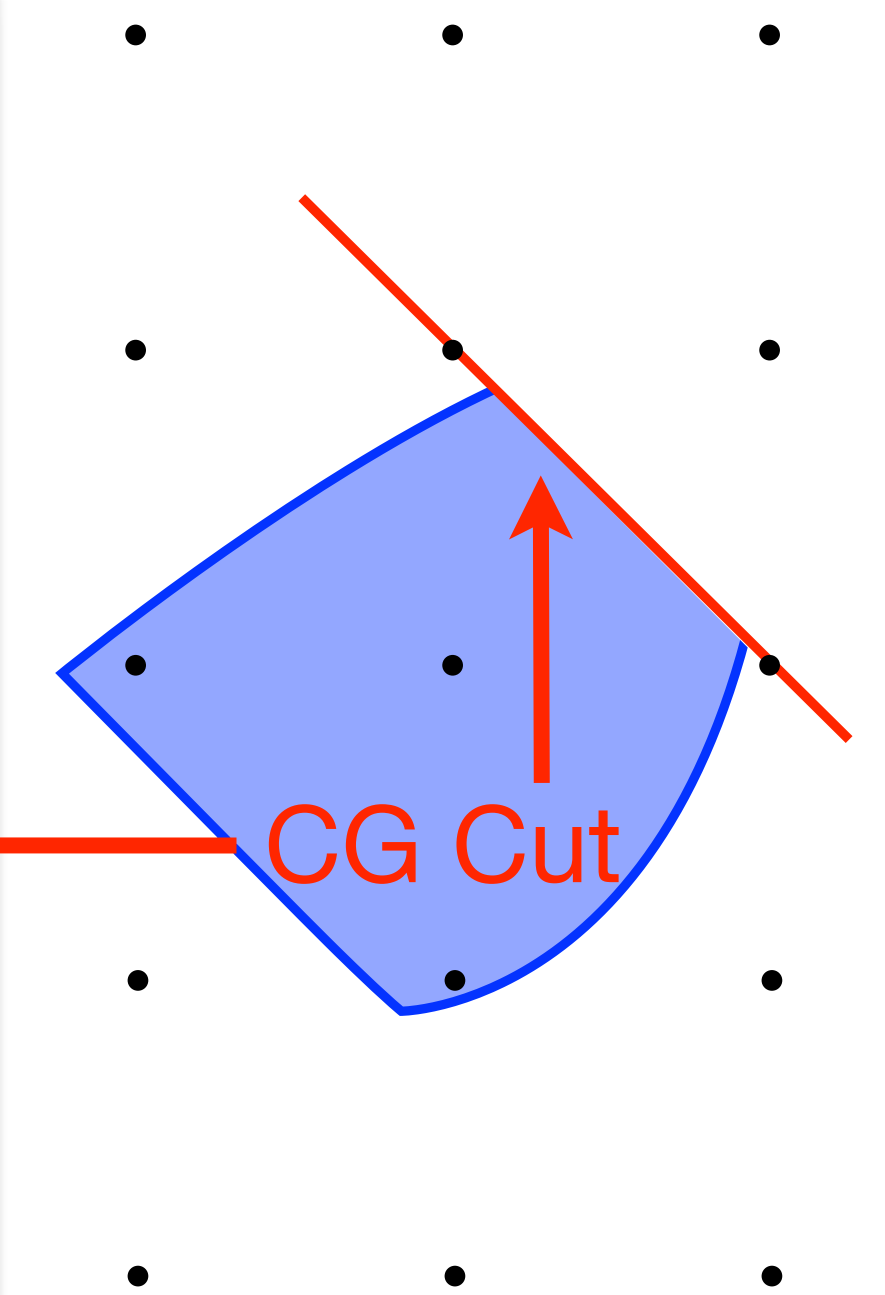
$$C \subseteq H := \{x \in \mathbb{R}^n : \underbrace{\langle a, x\rangle}_{\in \mathbb{Z}} \leq \sigma_C(a)\}$$

if $a,\ x \in \mathbb{Z}^n$

$$\langle a, x\rangle \leq \lfloor \sigma_C(a) \rfloor$$

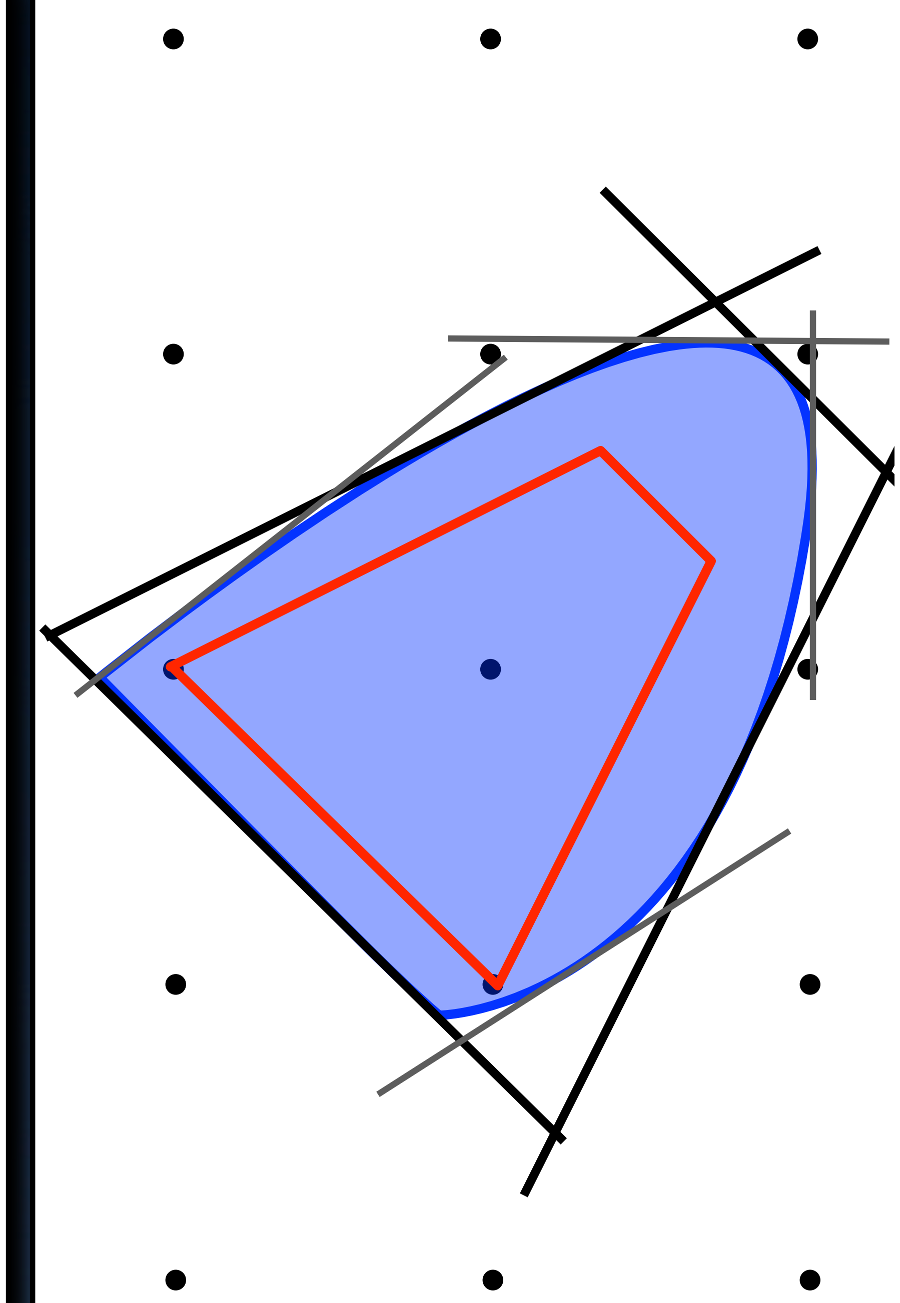
Valid for $H \cap \mathbb{Z}^n$

Valid for $C \cap \mathbb{Z}^n$



CG Closure = Add all CG Cuts

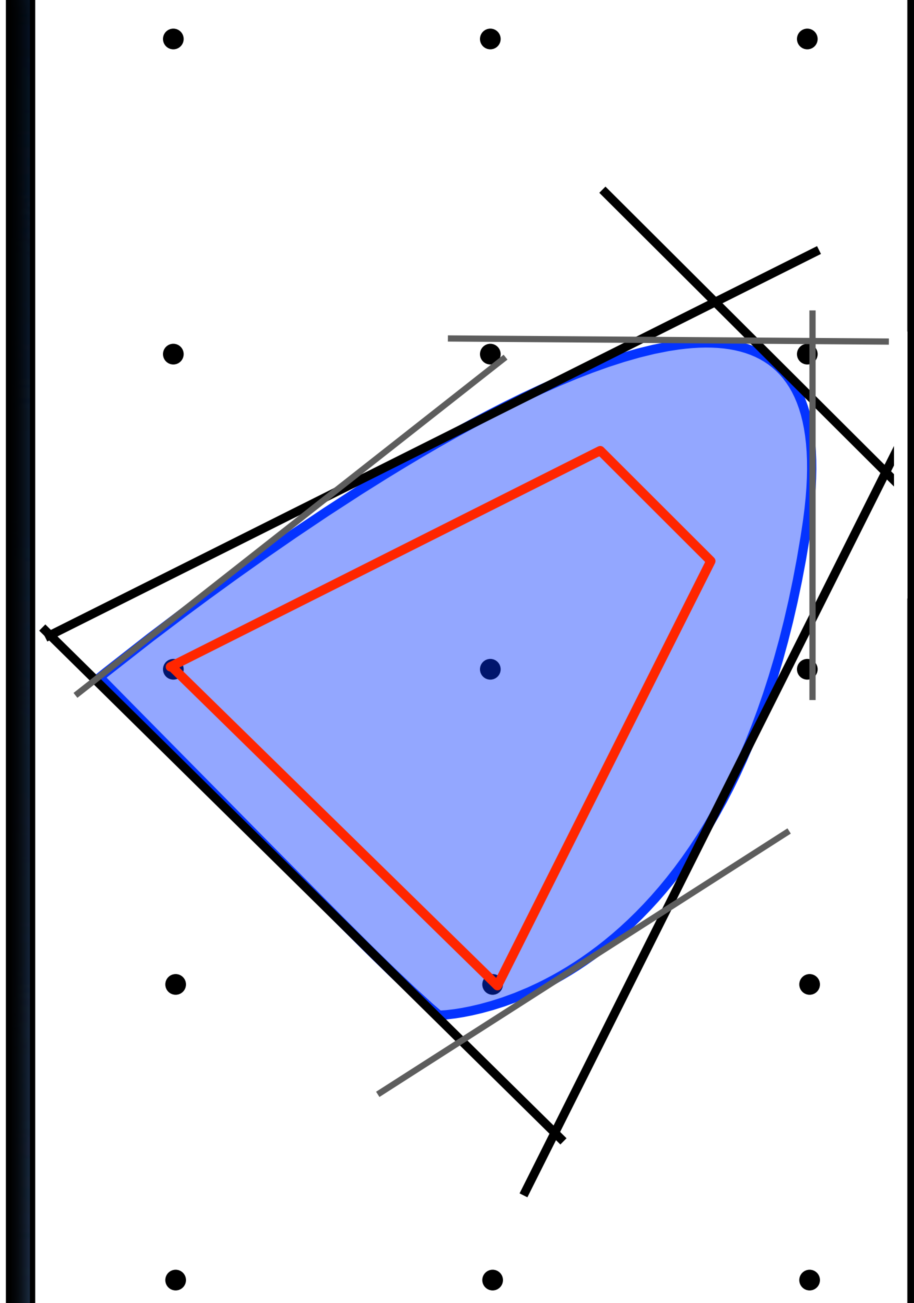
$$\text{CC}(C) := \bigcap_{a \in \mathbb{Z}^n} \{x \in \mathbb{R}^n \,:\, \langle a, x \rangle \leq \lfloor \sigma_C(a) \rfloor\}$$



CG Closure = Add all CG Cuts

$$\mathtt{CC}(C) := \bigcap_{a \in \mathbb{Z}^n} \{x \in \mathbb{R}^n : \langle a, x \rangle \leq \lfloor \sigma_C(a) \rfloor\}$$

Polyhedral?

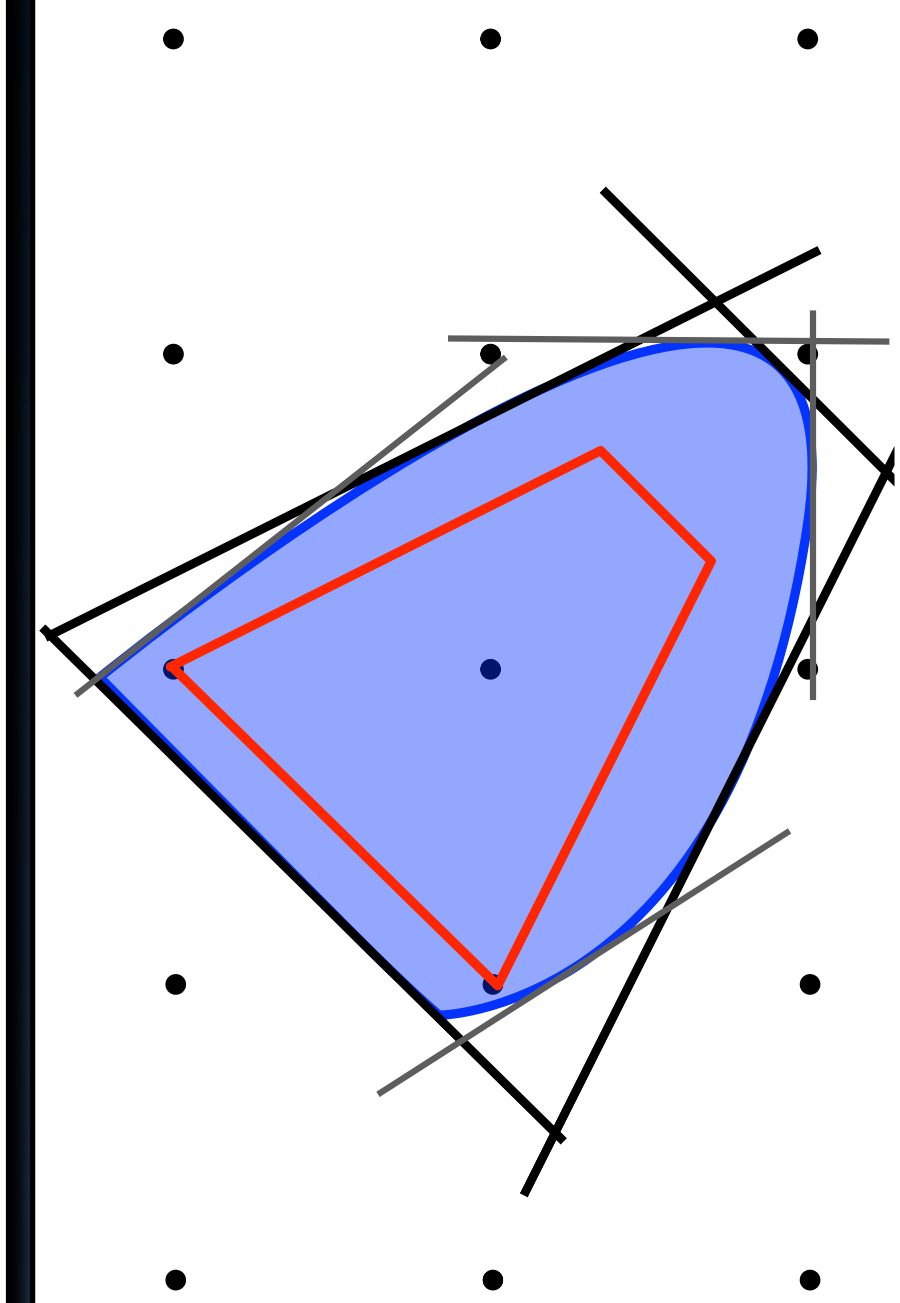


CG Closure = Add all CG Cuts

$$\text{CC}(C) := \bigcap_{a \in \mathbb{Z}^n} \{x \in \mathbb{R}^n : \langle a, x \rangle \leq \lfloor \sigma_C(a) \rfloor\}$$

Polyhedral?

$$C = \bigcap_{a \in \mathbb{Z}^n} \{x \in \mathbb{R}^n : \langle a, x \rangle \leq \sigma_C(a)\}$$

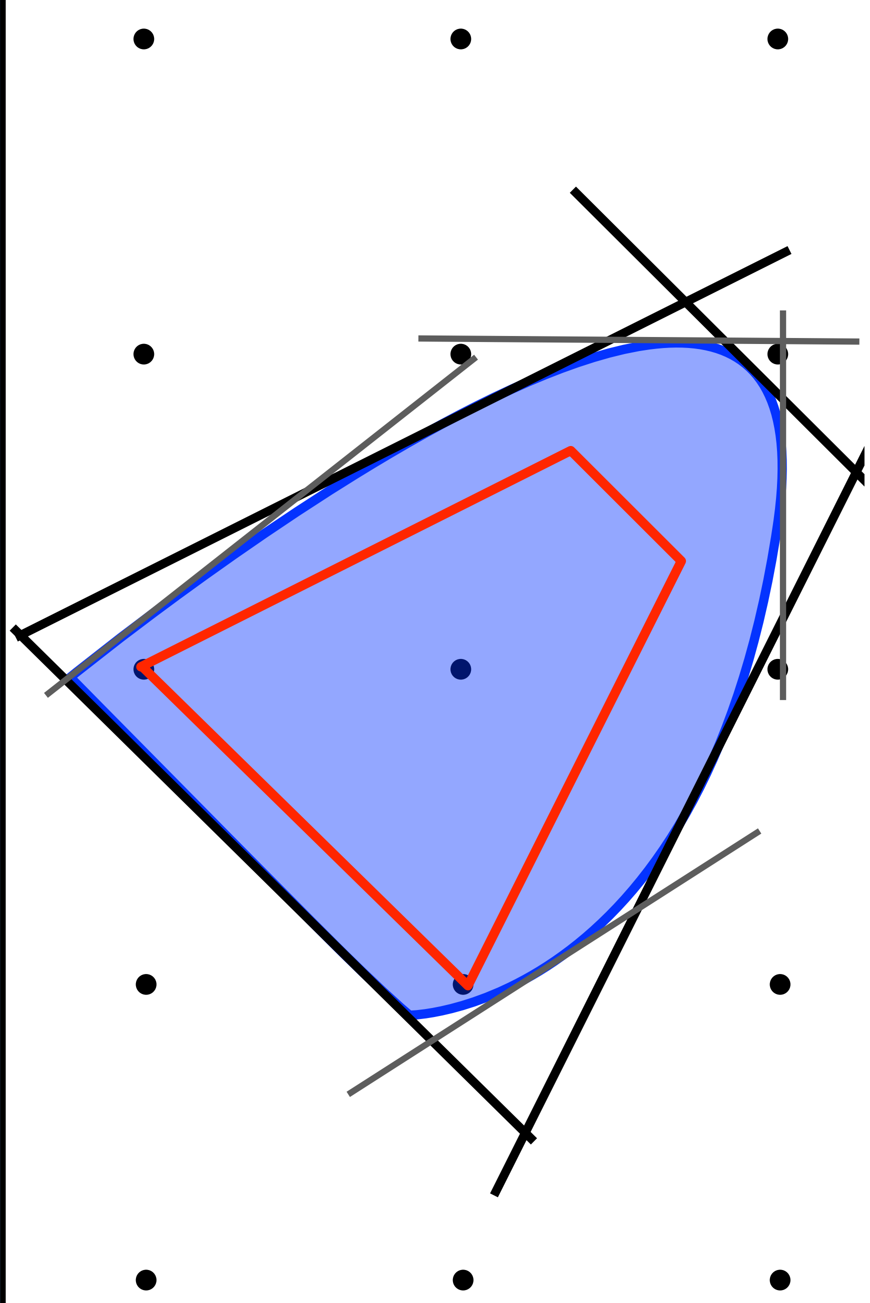


CG Closure = Add all CG Cuts

$$\mathrm{CC}(C) := \bigcap_{a \in \mathbb{Z}^n} \{x \in \mathbb{R}^n : \langle a, x \rangle \leq \lfloor \sigma_C(a) \rfloor\}$$

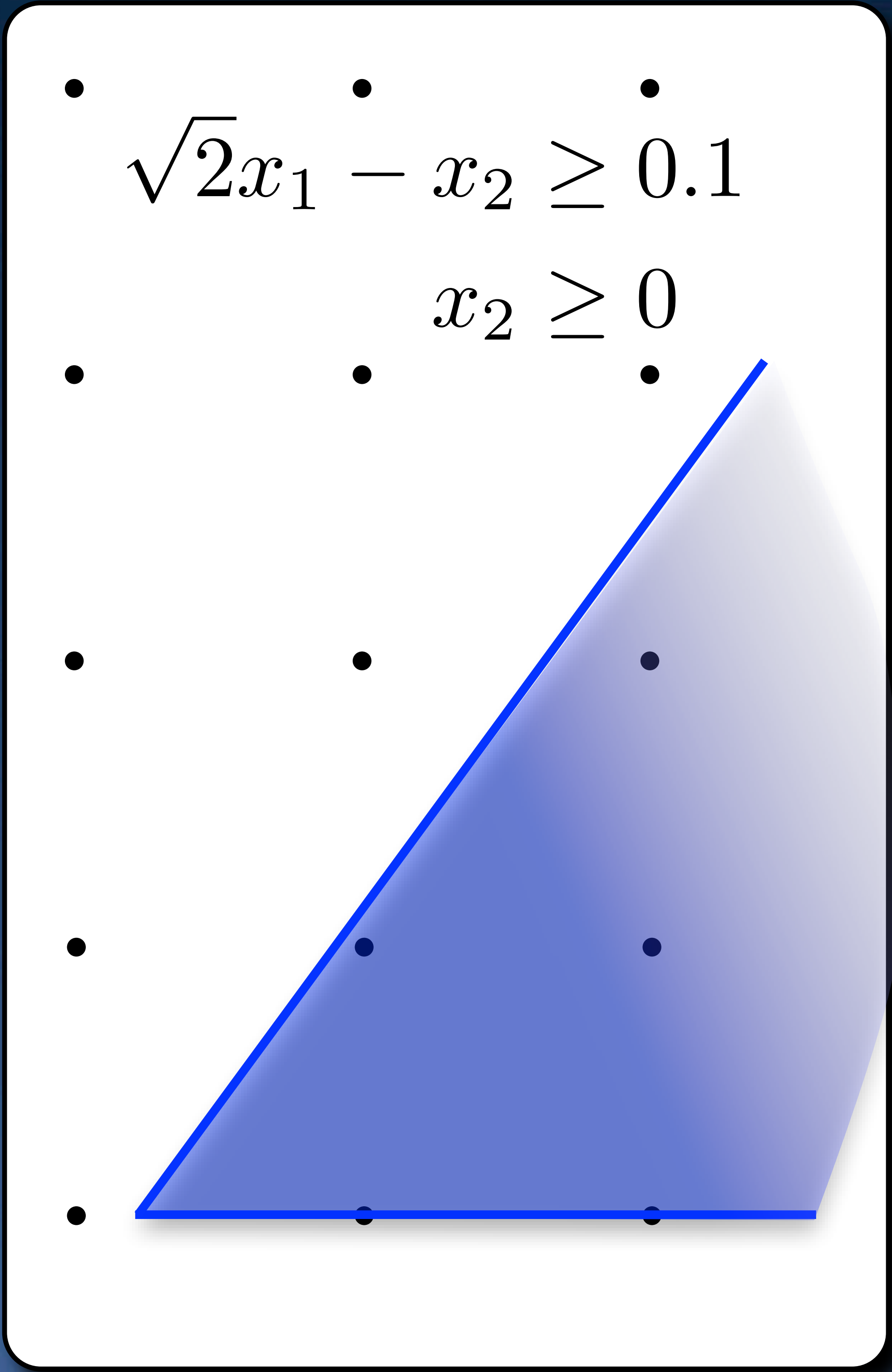
Polyhedral?

