My Research Topics on Competition Graphs and Competition Numbers

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DIMACS/CCICADA Interdisciplinary Seminar Series

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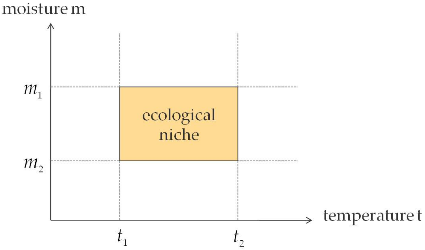
Ecological niche of a species

- A species' health environment is characterized by ranges of different important factors.
- Assuming that the ranges on the factors are independent, the ecological niche can be represented by a box in Euclidean space.



two dimensional case

Ecological niche of a species

In 1968, Cohen asked:

Can we assign to each species an ecological niche so that competition between two species corresponds to overlap of their ecological niches?

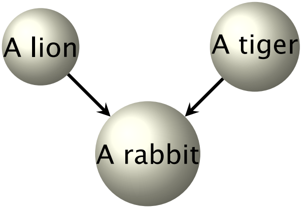
$\Rightarrow$ Cohen introduced the notion of a competition graph.

$\Rightarrow$ The research has been motivated by the observation that many competition graphs have the boxicity one.

- Any graph is the intersection graph of boxes in p-space for some p, the smallest such p is called the *boxicity* of a graph.

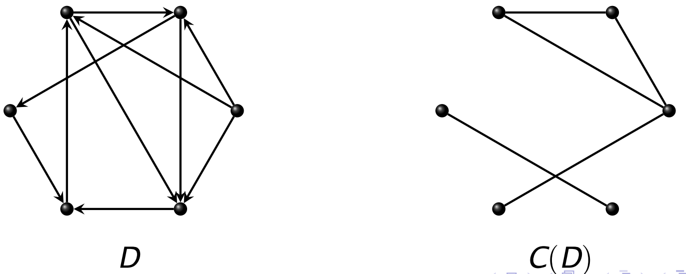
Competition graph

- Cohen [1968] introduced the notion of a competition graph.
- A food web can be considered as a digraph.



Competition graph

Given a digraph D, the competition graph $C(D)$ of D is a (simple) graph such that
$V(C(D)) = V(D)$;
$E(C(D)) = \{uv \mid \exists x \in V(D) \text{ s.t.} (u,x),(v,x) \in A(D)\}$.
A graph G is said to be a *competition graph* if there exists a digraph D such that $C(D) = G$.



Competition graph

It is applicable to ...

- Coding
- Channel assignment in communications
- Modeling of complex systems arising from study of energy and economic systems
- Spread of opinions/influence in decision-making situations
- Information transmission in computer and communication networks

(For more history and background, please refer to http://dimacs.rutgers.edu/Events/2011/abstracts/roberts.html)

Research on Competition graphs

What kind of research topics are people working on?

1. Characterize competition graphs or, study the properties or structures of competition graphs of some special digraphs.

   What is the characterization of the competition graph of a tournament?
   What kind of properties does the competition graph of a transitive digraph have?
   If the degrees of a digraph are restricted, then can you describe the property of the competition graph of the digraph?
   What are the properties of the digraphs whose competition graphs are interval?

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Competition graph of an acyclic digraph

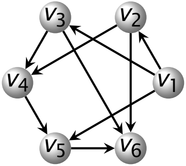
Is any graph the competition graph of an acyclic digraph....???

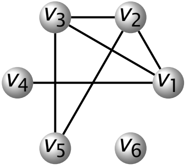
NO!

$*$ A digraph D is acyclic if and only if there exists an ordering $v_1, v_2, \ldots, v_n$ of the vertices such that if $(v_i, v_j) \in A(D)$ then $i < j$.

Observation

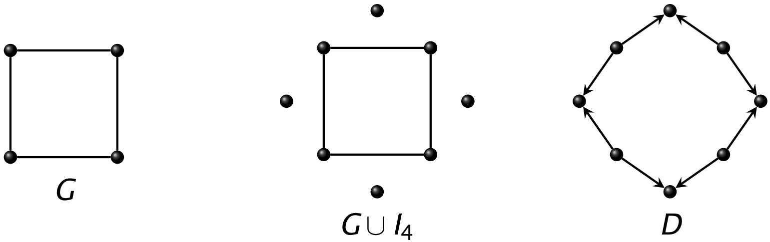
The competition graph of an acyclic digraph has an isolated vertex.





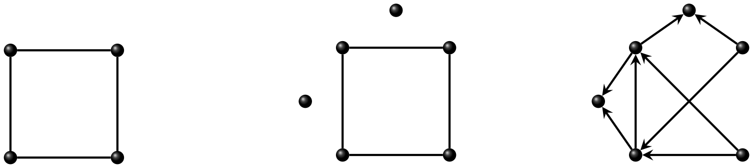
Competition graph of an acyclic digraph

Roberts [1978] observed that every graph can be made into the competition graph of an acyclic digraph by adding isolated vertices.