$$C = \bigcap_{a \in \mathbb{Z}^n} \{x \in \mathbb{R}^n : \langle a, x \rangle \leq \sigma_C(a)\}$$



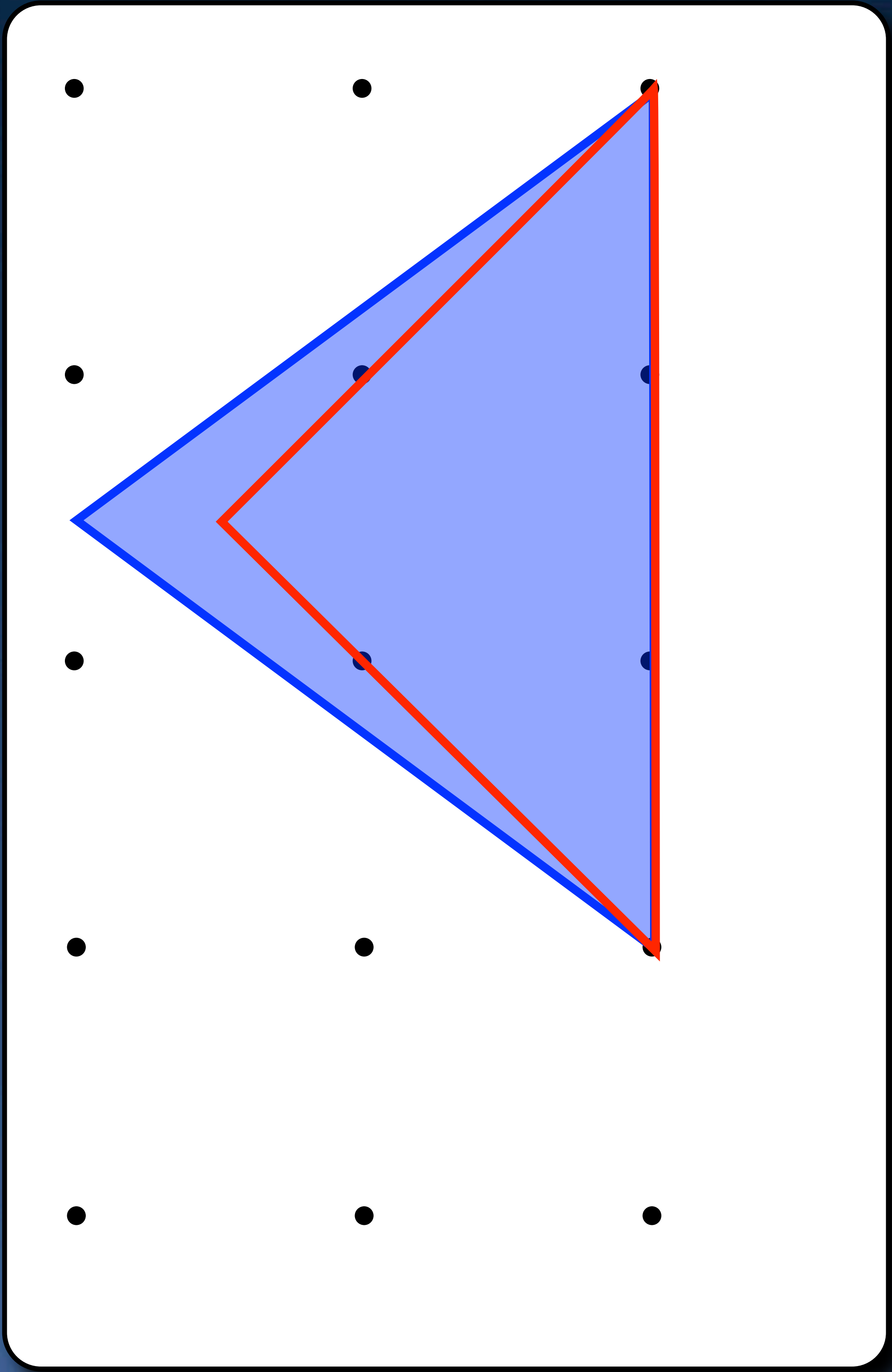
Polyhedrality of CG Closure

- Not always a polyhedron



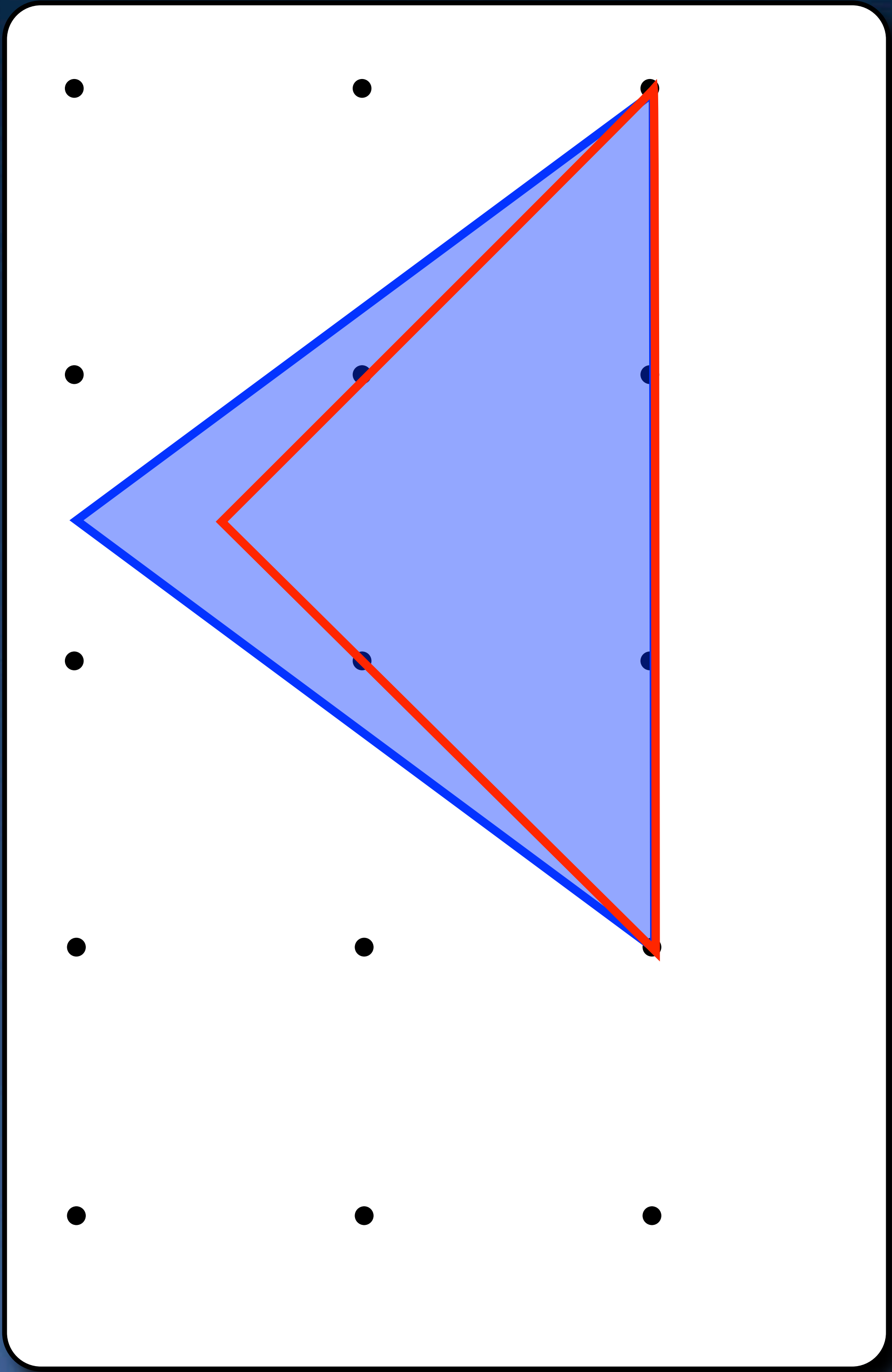
Polyhedrality of CG Closure

- Not always a polyhedron
- Shrijver 1980:
  - Theorem: If C is a rational polyhedron then $\mathrm{CC}(C)$ is too. (Constructive Proof)



Polyhedrality of CG Closure

- Not always a polyhedron
- Shrijver 1980:
  - Theorem: If C is a rational polyhedron then $\mathrm{CC}(C)$ is too. (Constructive Proof)
  - Question: What about for non-rational polytopes?



CG Closure is Finitely Generated

- Theorem (Dadush, Dey, V. 2011): If C is a compact convex set then there exists a finite $S \subseteq \mathbb{Z}^n$ such that:

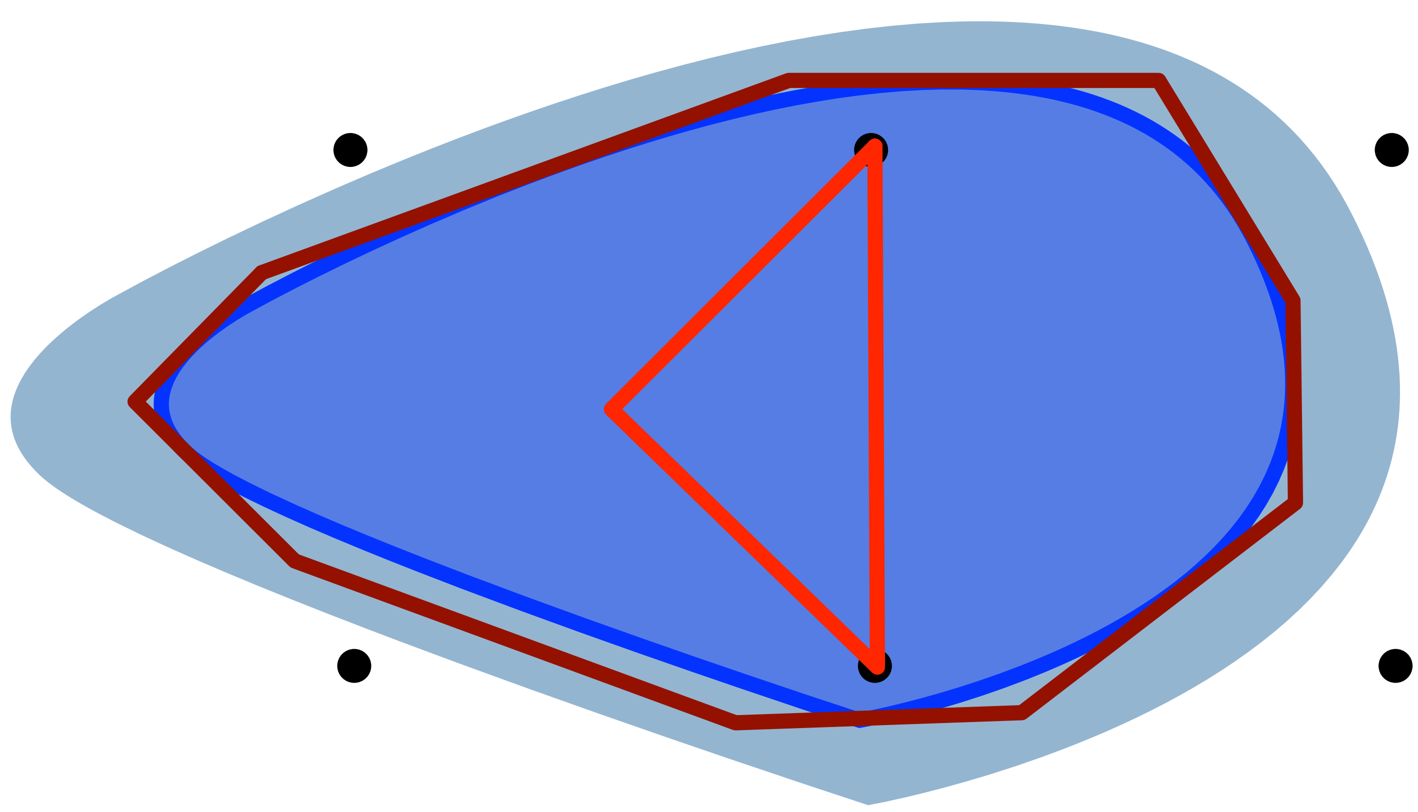
$$\mathrm{CC}(C) = \underbrace{\bigcap_{a \in S} \{x \in \mathbb{R}^n \ : \ \langle a, x \rangle \leq \lfloor \sigma_C(a) \rfloor\}}_{\mathrm{CC}(C,S)}$$

- Corollary: $\mathrm{CC}(C)$ is a rational polytope.
- In particular answers Shrijver's question.
  - Also answered by Dunkel and Schulz 2011.

Corollary: Stability of CG Closure

$\exists \varepsilon > 0$ such that:

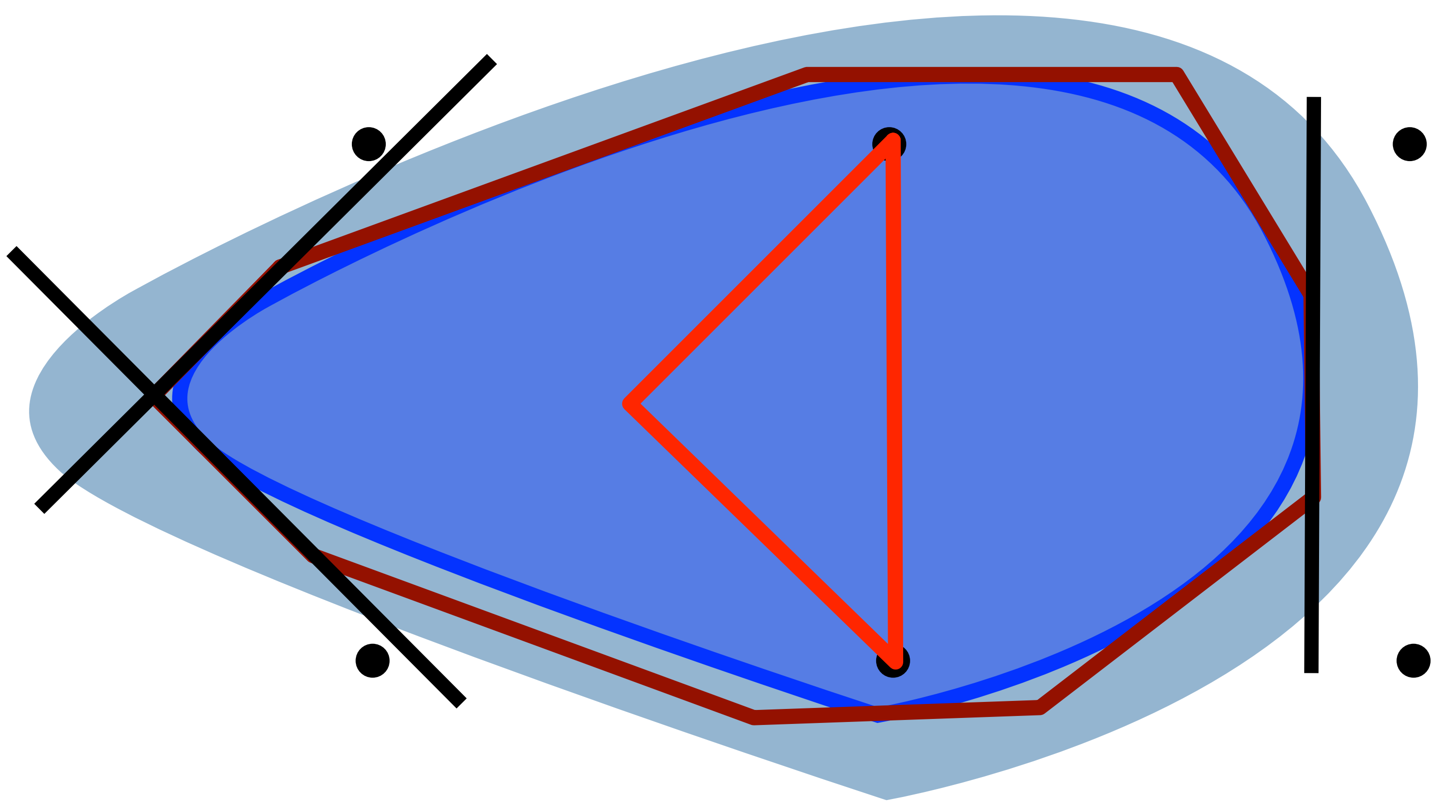
$$C \subseteq P \subseteq C + \varepsilon B_2 \Rightarrow \mathtt{CC}(C) = \mathtt{CC}(P)$$



Corollary: Stability of CG Closure

$\exists \varepsilon > 0$ such that:

$$C \subseteq P \subseteq C + \varepsilon B_2 \Rightarrow \mathtt{CC}(C) = \mathtt{CC}(P)$$



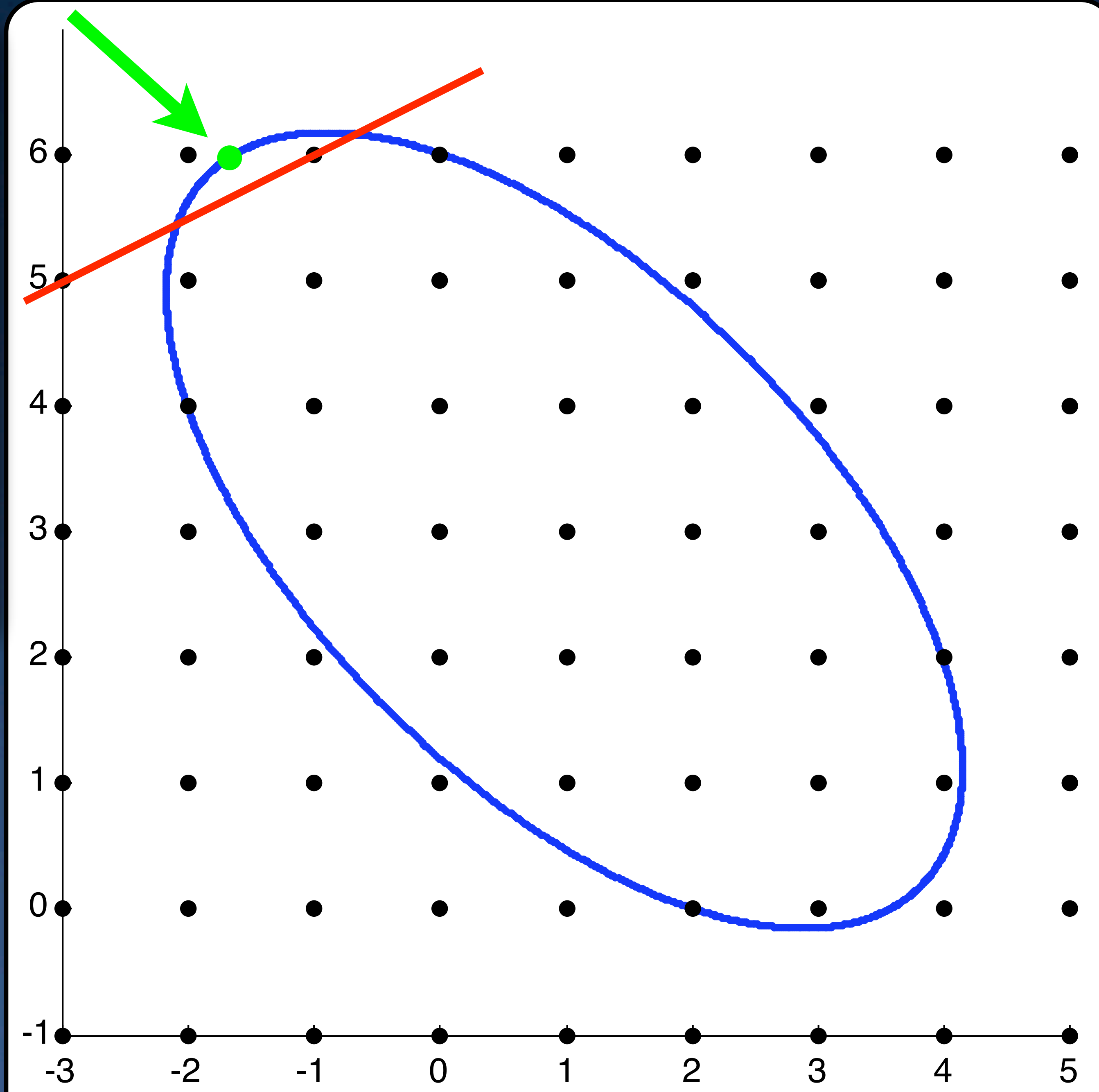
Proof Sketch of Theorem

- For strictly convex sets without integral points in boundary
- Proof Outline:
  - Step 1: Create finite S_1 s.t. cc$(C, S_1) \subseteq \mathrm{int}(C)$.
    - Separate points in boundary
    - Compactness argument
  - Step 2: Show only missed finite number of cuts

Separate points in $\mathrm{bd}(C)$

$u \in \mathrm{bd}(C) \setminus \mathbb{Z}^n$ $\quad \exists\, a^u \in \mathbb{Z}^n$

$$\langle a^u, u \rangle > \lfloor \sigma_C(a^u) \rfloor$$

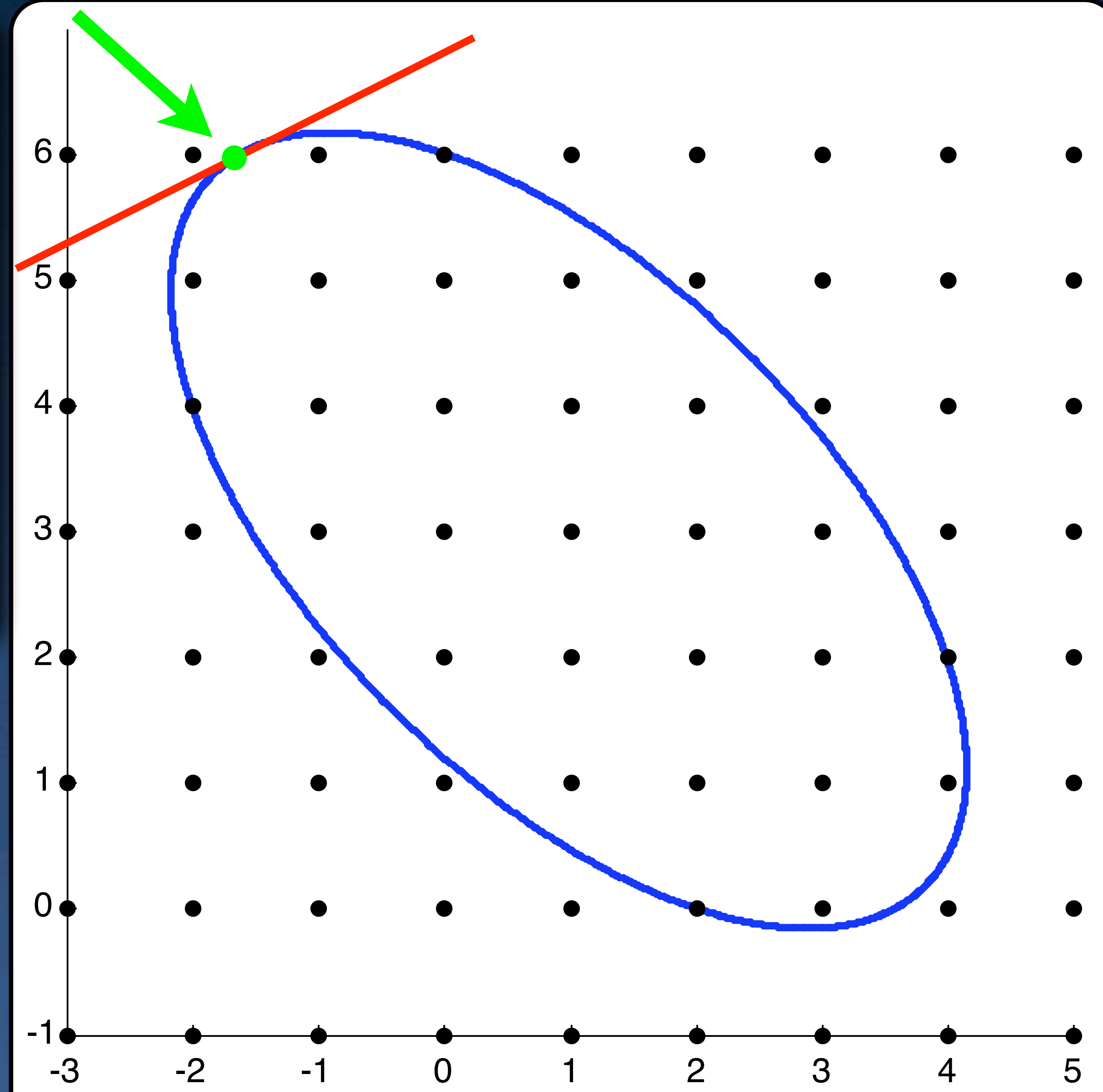


Separate points in $\mathrm{bd}(C)$

$u \in \mathrm{bd}(C) \setminus \mathbb{Z}^n$ $\quad \exists\, a^u \in \mathbb{Z}^n$

$$\langle a^u, u\rangle > \lfloor \sigma_C(a^u) \rfloor$$

$$\langle s(u), u\rangle = \sigma_C(s(u))$$

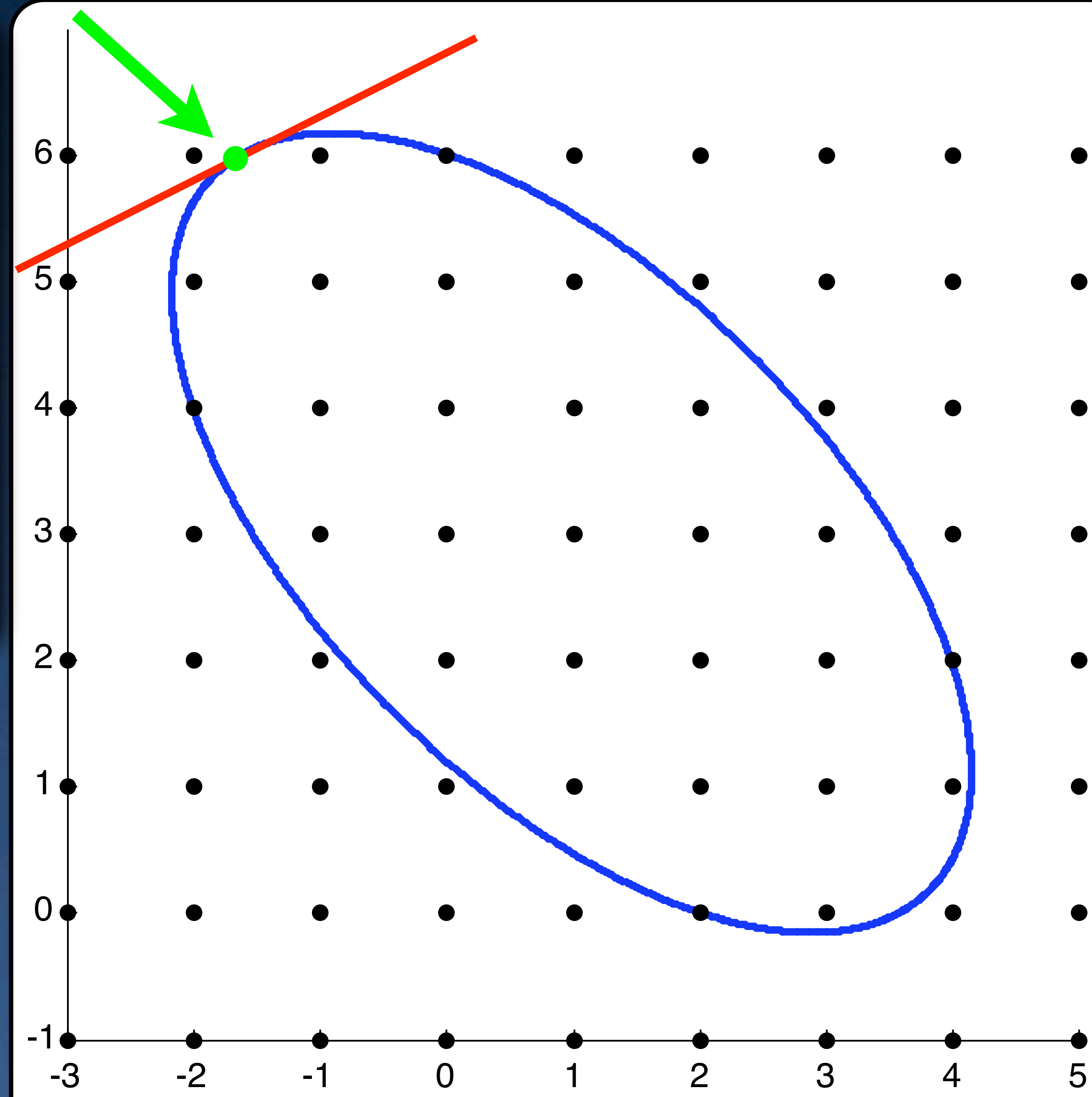


Separate points in $\mathrm{bd}(C)$

$u \in \mathrm{bd}(C) \setminus \mathbb{Z}^n$ $\quad \exists\, a^u \in \mathbb{Z}^n$

$$\langle a^u, u\rangle > \lfloor \sigma_C(a^u) \rfloor$$

$$\underbrace{\langle s(u), u\rangle}_{\in \mathbb{Z}^n} = \underbrace{\sigma_C(s(u))}_{\notin \mathbb{Z}}$$

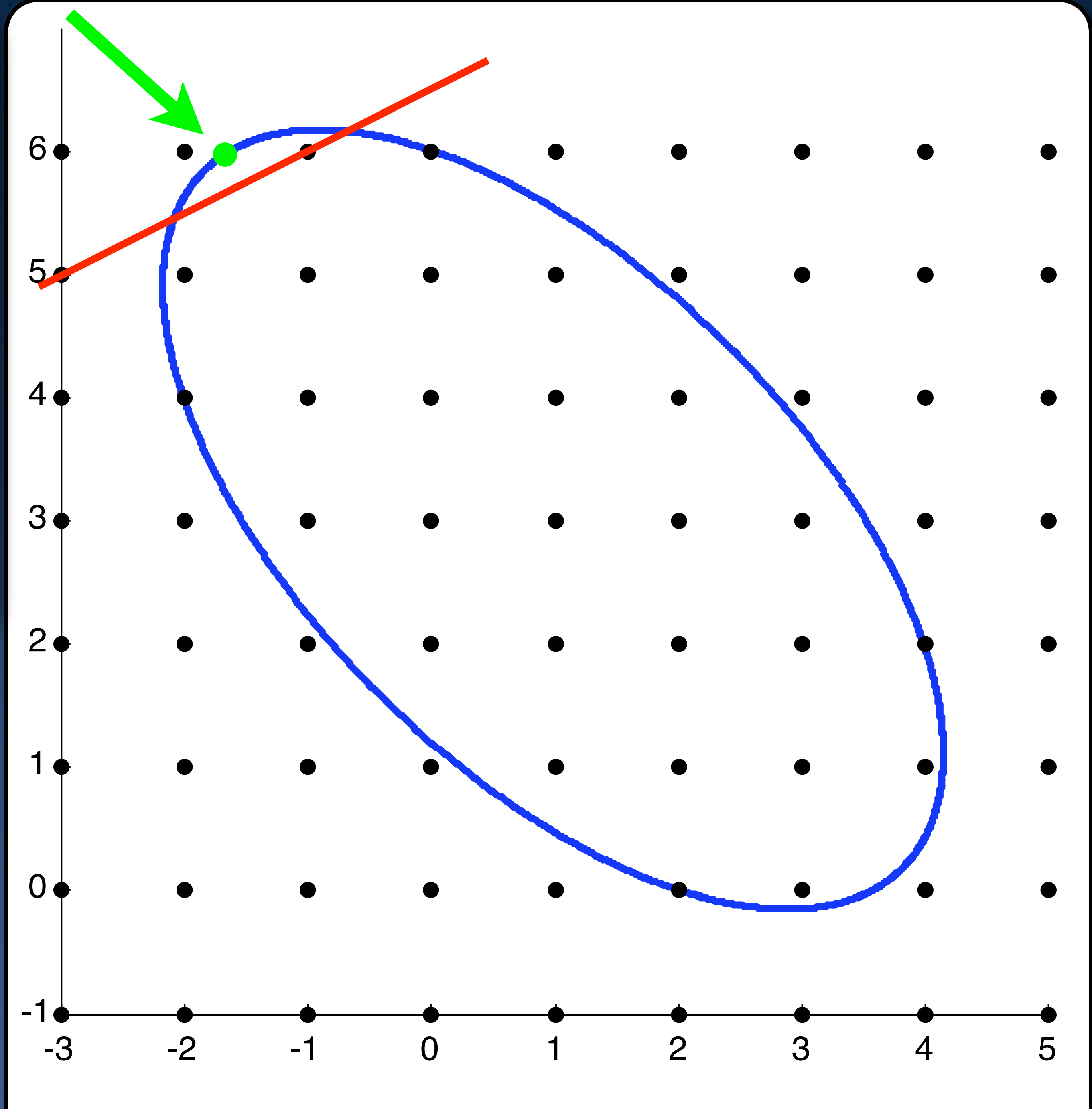


Separate points in $\mathrm{bd}(C)$

$u \in \mathrm{bd}(C) \setminus \mathbb{Z}^n \qquad \exists\, a^u \in \mathbb{Z}^n$

$$\langle a^u, u\rangle > \lfloor \sigma_C(a^u) \rfloor$$

$$\underbrace{\langle s(u), u\rangle}_{\in \mathbb{Z}^n} = \underbrace{\sigma_C(s(u))}_{\notin \mathbb{Z}}$$



Separate points in $\operatorname{bd}(C)$

$u \in \operatorname{bd}(C) \setminus \mathbb{Z}^n \qquad \exists\, a^u \in \mathbb{Z}^n$

$$\langle a^u, u \rangle > \lfloor \sigma_C(a^u) \rfloor$$

$$\underbrace{\langle s(u)}_{\in \mathbb{Z}^n}, u\rangle = \underbrace{\sigma_C(s(u))}_{\notin \mathbb{Z}}$$

$\lambda s(u) \notin \mathbb{Z}^n \quad \forall \lambda > 0 :$

$\lambda s(u) \in \mathbb{Z}^n \Rightarrow \sigma_C(\lambda s(u)) \in \mathbb{Z} :$

Separate points in $\mathrm{bd}(C)$

$u \in \mathrm{bd}(C) \setminus \mathbb{Z}^n \qquad \exists\, a^u \in \mathbb{Z}^n$

$$\langle a^u, u \rangle > \lfloor \sigma_C(a^u) \rfloor$$

$$\underbrace{\langle s(u), u \rangle}_{\in \mathbb{Z}^n} = \underbrace{\sigma_C(s(u))}_{\notin \mathbb{Z}}$$

$\lambda s(u) \notin \mathbb{Z}^n \quad \forall \lambda > 0:$

$$C = \left\{ x \in \mathbb{R}^2 \;:\; \sqrt{x_1^2 + x_2^2} \leq 1 \right\}$$

$$u = (1/2, \sqrt{3}/2)^T \in \mathrm{bd}(C)$$

$$s(u) = (1/2, \sqrt{3}/2)^T$$

$\lambda s(u) \in \mathbb{Z}^n \Rightarrow \sigma_C(\lambda s(u)) \in \mathbb{Z}:$

Separate points in $\mathrm{bd}(C)$

$u \in \mathrm{bd}(C) \setminus \mathbb{Z}^n \qquad \exists\, a^u \in \mathbb{Z}^n$

$$\langle a^u, u \rangle > \lfloor \sigma_C(a^u) \rfloor$$

$$\underbrace{\langle s(u), u \rangle}_{\in \mathbb{Z}^n} = \underbrace{\sigma_C(s(u))}_{\notin \mathbb{Z}}$$

$\lambda s(u) \notin \mathbb{Z}^n \quad \forall \lambda > 0:$

$$C = \left\{ x \in \mathbb{R}^2 \; : \; \sqrt{x_1^2 + x_2^2} \leq 1 \right\}$$

$$u = (1/2, \sqrt{3}/2)^T \in \mathrm{bd}(C)$$

$$s(u) = (1/2, \sqrt{3}/2)^T$$

$\lambda s(u) \in \mathbb{Z}^n \Rightarrow \sigma_C(\lambda s(u)) \in \mathbb{Z}:$

$$C = \left\{ x \in \mathbb{R}^2 \; : \; \sqrt{x_1^2 + x_2^2} \leq 5 \right\}$$

$$u = (25/13, 60/13)^T \in \mathrm{bd}(C)$$

$$s(u) = (5, 12)^T, \; \sigma_C(s(u)) = 65$$

Separate points in $\mathrm{bd}(C)$

$u \in \mathrm{bd}(C) \setminus \mathbb{Z}^n \qquad \exists\, a^u \in \mathbb{Z}^n$

$$\langle a^u, u \rangle > \lfloor \sigma_C(a^u) \rfloor$$

$$\underbrace{\langle s(u), u \rangle}_{\in \mathbb{Z}^n} = \underbrace{\sigma_C(s(u))}_{\notin \mathbb{Z}}$$

$$\frac{s^i}{\|s^i\|} \xrightarrow{i \to \infty} \frac{s(u)}{\|s(u)\|}$$

$$\lim_{i \to \infty} \langle s^i, u \rangle - \lfloor \sigma_C(s^i) \rfloor > 0$$

Diophantine approx. of $s(u)$

$\lambda s(u) \notin \mathbb{Z}^n \quad \forall \lambda > 0:$

$$C = \left\{ x \in \mathbb{R}^2 \;:\; \sqrt{x_1^2 + x_2^2} \leq 1 \right\}$$

$$u = (1/2, \sqrt{3}/2)^T \in \mathrm{bd}(C)$$

$$s(u) = (1/2, \sqrt{3}/2)^T$$

$\lambda s(u) \in \mathbb{Z}^n \Rightarrow \sigma_C(\lambda s(u)) \in \mathbb{Z}:$

$$C = \left\{ x \in \mathbb{R}^2 \;:\; \sqrt{x_1^2 + x_2^2} \leq 5 \right\}$$

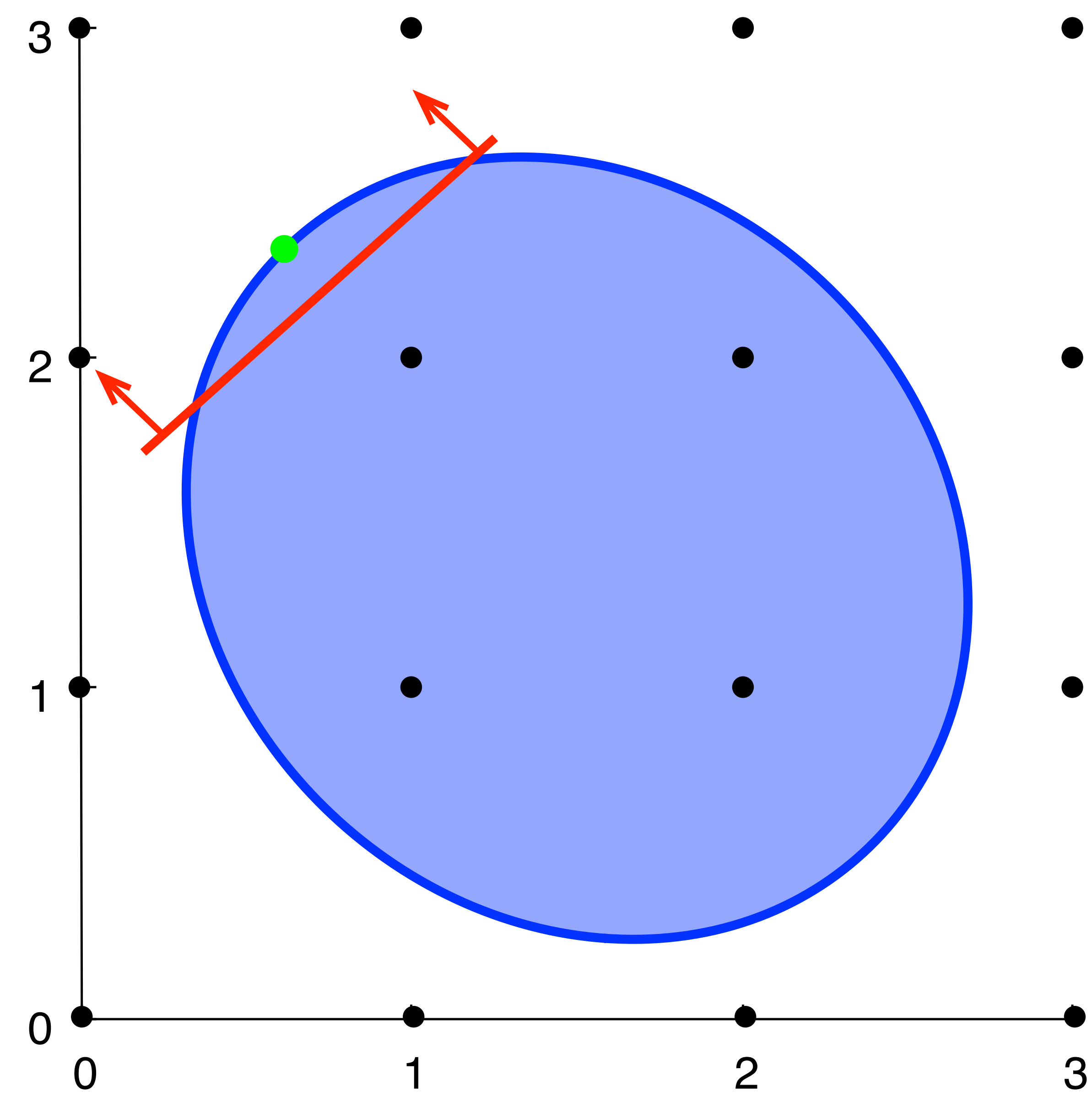
$$u = (25/13, 60/13)^T \in \mathrm{bd}(C)$$