Competition number

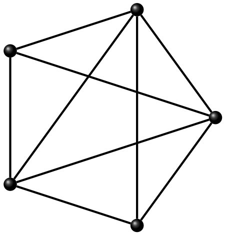
The competition number $k(G)$ of a graph G is defined to be the smallest nonnegative integer k such that there exists an acyclic digraph D satisfying $C(D) = G \cup I_k$.
(I_k: the set of k isolated vertices)



G $\qquad$ $G \cup I_2$ $\qquad$ D

Edge clique cover

- A *clique* of a graph G is a subset of $V(G)$ such that its induced subgraph of G is a complete graph.
- An *edge clique cover* of a graph G is a (multi) family of cliques of G such that the endpoints of each edge of G are contained in some clique in the family.
- The *edge clique cover number* $\theta_e(G)$ of a graph G is the minimum size of an edge clique cover of G.



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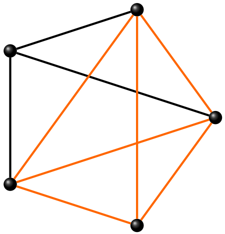

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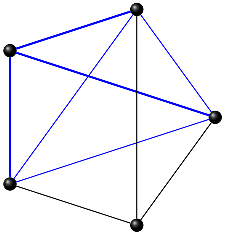

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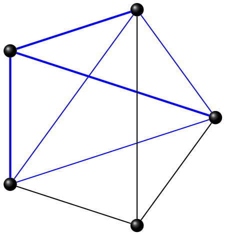
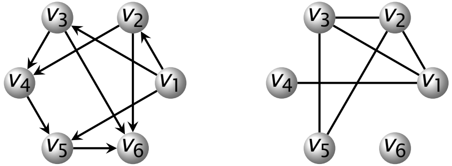


A graph G with $\theta_e(G) = 2$.

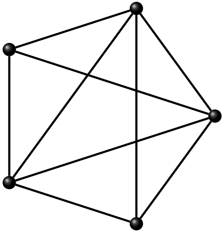
Observation

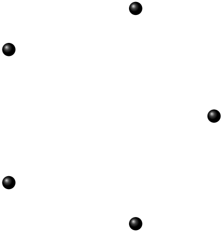
Given a digraph D, the in-neighborhood of a vertex in D forms a clique of $C(D)$.



Competition graph of an acyclic digraph

Observation





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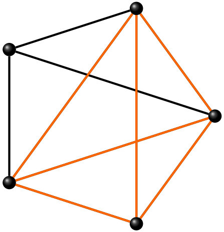

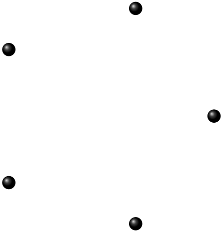

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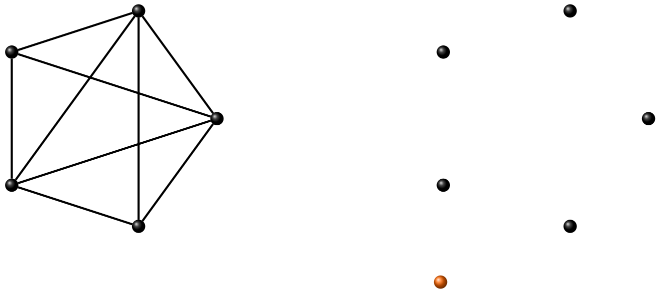

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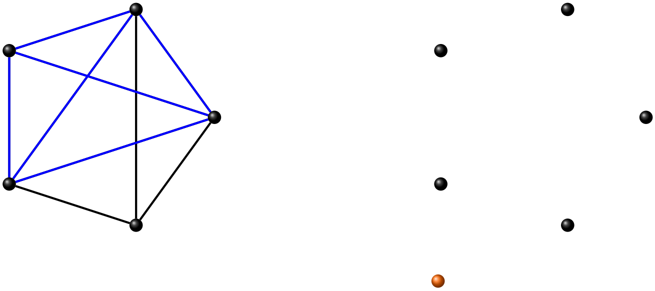

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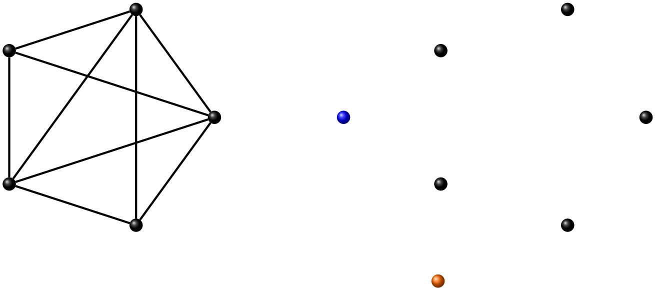