$$s(u) = (5, 12)^T,\ \sigma_C(s(u)) = 65$$

Compactness Argument

$K := \mathrm{bd}(C)$

$S_u := \{x : \langle a^u, x\rangle > \lfloor \sigma_C(a^u) \rfloor\}$

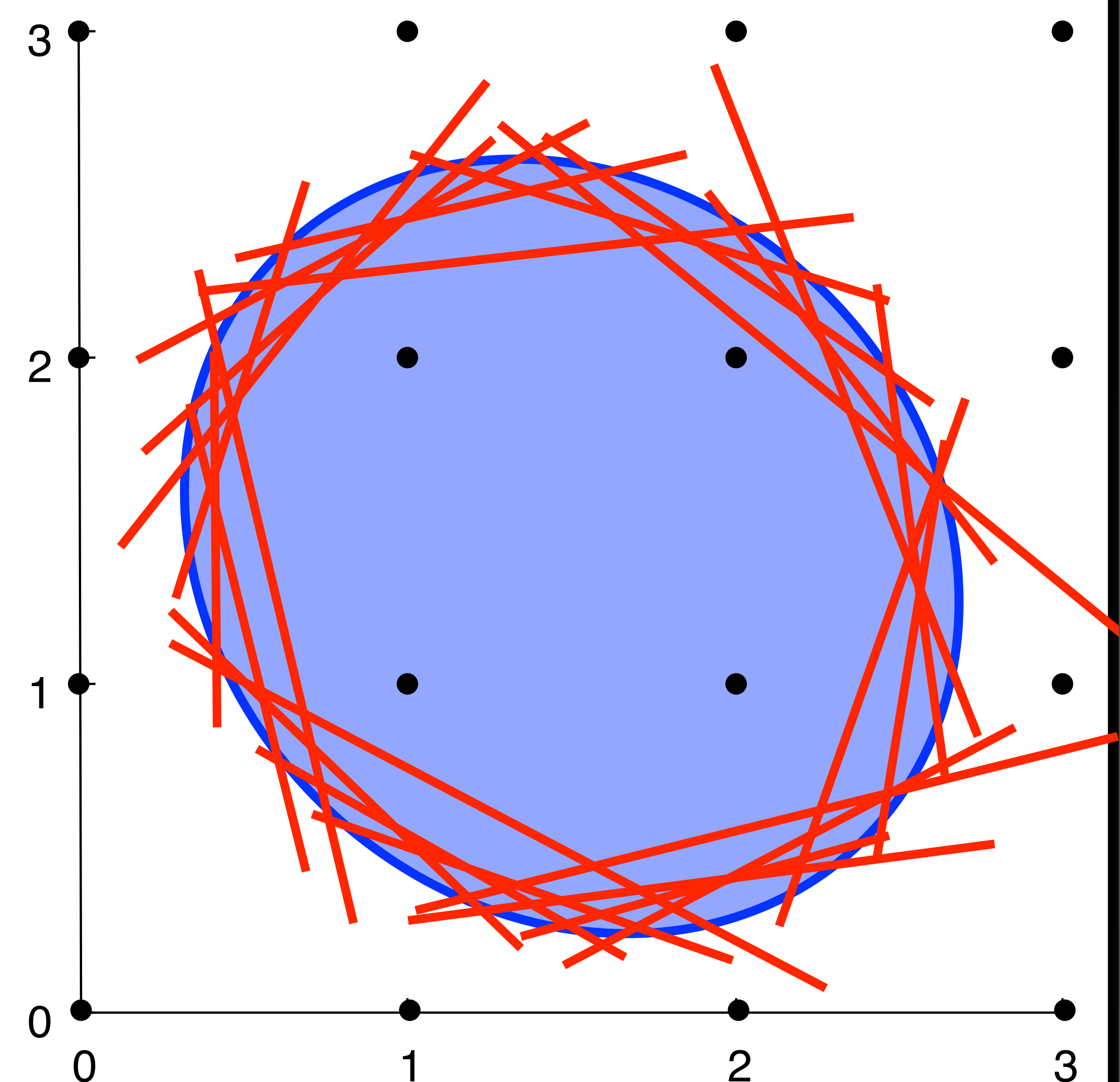


Compactness Argument

$K := \mathrm{bd}(C)$

$S_u := \{x : \langle a^u, x\rangle > \lfloor \sigma_C(a^u) \rfloor\}$

$$K \subset \bigcup_{u \in K} S_u$$

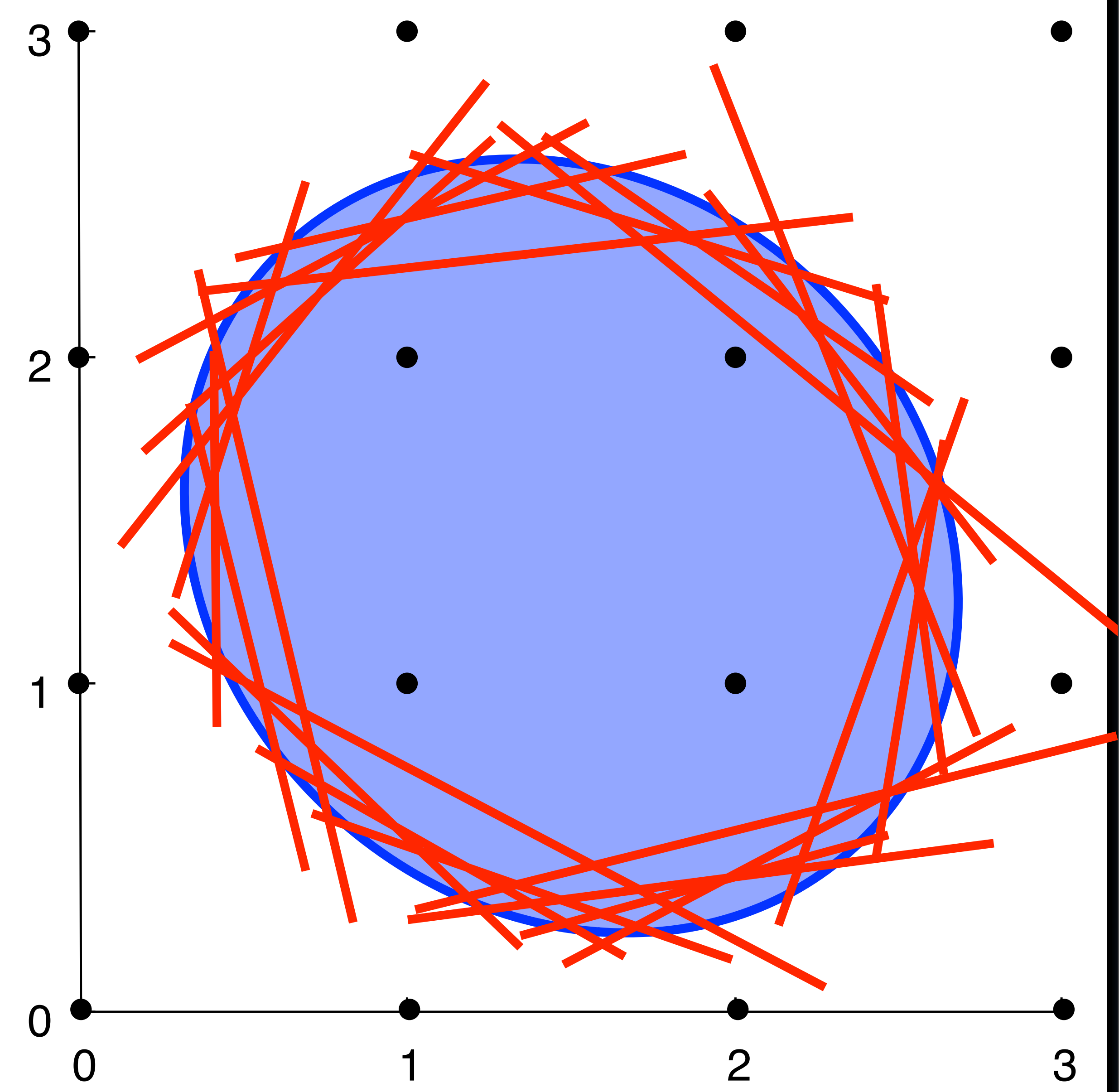


Compactness Argument

$K := \mathrm{bd}(C)$

$S_u := \{x : \langle a^u, x\rangle > \lfloor \sigma_C(a^u) \rfloor\}$

compact → $K \subset \bigcup_{u \in K} S_u$



Compactness Argument

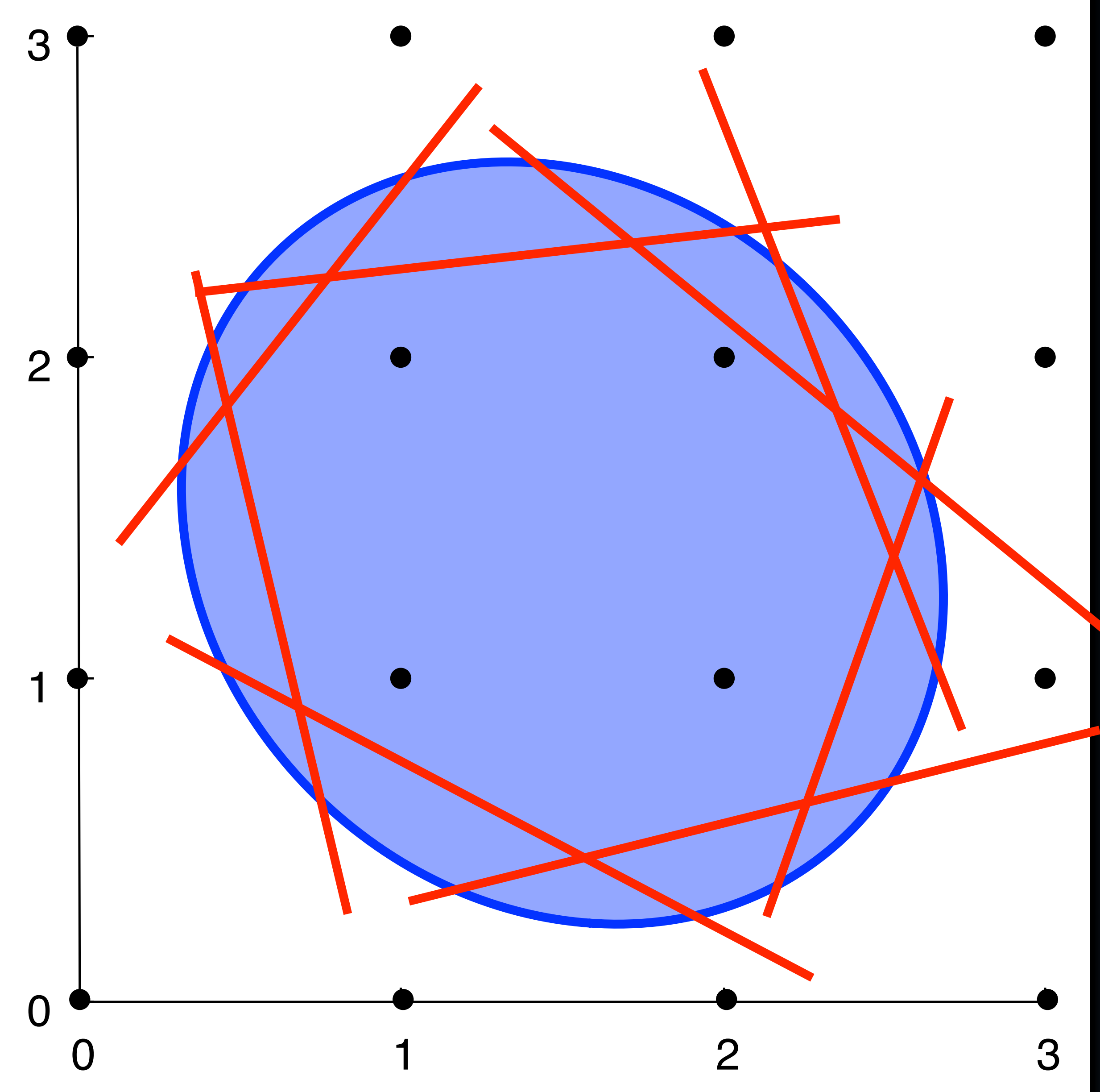
$K := \mathrm{bd}(C)$

$S_u := \{x : \langle a^u, x\rangle > \lfloor \sigma_C(a^u) \rfloor\}$

compact $\longrightarrow$ $K \subset \bigcup_{u \in K} S_u$

$\downarrow$

$K \subset \bigcup_{i=1}^{m} S_{u^i}$



Compactness Argument

$K := \mathrm{bd}(C)$

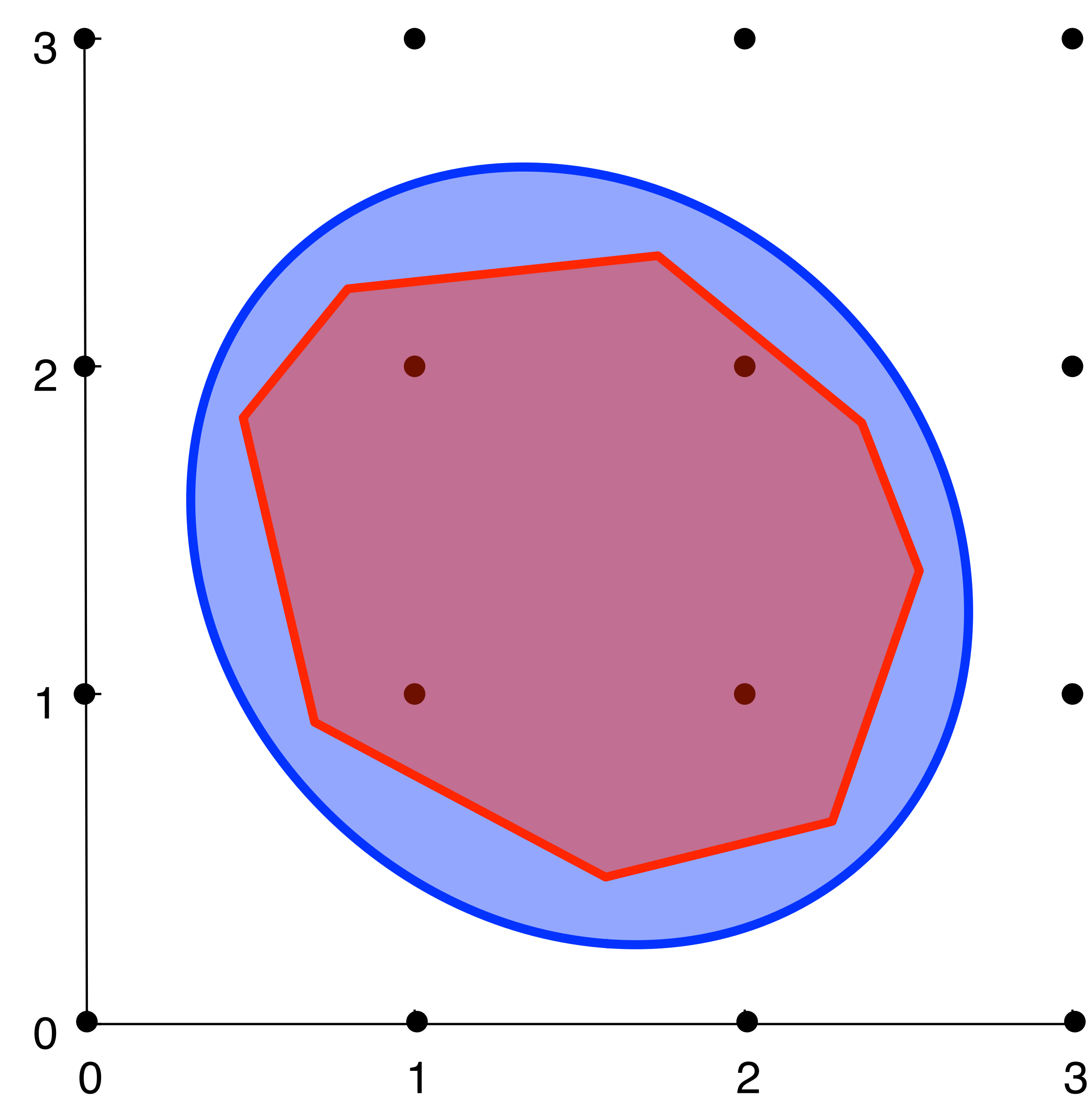
$S_u := \{x : \langle a^u, x\rangle > \lfloor \sigma_C(a^u)\rfloor\}$

compact → $K \subset \bigcup_{u \in K} S_u$

↓

$K \subset \bigcup_{i=1}^{m} S_{u^i}$

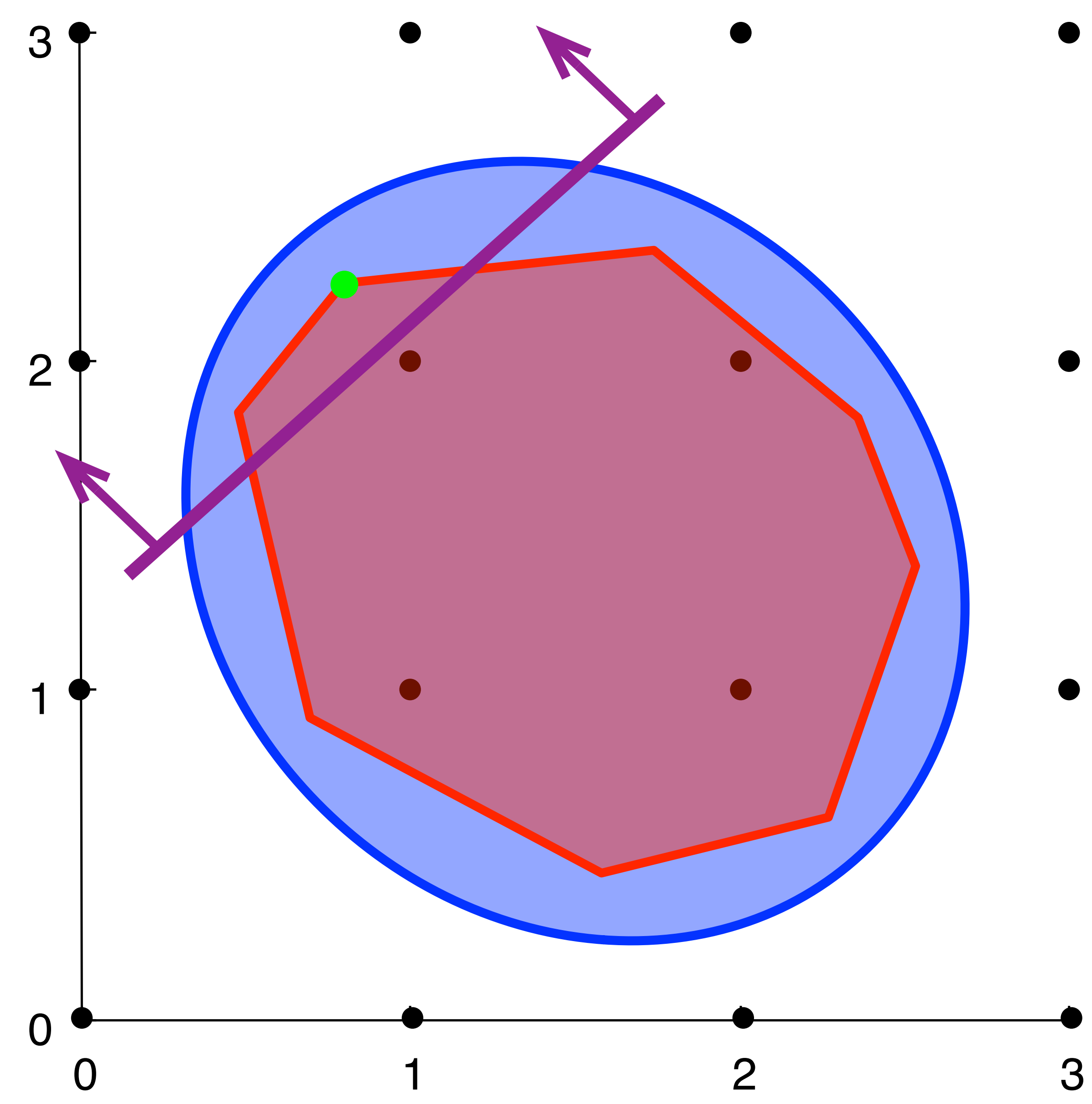
$$S^1 = \bigcup_{i=1}^{m} \left\{a^{u^i}\right\}$$



Step 2 : Separate $\mathrm{cc}(S^1, C) \setminus \mathrm{cc}(C)$

$$V := \mathrm{ext}\left(\mathrm{cc}(S^1, C)\right)$$

$$\langle a, v\rangle > \lfloor \sigma_C(a) \rfloor$$

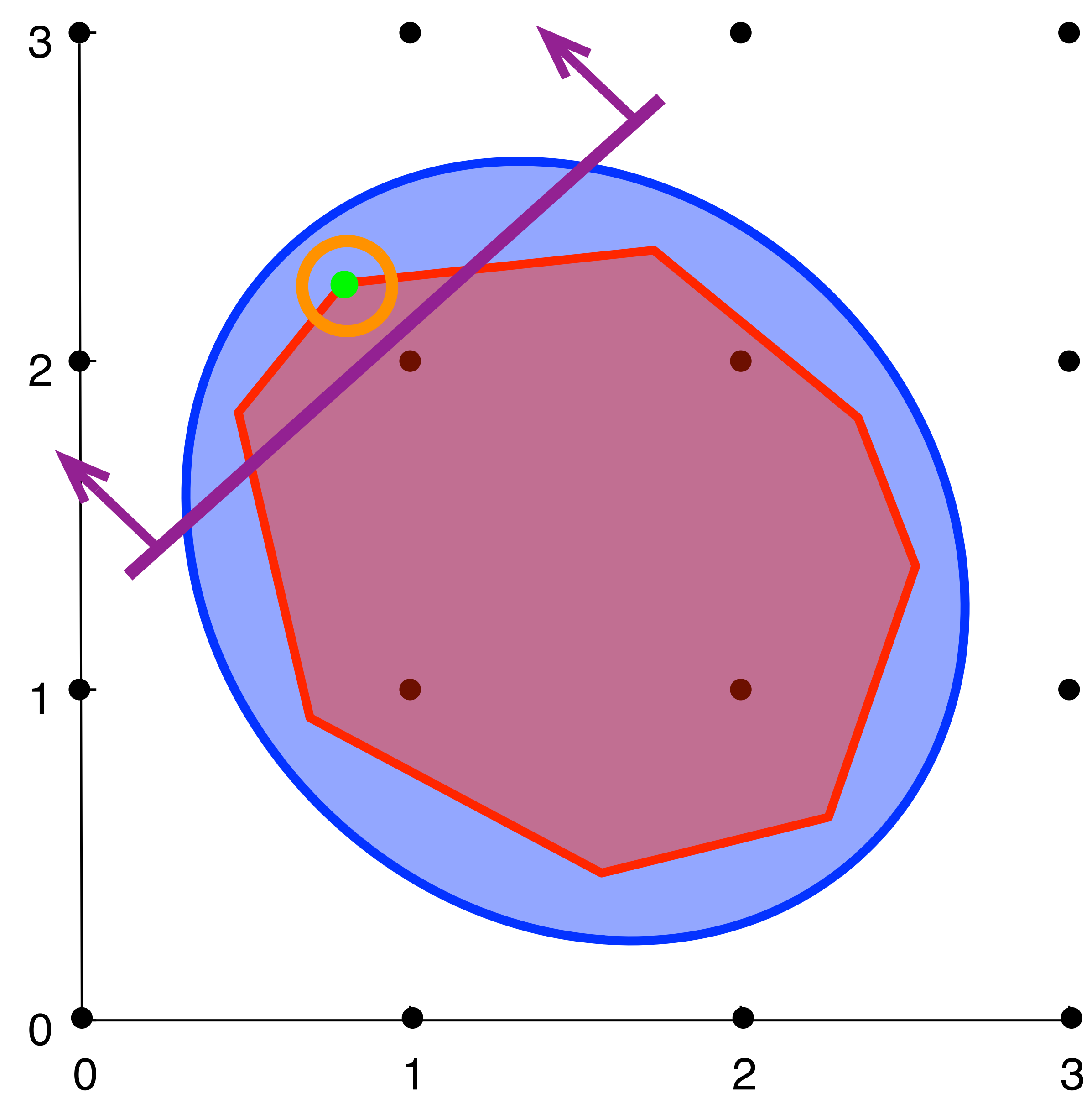


Step 2 : Separate $\mathrm{cc}(S^1, C) \setminus \mathrm{cc}(C)$

$$V := \mathrm{ext}\left(\mathrm{cc}(S^1, C)\right)$$

$$\langle a, v\rangle > \lfloor \sigma_C(a) \rfloor$$

$$\exists \varepsilon > 0 \quad \varepsilon B^n + v \subset C \quad \forall v \in V$$



Step 2 : Separate $\mathrm{CC}(S^1, C) \setminus \mathrm{CC}(C)$

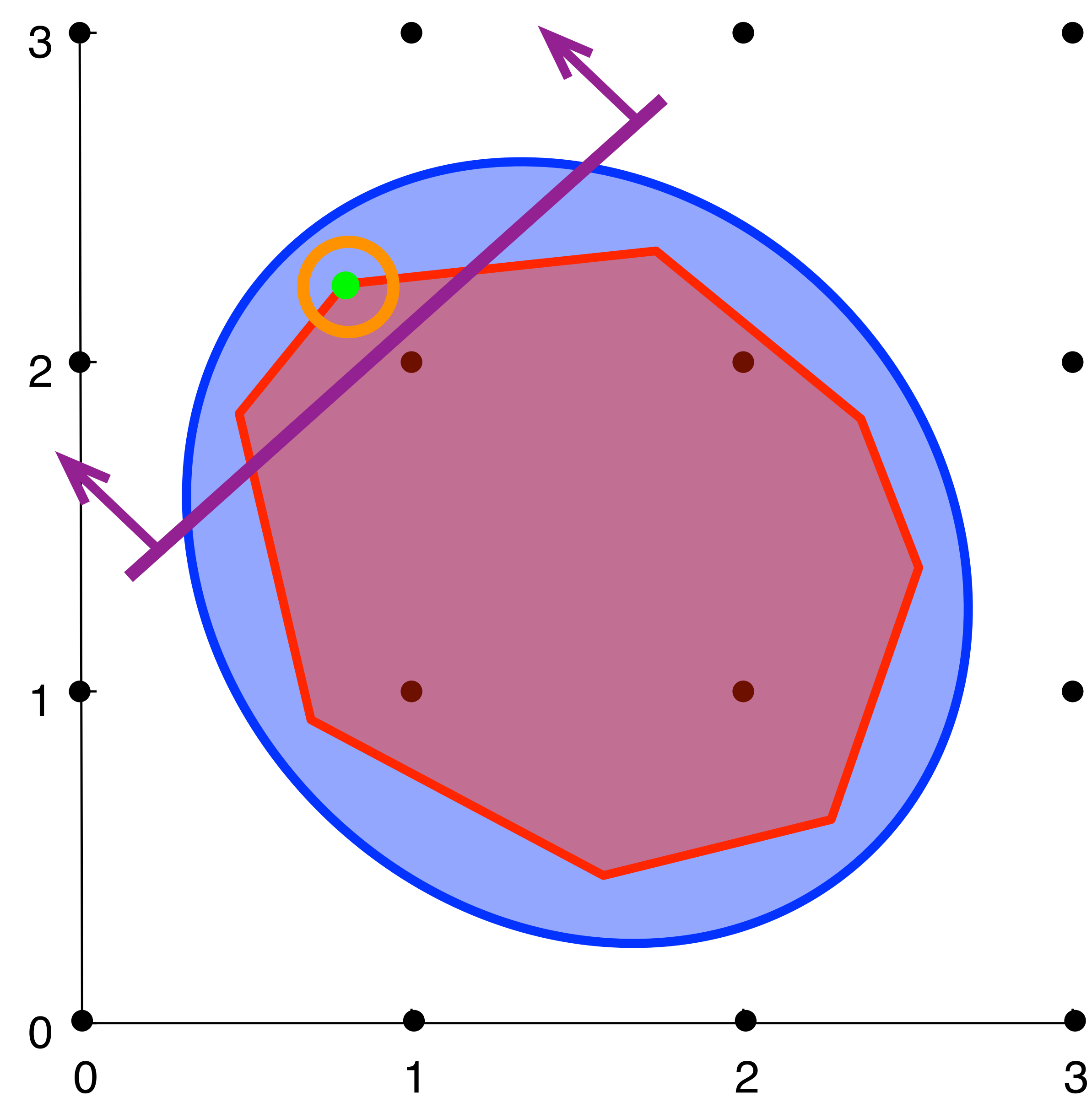
$$V := \mathrm{ext}\left(\mathrm{CC}(S^1, C)\right)$$

$$\langle a, v\rangle > \lfloor \sigma_C(a) \rfloor$$

$$\exists \varepsilon > 0 \quad \varepsilon B^n + v \subset C \quad \forall v \in V$$

$$\|a\| \geq \frac{1}{\varepsilon} \Rightarrow$$

$$\begin{aligned} \lfloor \sigma_C(a) \rfloor &\geq \sigma_C(a) - 1 \\ &\geq \sigma_{v+\varepsilon B^n}(a) - 1 \\ &= \langle v, a\rangle + \varepsilon\|a\| - 1 \\ &\geq \langle v, a\rangle \end{aligned}$$



Step 2 : Separate $\mathrm{CC}(S^1, C) \setminus \mathrm{CC}(C)$

$$V := \mathrm{ext}\left(\mathrm{CC}(S^1, C)\right)$$

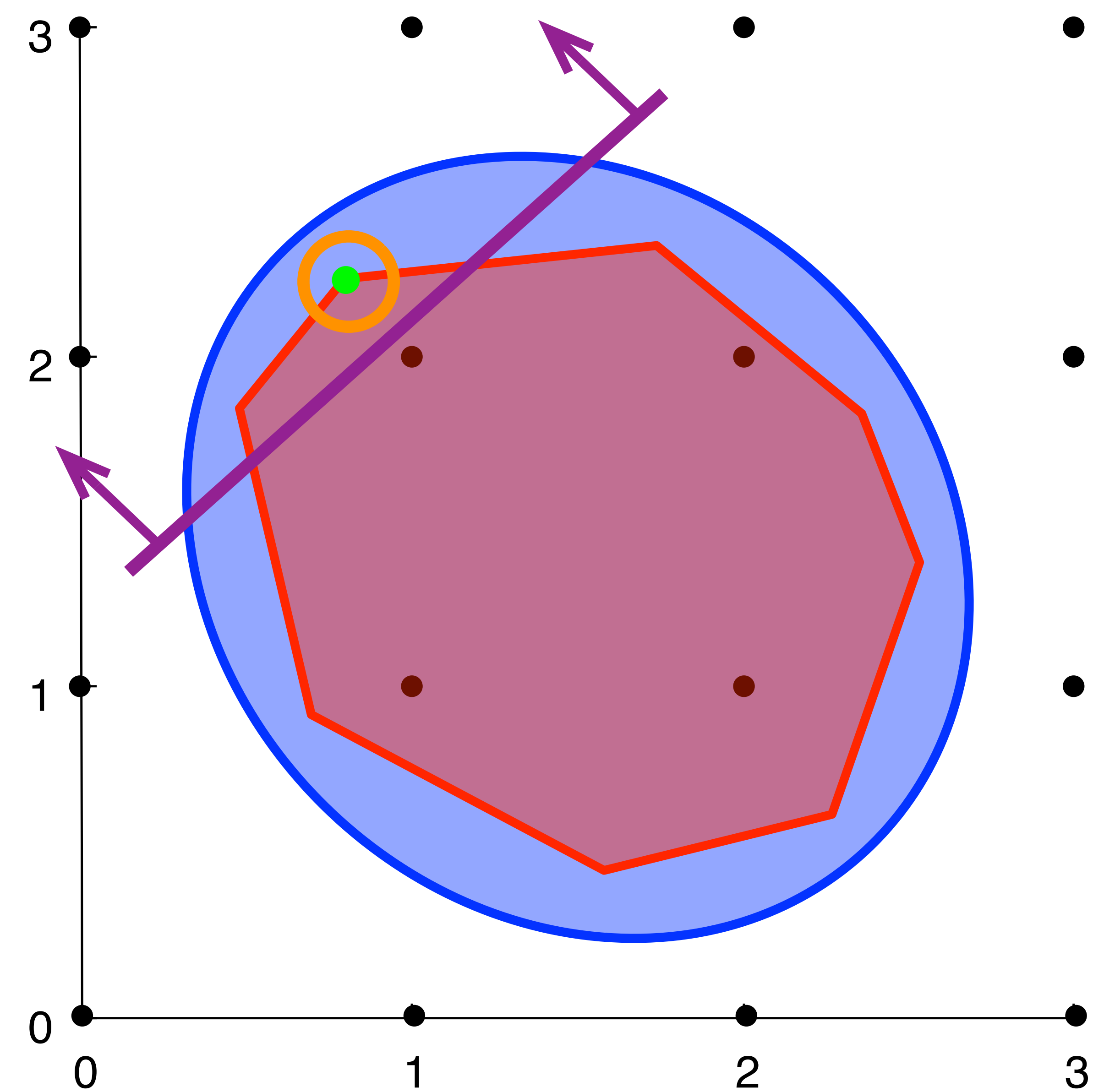
$$\langle a, v\rangle > \lfloor \sigma_C(a) \rfloor$$

$$\exists \varepsilon > 0 \quad \varepsilon B^n + v \subset C \quad \forall v \in V$$

$$\|a\| \geq \frac{1}{\varepsilon} \Rightarrow$$

$$\begin{aligned} \lfloor \sigma_C(a) \rfloor &\geq \sigma_C(a) - 1 \\ &\geq \sigma_{v+\varepsilon B^n}(a) - 1 \\ &= \langle v, a\rangle + \varepsilon\|a\| - 1 \\ &\geq \langle v, a\rangle \end{aligned}$$

$$S^2 = (1/\varepsilon)B \cap \mathbb{Z}^n$$



Main Tool: Lifting Cuts for Faces

P polyhedron, F face of P

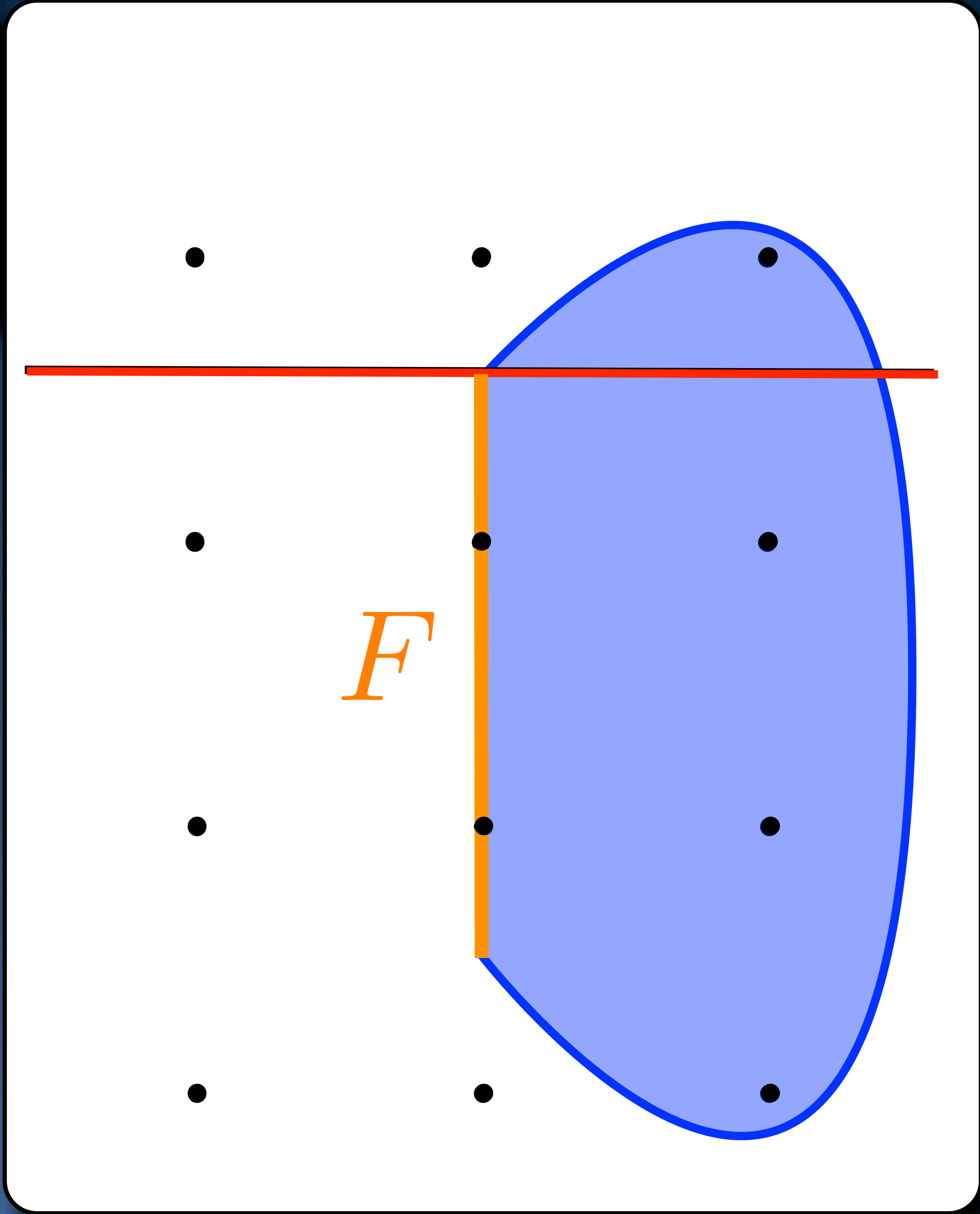
$\mathtt{CC}(F) = \mathtt{CC}(P) \cap F$ (Schrijver, '86)

(Convex Sets: Dadush, Dey, V. (2011)

- Part 1: Kill Irrationality:

  $\mathrm{aff}_I(F) := \mathrm{aff}\,(\mathrm{aff}(F) \cap \mathbb{Z}^n)$

  - Kronecker's approx.
- Part 2: Lift inside
  - Dirichlet's approx.



Main Tool: Lifting Cuts for Faces

P polyhedron, F face of P

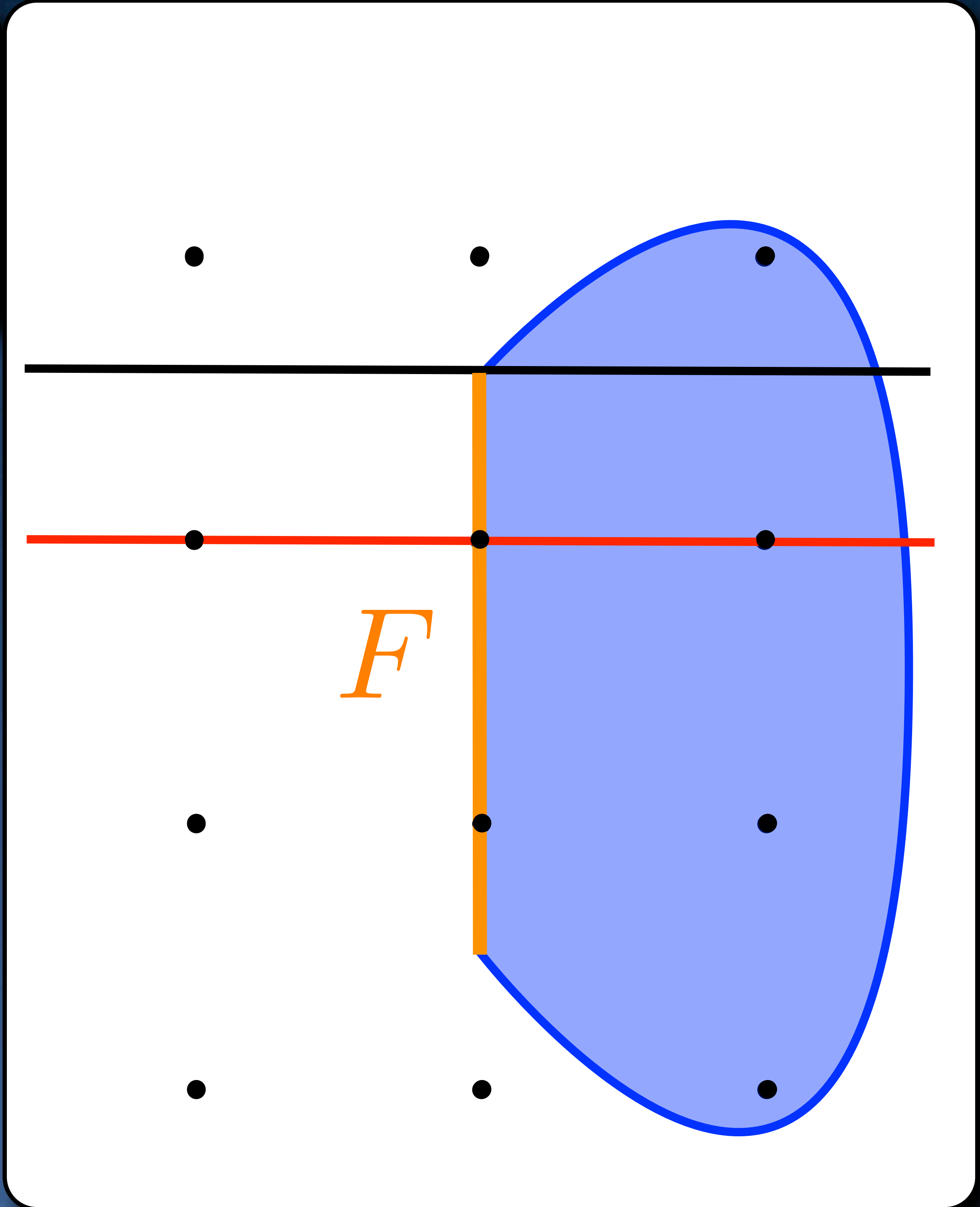
$\mathtt{CC}(F) = \mathtt{CC}(P) \cap F$ (Schrijver, '86)

(Convex Sets: Dadush, Dey, V. (2011))

- Part 1: Kill Irrationality:

  $\text{aff}_I(F) := \text{aff}\,(\text{aff}(F) \cap \mathbb{Z}^n)$

  - Kronecker's approx.
- Part 2: Lift inside
  - Dirichlet's approx.



Main Tool: Lifting Cuts for Faces

P polyhedron, F face of P

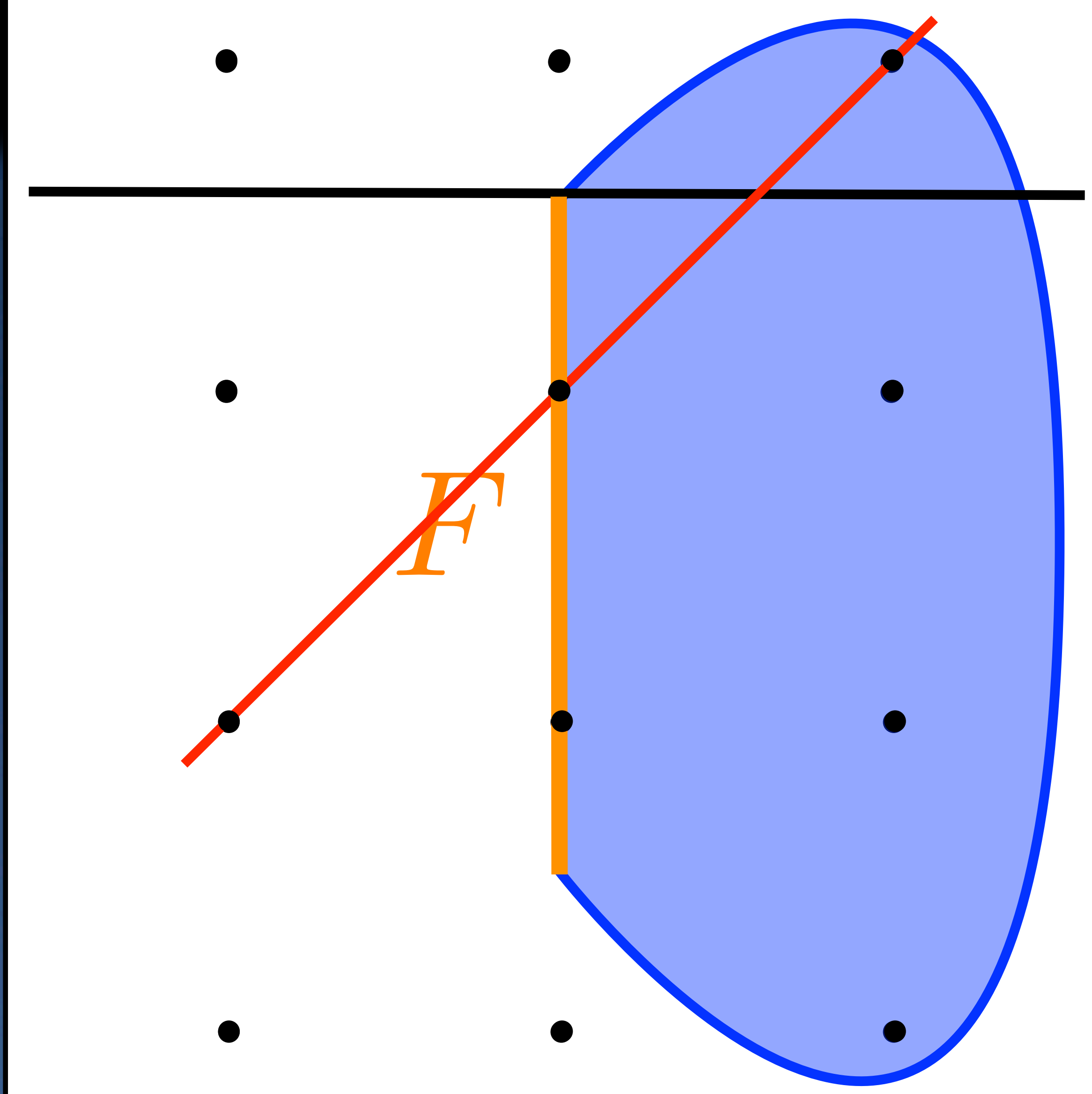
$\mathrm{CC}(F) = \mathrm{CC}(P) \cap F$ (Schrijver, '86)

(Convex Sets: Dadush, Dey, V. (2011)

- Part 1: Kill Irrationality:

  $\mathrm{aff}_I(F) := \mathrm{aff}\,(\mathrm{aff}(F) \cap \mathbb{Z}^n)$

  - Kronecker's approx.
- Part 2: Lift inside
  - Dirichlet's approx.



Main Tool: Lifting Cuts for Faces

P polyhedron, F face of P

$\mathtt{CC}(F) = \mathtt{CC}(P) \cap F$ (Schrijver, '86)

(Convex Sets: Dadush, Dey, V. (2011))

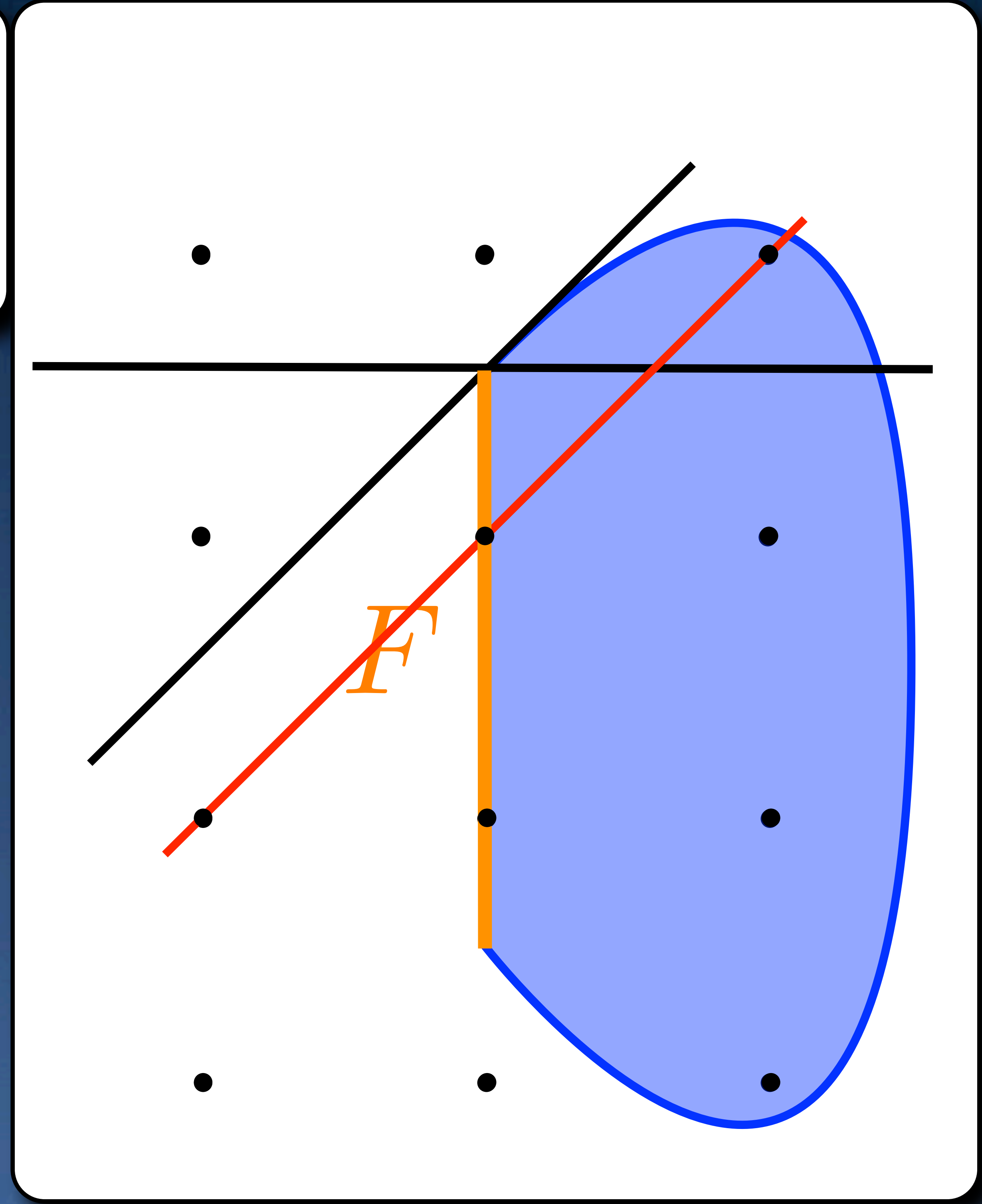
- Part 1: Kill Irrationality:

  $\mathrm{aff}_I(F) := \mathrm{aff}\,(\mathrm{aff}(F) \cap \mathbb{Z}^n)$

  - Kronecker's approx.

- Part 2: Lift inside

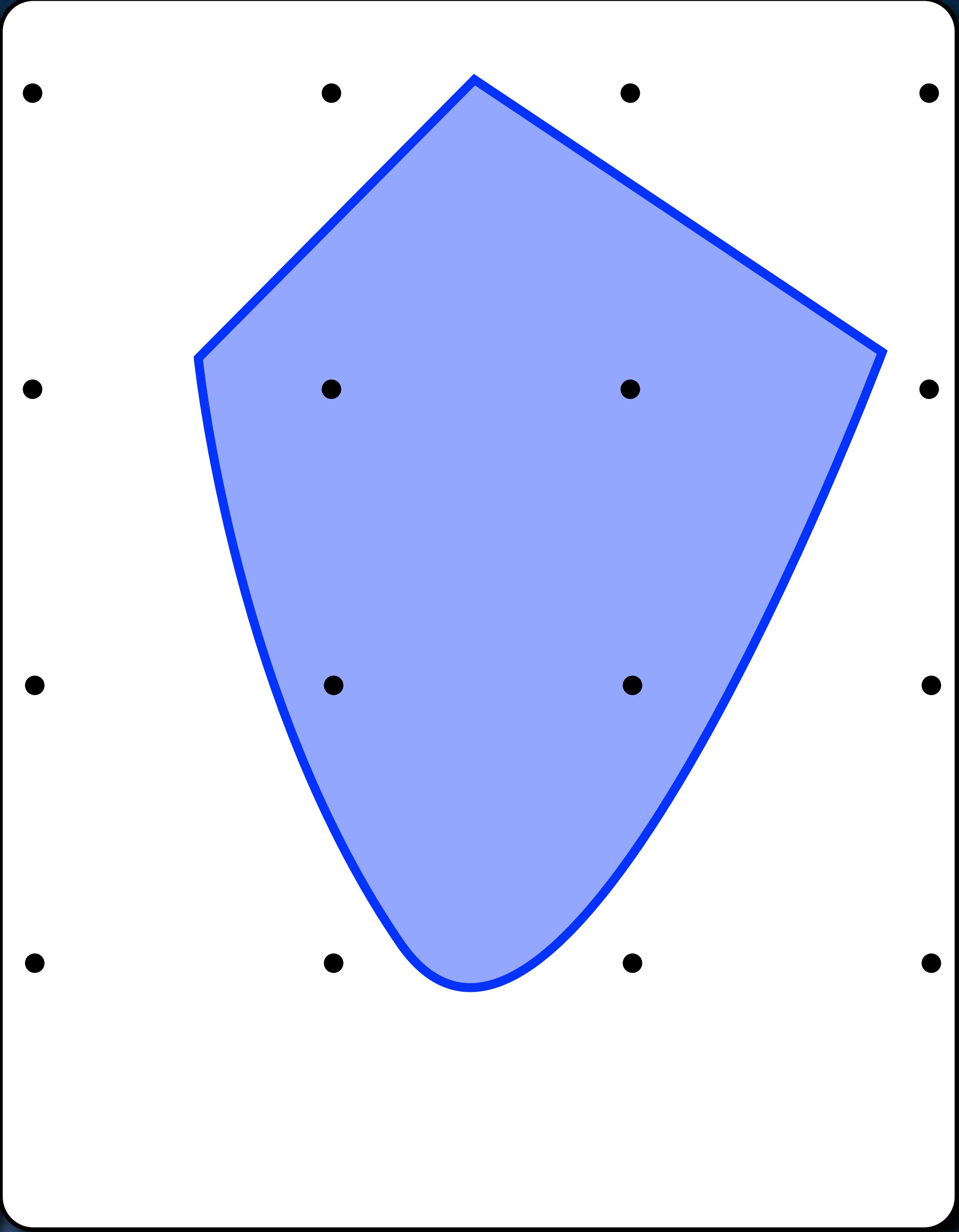
  - Dirichlet's approx.



Split Cuts

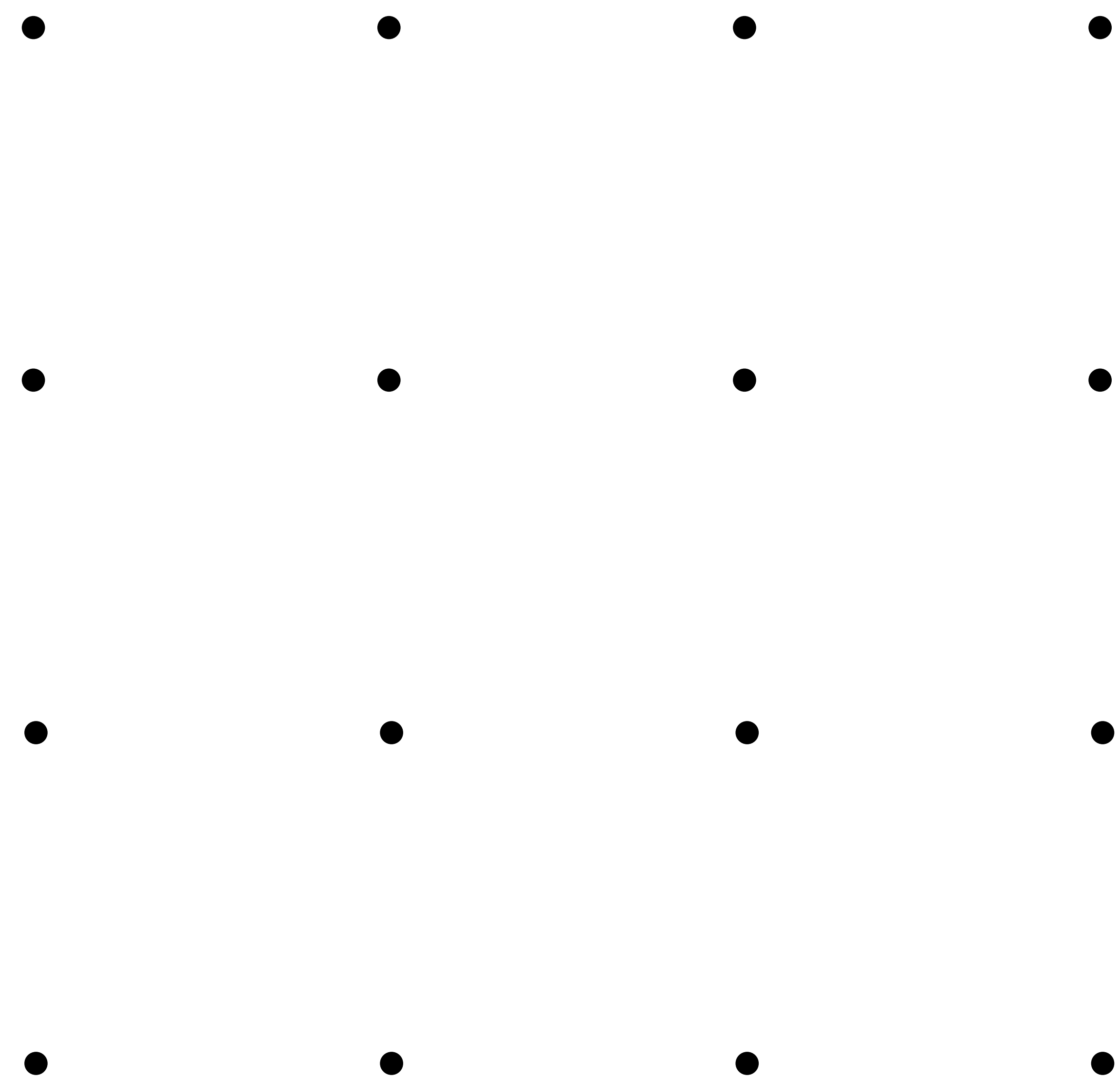
Split Disjunctions and Split Cuts

$\pi \in \mathbb{Z}^n$, $\pi_0 \in \mathbb{Z}$



Split Disjunctions and Split Cuts

$\pi \in \mathbb{Z}^n$, $\pi_0 \in \mathbb{Z}$



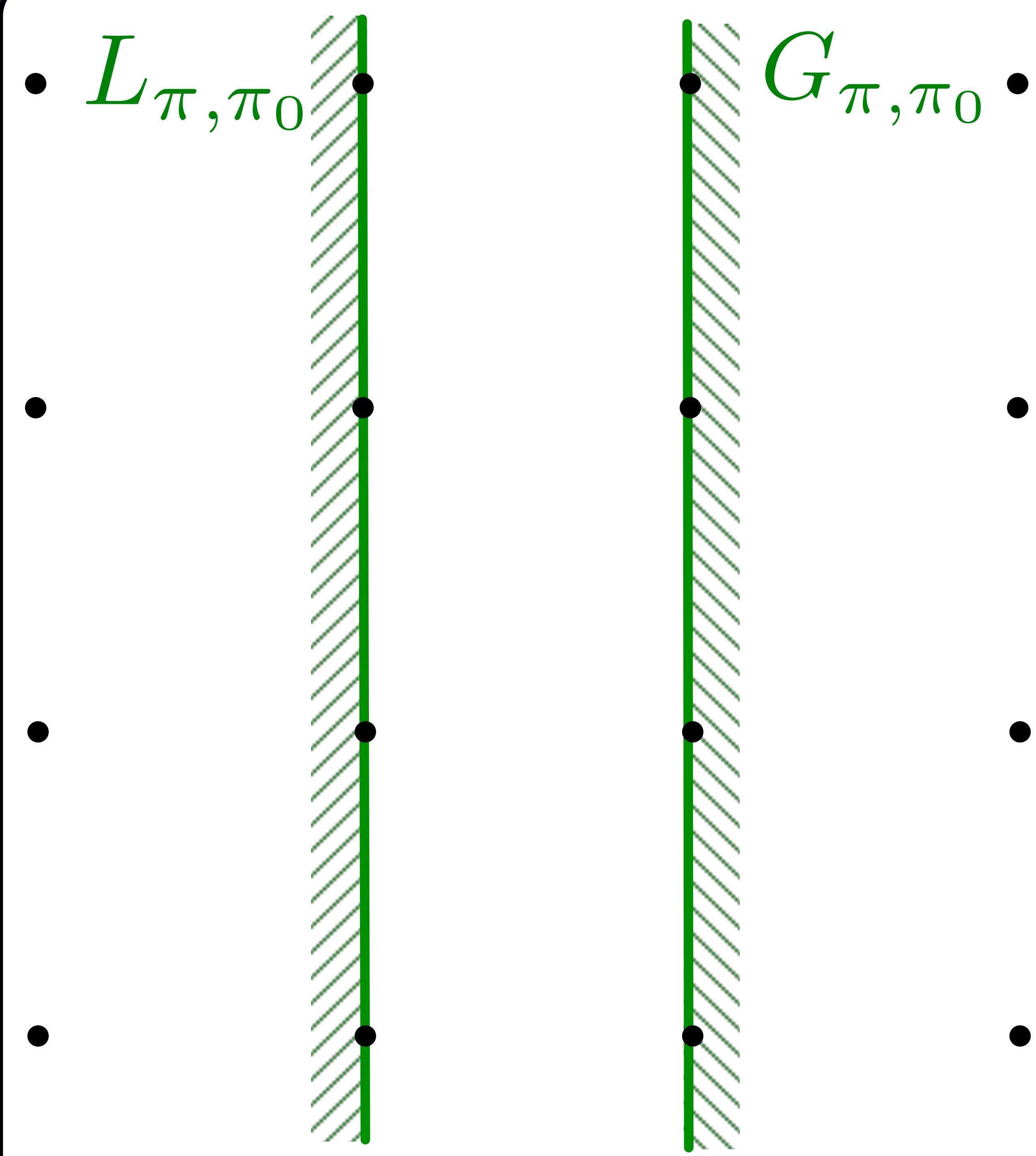
Split Disjunctions and Split Cuts

$\pi \in \mathbb{Z}^n,\ \pi_0 \in \mathbb{Z}$ Split Disjunction

$$L_{\pi,\pi_0} = \{x \in \mathbb{R}^n \ : \ \langle \pi, x \rangle \leq \pi_0\}$$

$$R_{\pi,\pi_0} = \{x \in \mathbb{R}^n \ : \ \langle \pi, x \rangle \geq \pi_0 + 1\}$$

$$\mathbb{Z}^n \subseteq L_{\pi,\pi_0} \cup R_{\pi,\pi_0}$$



Split Disjunctions and Split Cuts

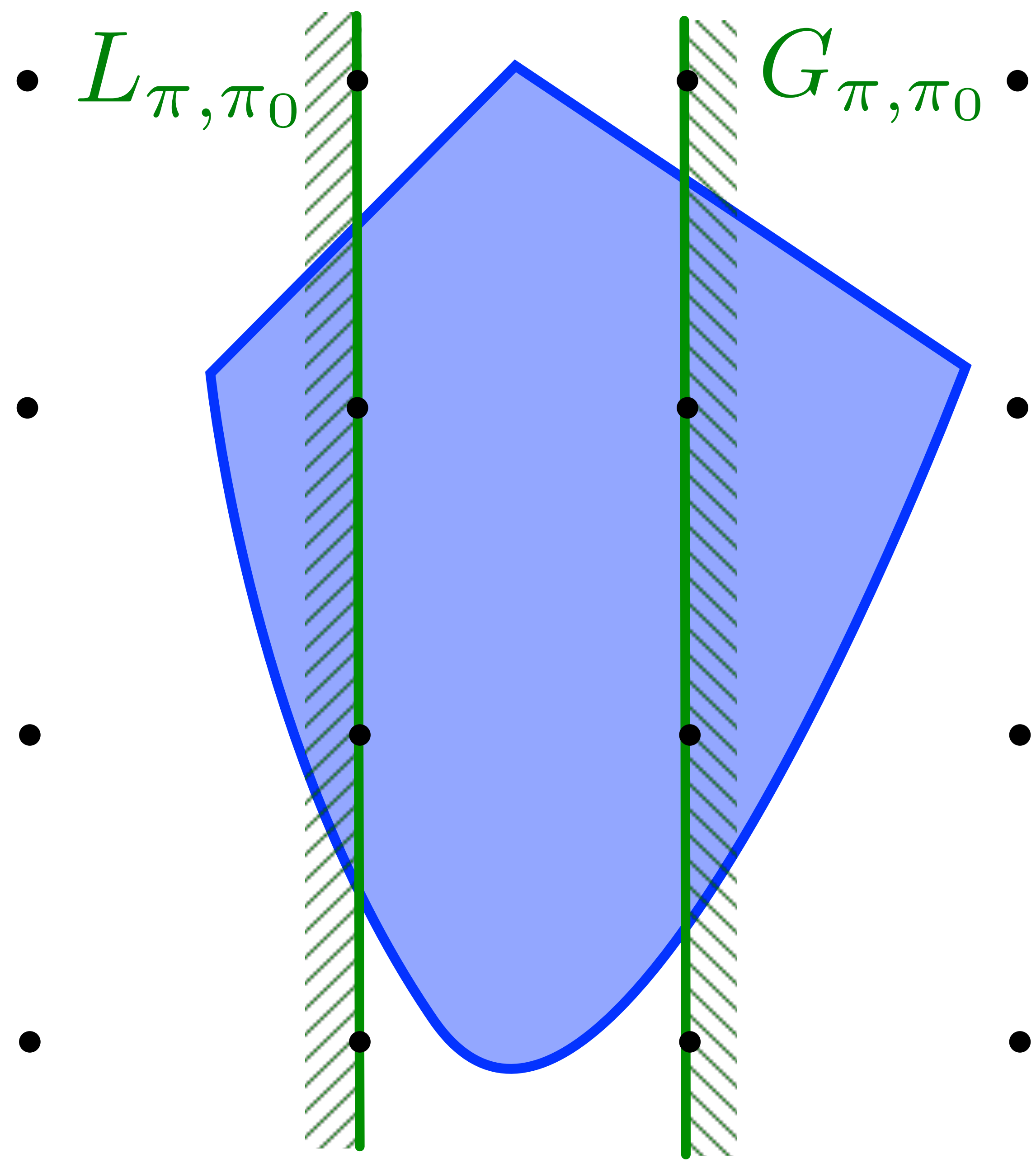
$\pi \in \mathbb{Z}^n, \ \pi_0 \in \mathbb{Z}$ Split Disjunction

$$L_{\pi,\pi_0} = \{x \in \mathbb{R}^n \ : \ \langle \pi, x \rangle \leq \pi_0\}$$

$$R_{\pi,\pi_0} = \{x \in \mathbb{R}^n \ : \ \langle \pi, x \rangle \geq \pi_0 + 1\}$$

$$\mathbb{Z}^n \subseteq L_{\pi,\pi_0} \cup R_{\pi,\pi_0}$$

$$C \cap \mathbb{Z}^n \subseteq C \cap (L_{\pi,\pi_0} \cup R_{\pi,\pi_0})$$



Split Disjunctions and Split Cuts

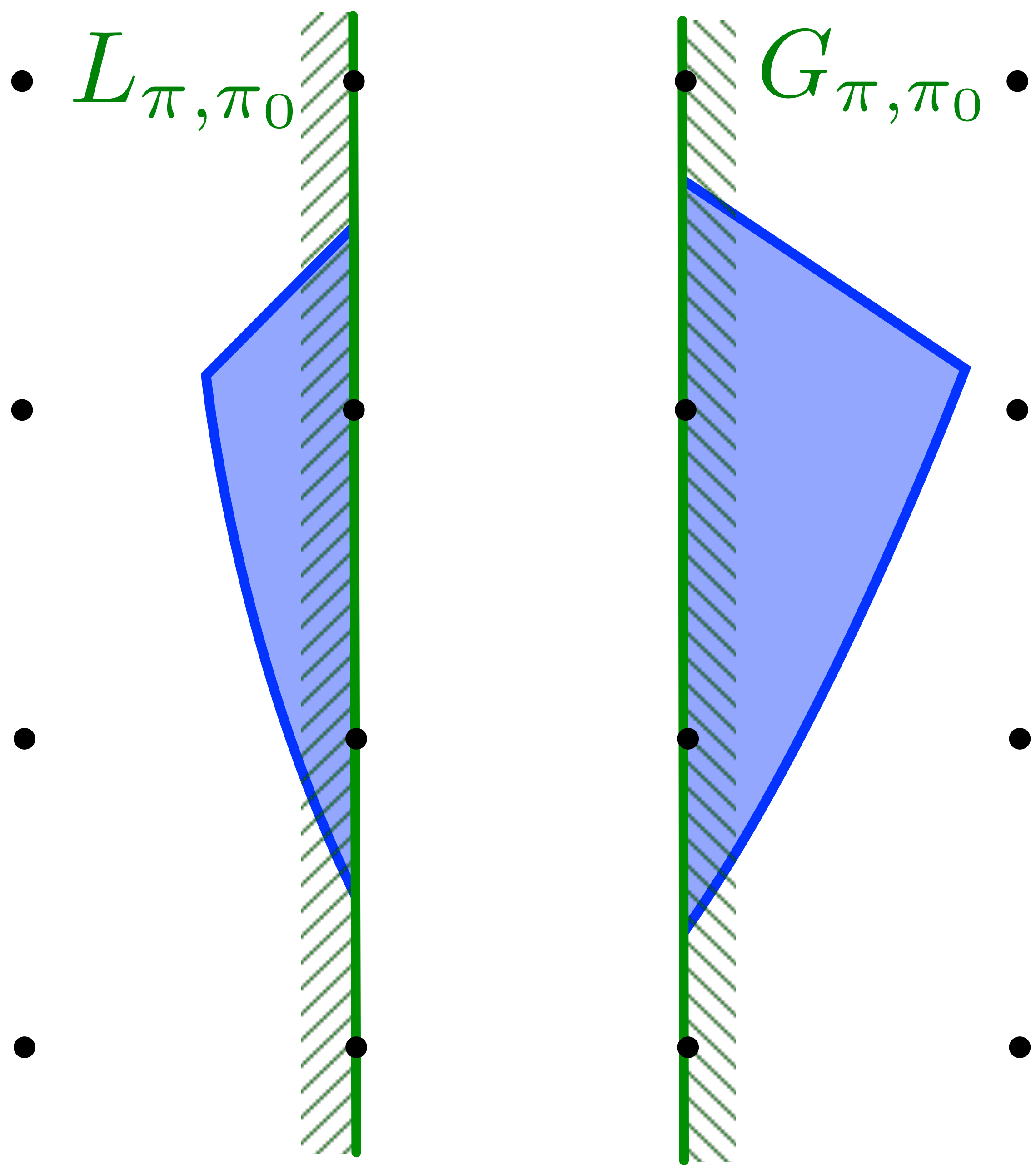
$\pi \in \mathbb{Z}^n,\ \pi_0 \in \mathbb{Z}$ Split Disjunction

$$L_{\pi,\pi_0} = \{x \in \mathbb{R}^n \ : \ \langle \pi, x \rangle \leq \pi_0\}$$

$$R_{\pi,\pi_0} = \{x \in \mathbb{R}^n \ : \ \langle \pi, x \rangle \geq \pi_0 + 1\}$$

$$\mathbb{Z}^n \subseteq L_{\pi,\pi_0} \cup R_{\pi,\pi_0}$$

$$C \cap \mathbb{Z}^n \subseteq C \cap (L_{\pi,\pi_0} \cup R_{\pi,\pi_0})$$



Split Disjunctions and Split Cuts

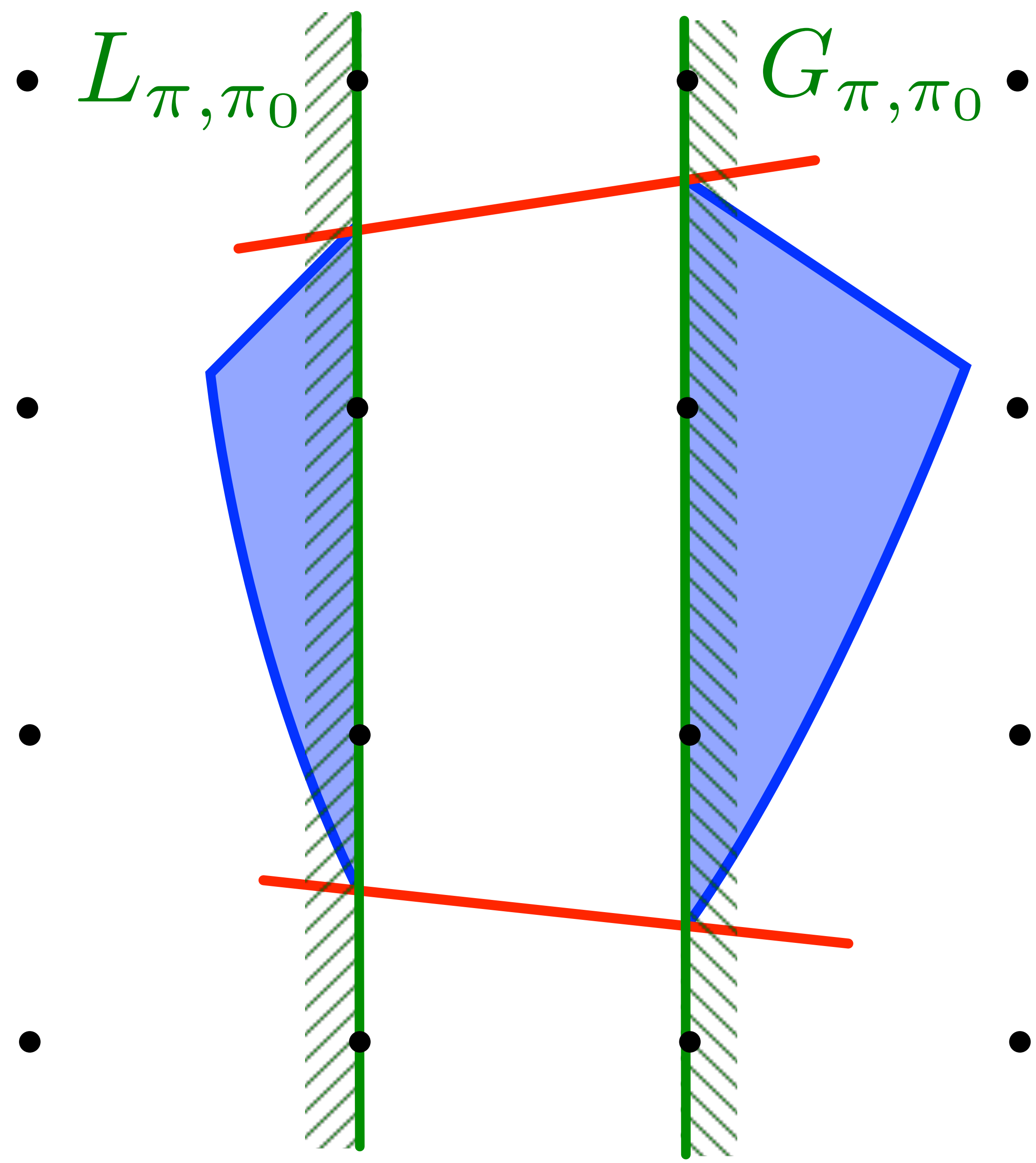
$\pi \in \mathbb{Z}^n,\ \pi_0 \in \mathbb{Z}$ Split Disjunction

$$L_{\pi,\pi_0} = \{x \in \mathbb{R}^n : \langle \pi, x \rangle \le \pi_0\}$$

$$R_{\pi,\pi_0} = \{x \in \mathbb{R}^n : \langle \pi, x \rangle \ge \pi_0 + 1\}$$

$$\mathbb{Z}^n \subseteq L_{\pi,\pi_0} \cup R_{\pi,\pi_0}$$

$$C \cap \mathbb{Z}^n \subseteq C \cap (L_{\pi,\pi_0} \cup R_{\pi,\pi_0})$$



Split Disjunctions and Split Cuts

$\pi \in \mathbb{Z}^n$, $\pi_0 \in \mathbb{Z}$ Split Disjunction

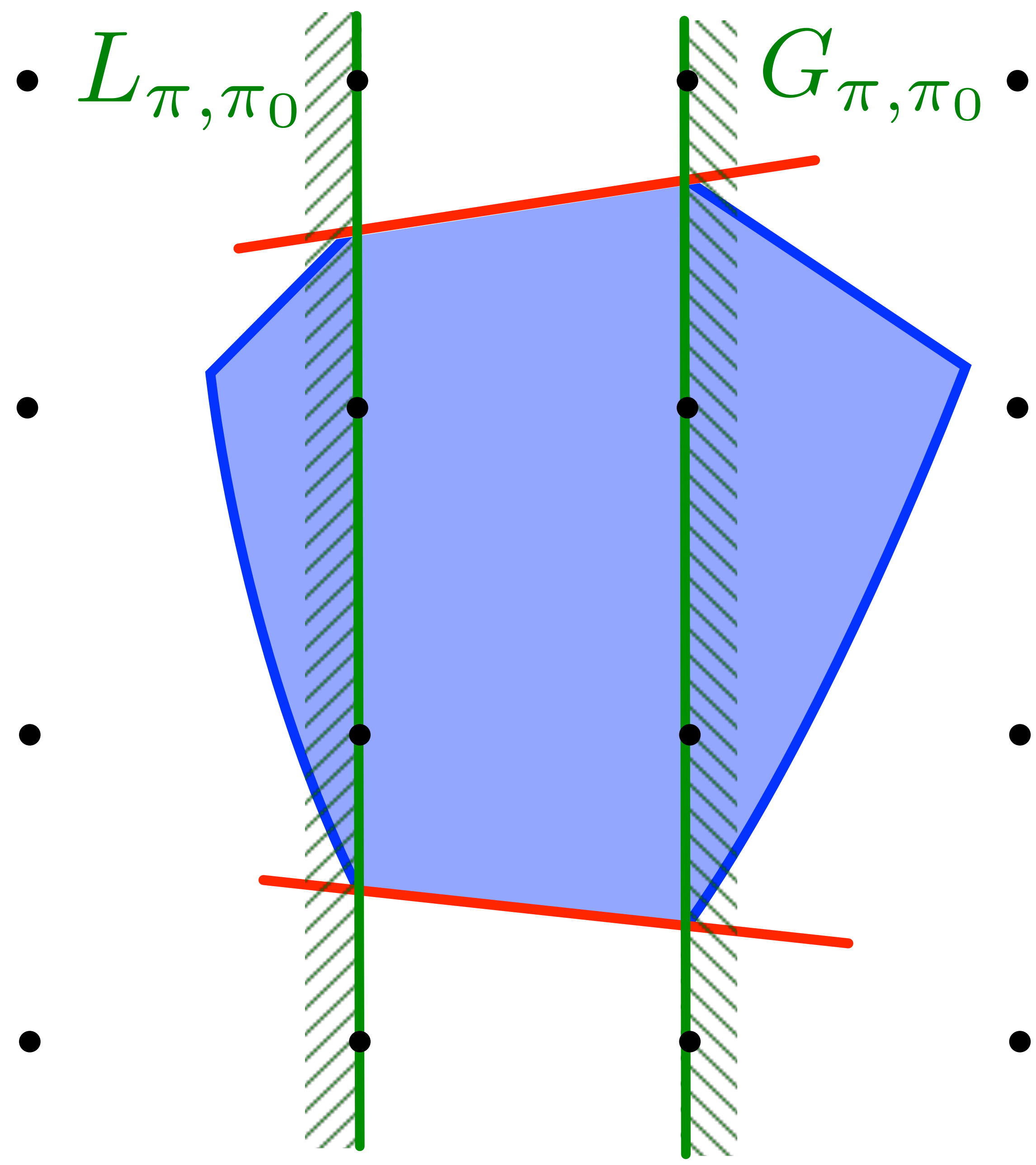
$$L_{\pi,\pi_0} = \{x \in \mathbb{R}^n \ : \ \langle \pi, x \rangle \leq \pi_0\}$$

$$R_{\pi,\pi_0} = \{x \in \mathbb{R}^n \ : \ \langle \pi, x \rangle \geq \pi_0 + 1\}$$

$$\mathbb{Z}^n \subseteq L_{\pi,\pi_0} \cup R_{\pi,\pi_0}$$

$$C \cap \mathbb{Z}^n \subseteq C \cap (L_{\pi,\pi_0} \cup R_{\pi,\pi_0})$$

$$C^{\pi,\pi_0} := \operatorname{conv}\left(C \cap (L_{\pi,\pi_0} \cup R_{\pi,\pi_0})\right)$$



Split Disjunctions and Split Cuts

$\pi \in \mathbb{Z}^n,\ \pi_0 \in \mathbb{Z}$ **Split Disjunction**

$$L_{\pi,\pi_0} = \{x \in \mathbb{R}^n \ :\ \langle \pi, x\rangle \le \pi_0\}$$

$$R_{\pi,\pi_0} = \{x \in \mathbb{R}^n \ :\ \langle \pi, x\rangle \ge \pi_0 + 1\}$$

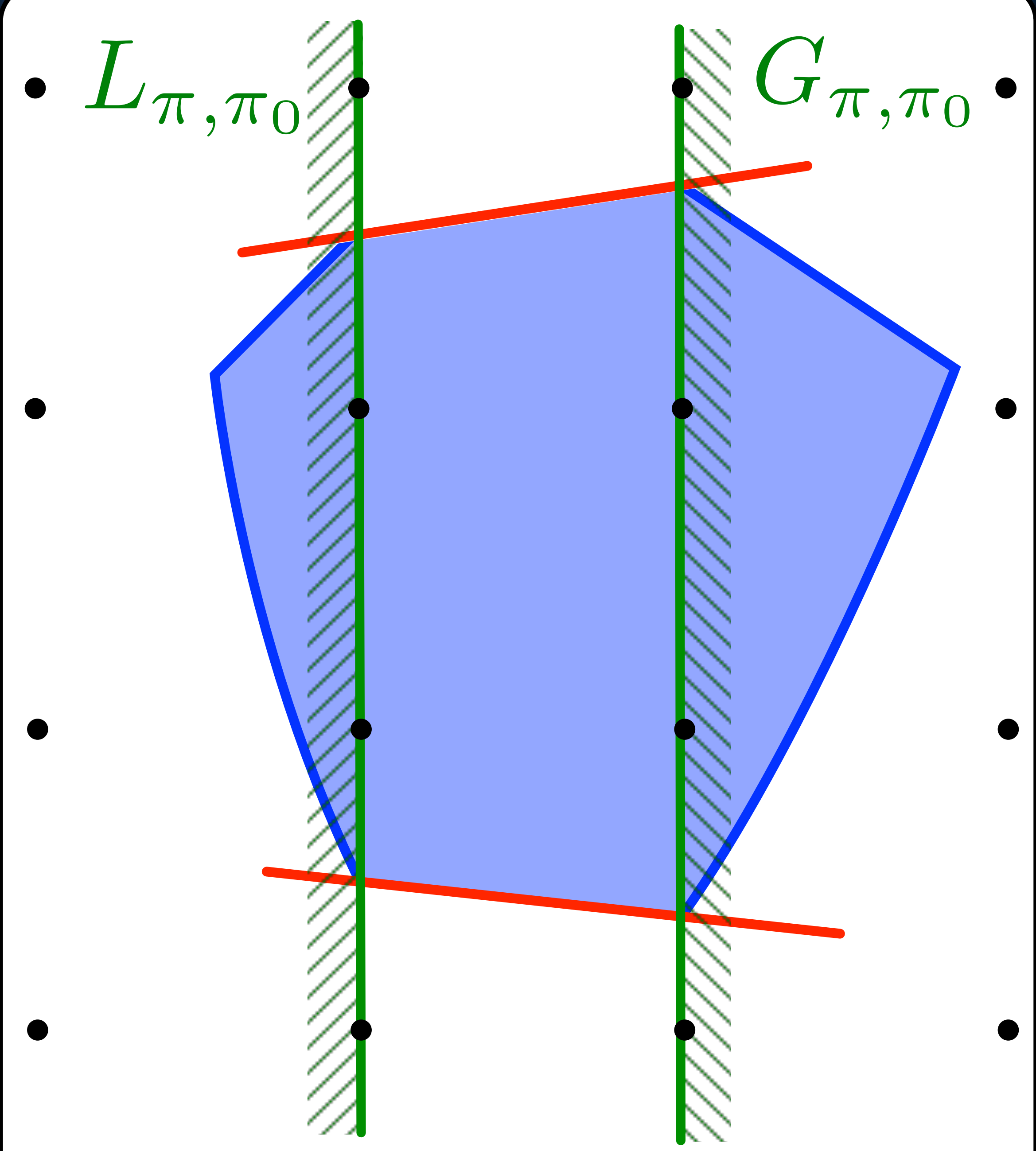
$$\mathbb{Z}^n \subseteq L_{\pi,\pi_0} \cup R_{\pi,\pi_0}$$

$$C \cap \mathbb{Z}^n \subseteq C \cap (L_{\pi,\pi_0} \cup R_{\pi,\pi_0})$$

$$C^{\pi,\pi_0} := \operatorname{conv}\big(C \cap (L_{\pi,\pi_0} \cup R_{\pi,\pi_0})\big)$$

$$||$$

$$\{x \in C \ :\ g_i(x) \le 0,\ i \in I\}$$



Split Disjunctions and Split Cuts

$\pi \in \mathbb{Z}^n,\ \pi_0 \in \mathbb{Z}$ Split Disjunction

$$L_{\pi,\pi_0} = \{x \in \mathbb{R}^n \ :\ \langle \pi, x\rangle \leq \pi_0\}$$

$$R_{\pi,\pi_0} = \{x \in \mathbb{R}^n \ :\ \langle \pi, x\rangle \geq \pi_0 + 1\}$$

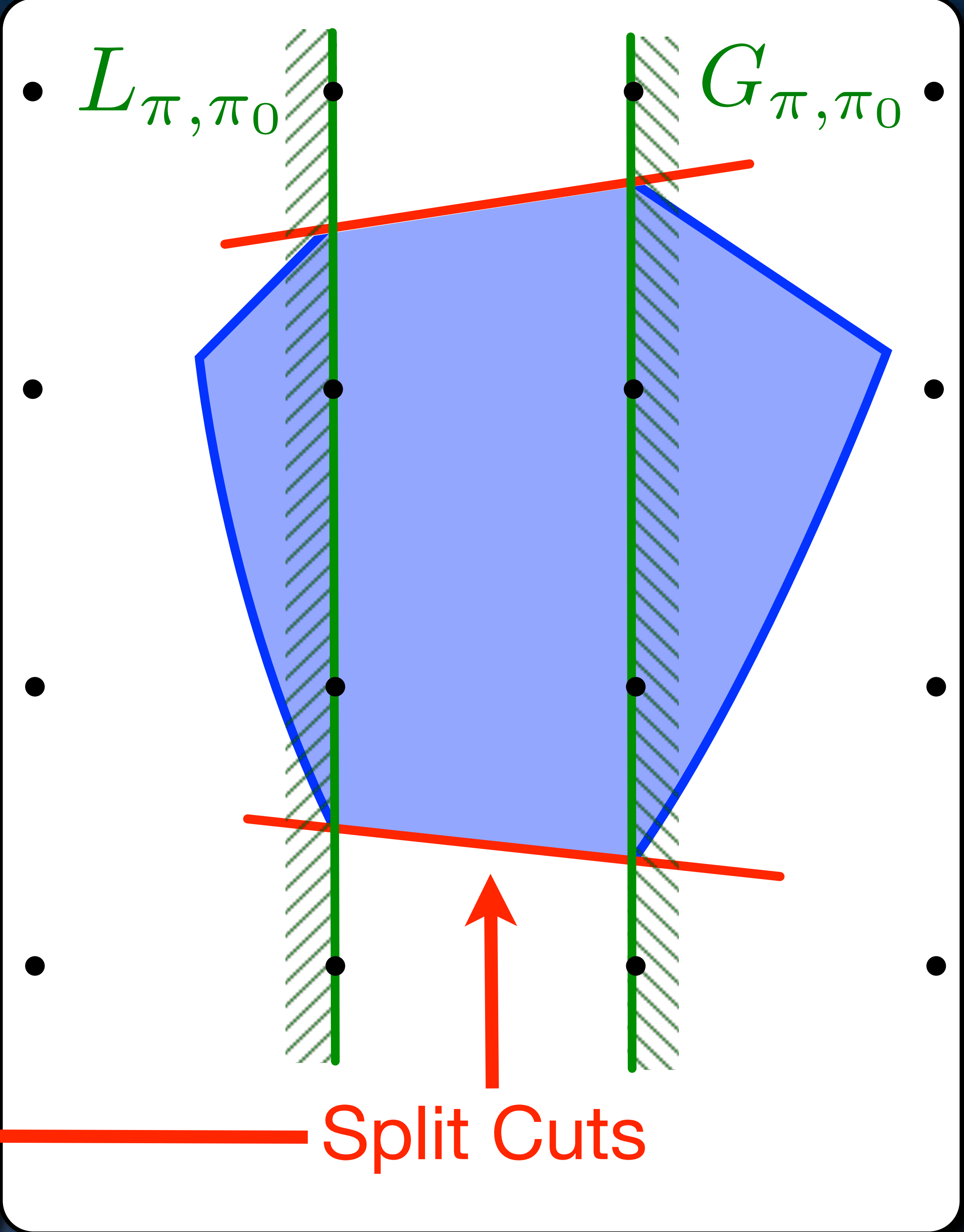
$$\mathbb{Z}^n \subseteq L_{\pi,\pi_0} \cup R_{\pi,\pi_0}$$

$$C \cap \mathbb{Z}^n \subseteq C \cap (L_{\pi,\pi_0} \cup R_{\pi,\pi_0})$$

$$C^{\pi,\pi_0} := \operatorname{conv}\left(C \cap (L_{\pi,\pi_0} \cup R_{\pi,\pi_0})\right)$$

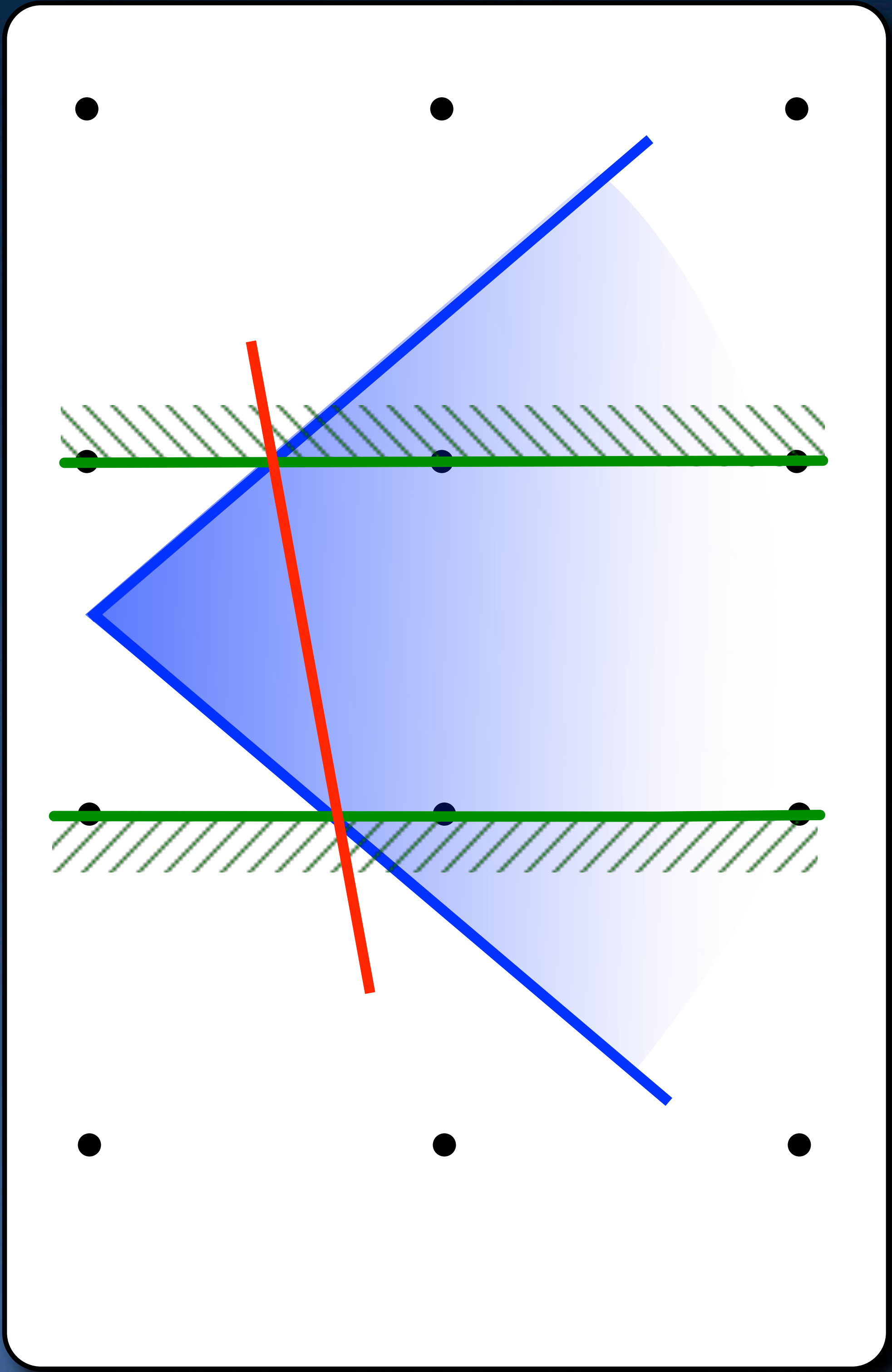
$$||$$

$$\{x \in C \ :\ g_i(x) \leq 0,\ i \in I\}$$



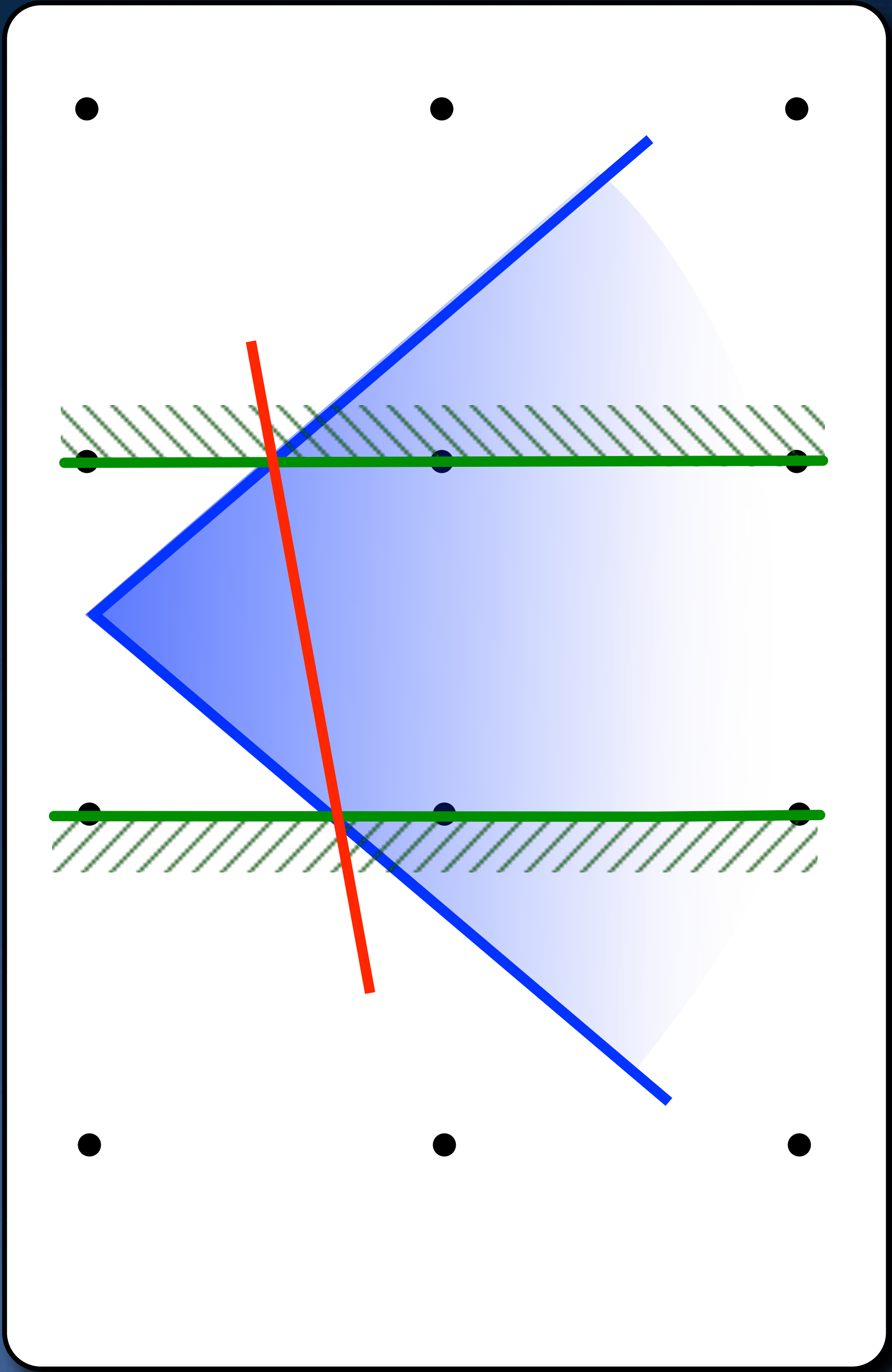
Known Facts for Rational Polyhedra

- Formulas for simplicial cones:
  - MIG (Gomory 1960) and MIR (Nemhauser and Wolsey 1988)



Known Facts for Rational Polyhedra

- Formulas for simplicial cones:
  - MIG (Gomory 1960) and MIR (Nemhauser and Wolsey 1988)
- Split Closure $\bigcap_{(\pi,\pi_0)\in\mathbb{Z}^n\times\mathbb{Z}} C^{\pi,\pi_0}$:
  - Rational Polyhedron (Cook, Kannan and Shrijver 1990)
  - Constructive Proofs:
    - Dash, Günlük and Lodi 2007; V. 2007.

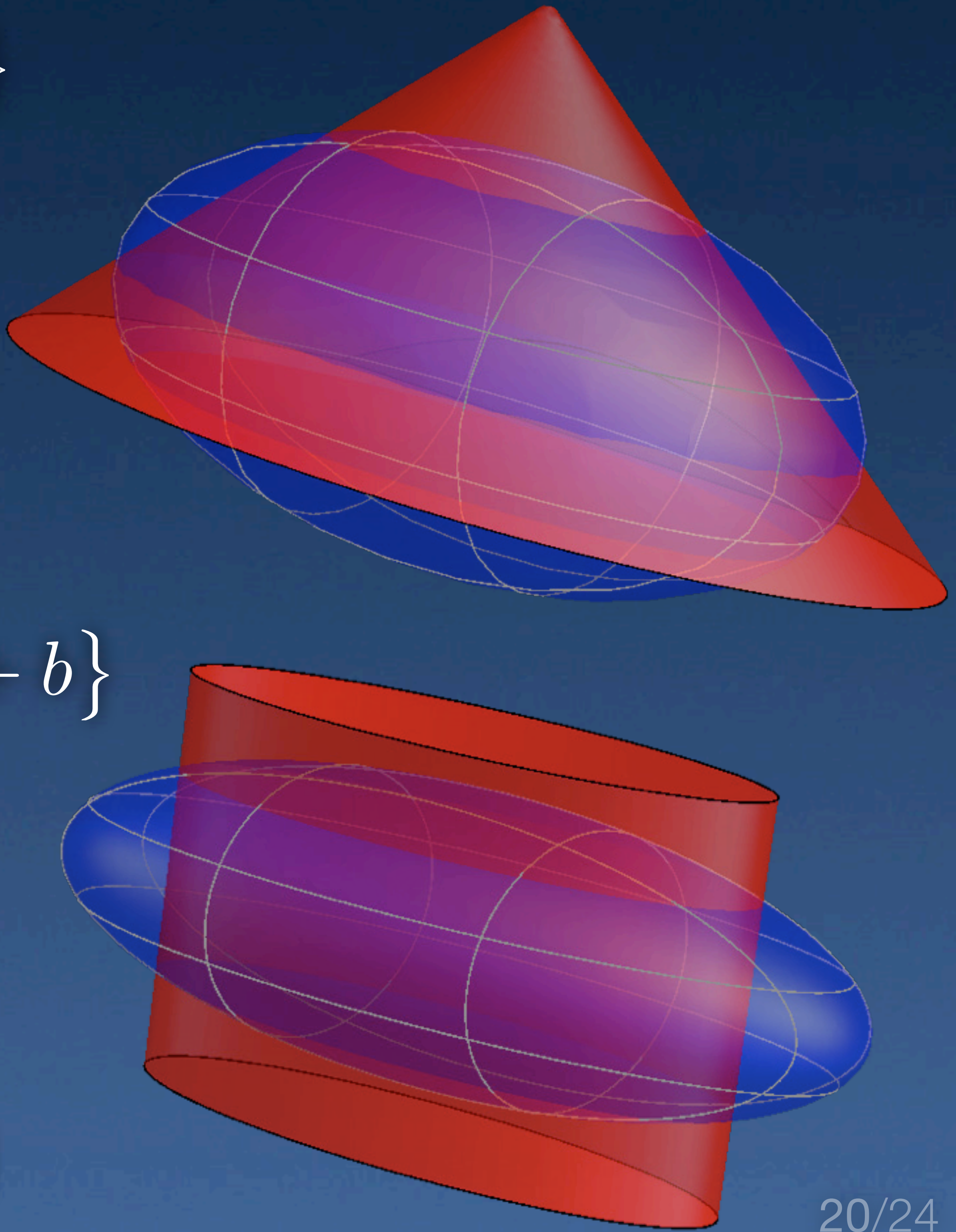


Split Cut for Ellipsoids

- $C = \{x \in \mathbb{R}^n \, : \, \|A(x-c)\|_2 \leq 1\}$

- Dadush, Dey and V. 2011:

$$C^{\pi,\pi_0} = \Big\{x \in \mathbb{R}^n \, : \, \|A(x-c)\|_2 \leq 1, \quad \|B(x-c)\|_2 \leq a\langle \pi, x\rangle + b\Big\}$$

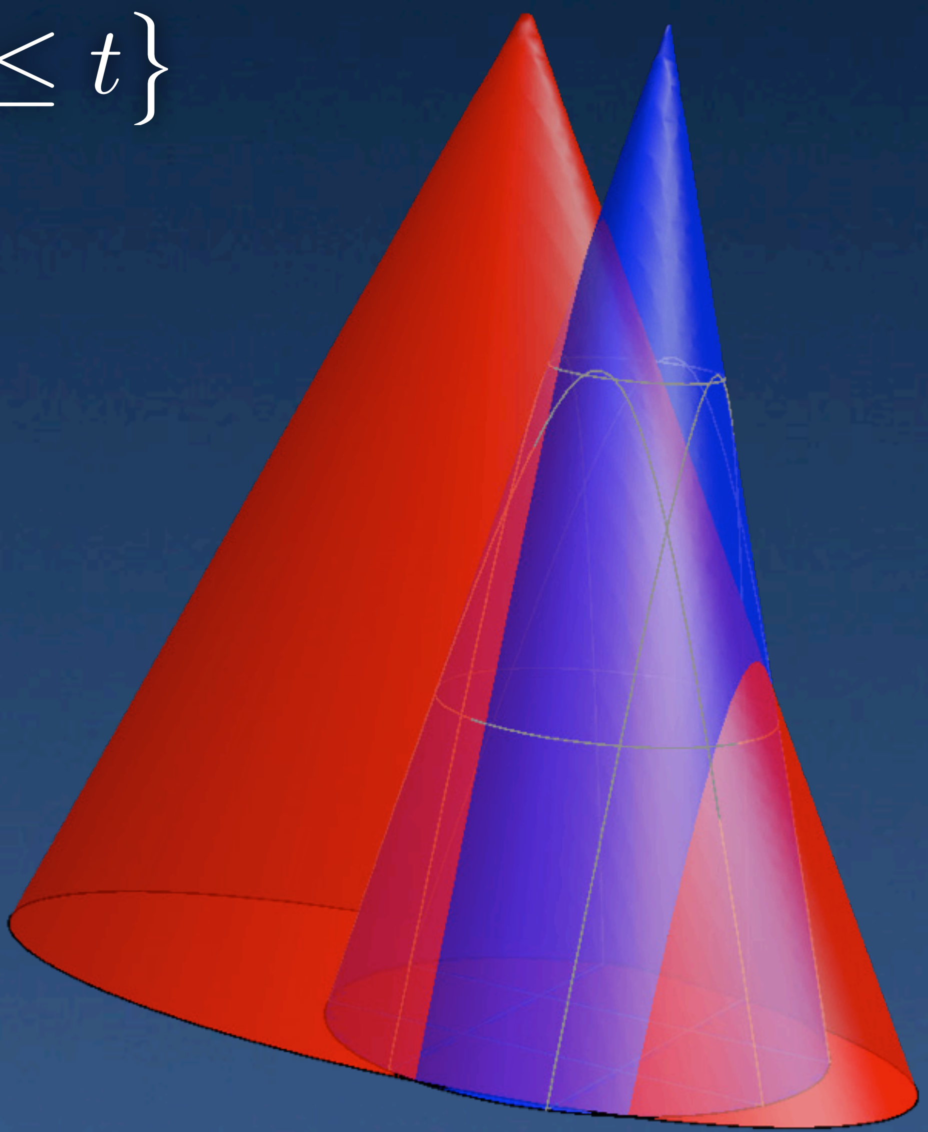


- Also see Belotti, Góez, Polik, Ralphs, Terlaky 2011

Split Cut for Quadratic Cones

- $C = \{(x,t) \in \mathbb{Z}^n \times \mathbb{R} \,:\, \|A(x-c)\|_2 \leq t\}$
- Modaresi, Kılınç, V. 2011:

$$C^{\pi,\pi_0} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \,: \\ \|A(x-c)\|_2 \leq t \\ \|Bx-d\|_2 \leq t\}$$



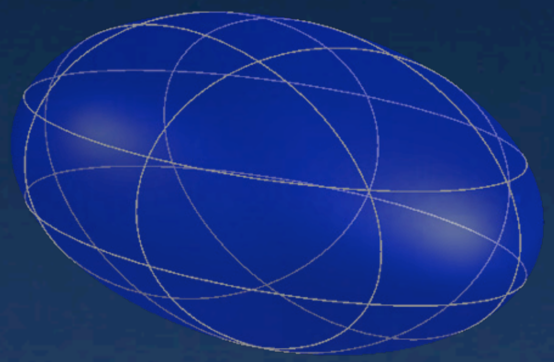
Split Closure is Finitely Generated

- Theorem (Dadush, Dey, V. 2011): If C is a strictly convex set then there exists a finite $D \subseteq \mathbb{Z}^n \times \mathbb{Z}$ such that:

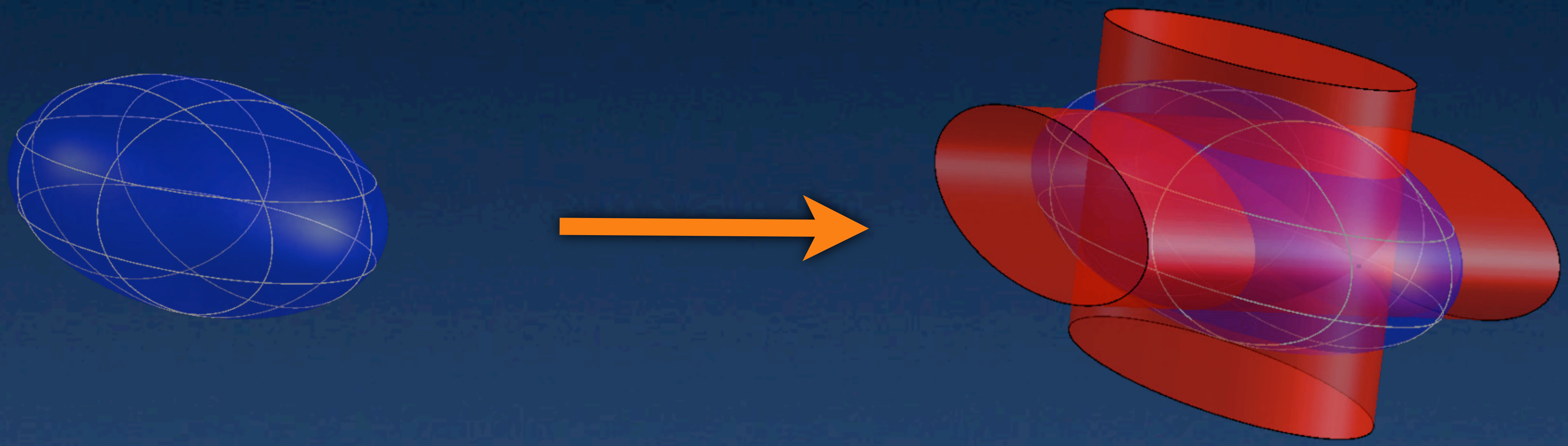
$$\bigcap_{(\pi,\pi_0)\in\mathbb{Z}^n\times\mathbb{Z}} C^{\pi,\pi_0} = \bigcap_{(\pi,\pi_0)\in D} C^{\pi,\pi_0}$$

- Does not imply polyhedrality of split closure.
- Split Closure is not stable

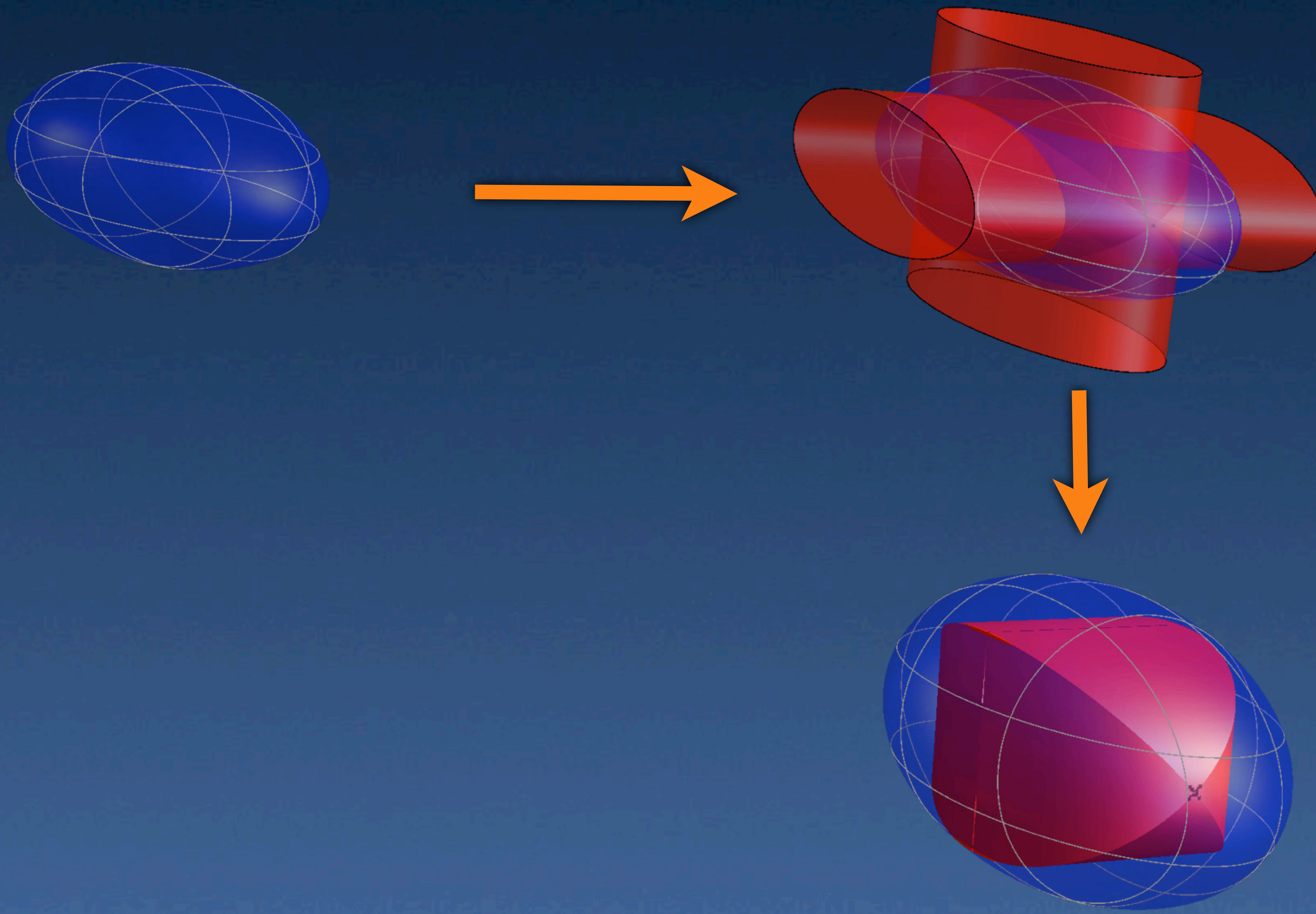
Split Closure Can Be Non-Polyhedral



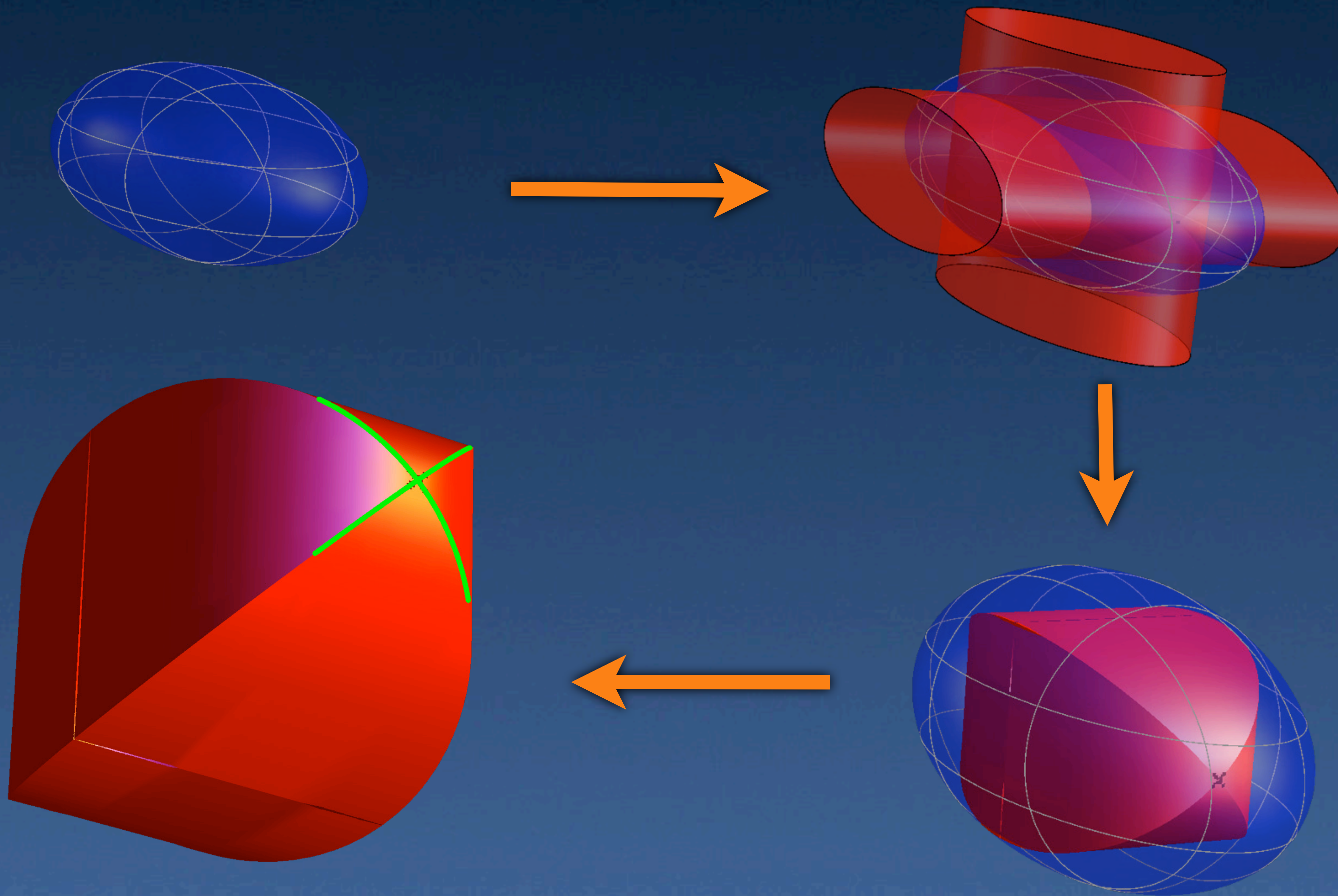
Split Closure Can Be Non-Polyhedral



Split Closure Can Be Non-Polyhedral



Split Closure Can Be Non-Polyhedral



Other Results and Open Questions

- CG closure is polyhedron for a class of unbounded sets:
  - Class includes rational polyhedra = True generalization of Schrijver theorem.
- Open Questions:
  - Constructive characterization of CG closure.
  - Algorithms to separate CG/Split cuts.