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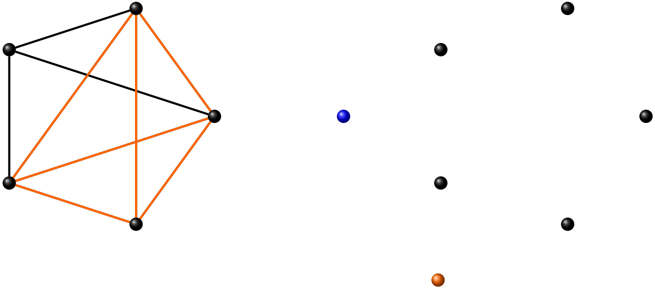

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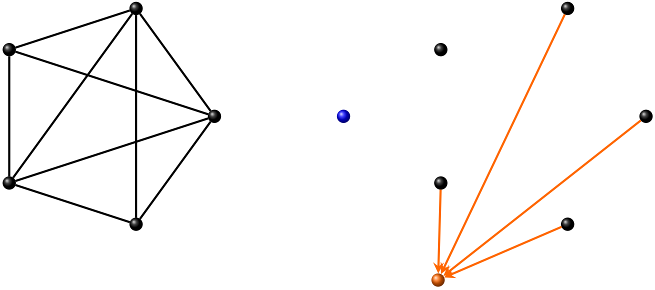

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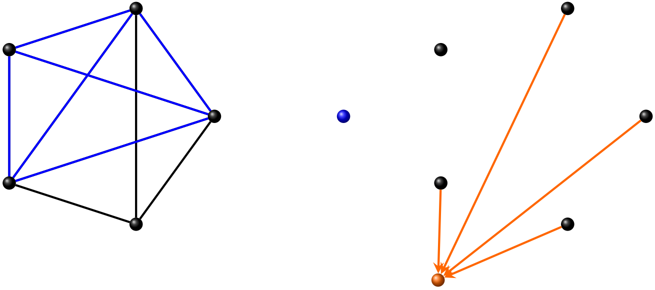

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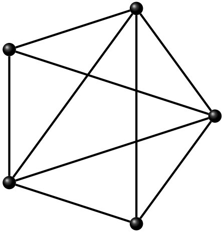

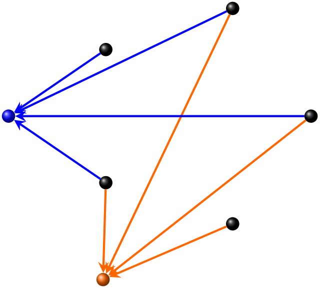

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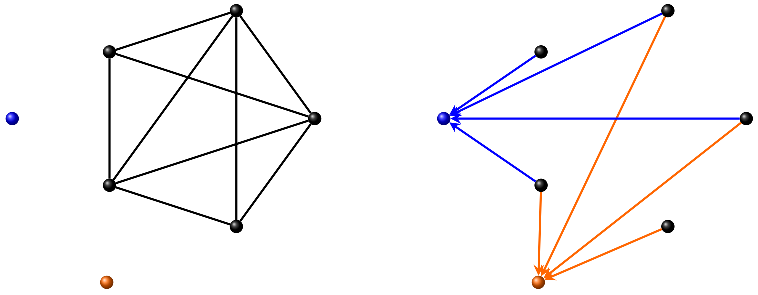

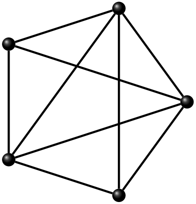

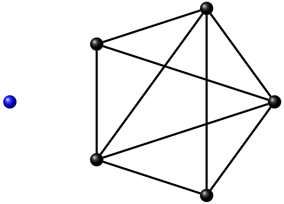

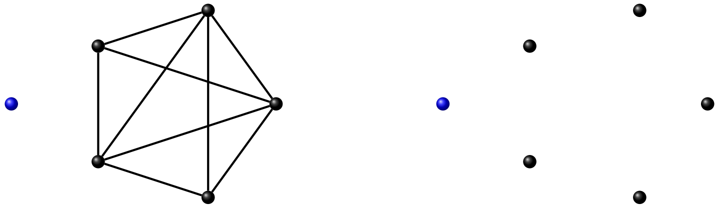

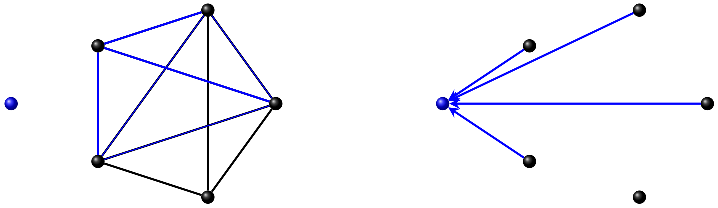

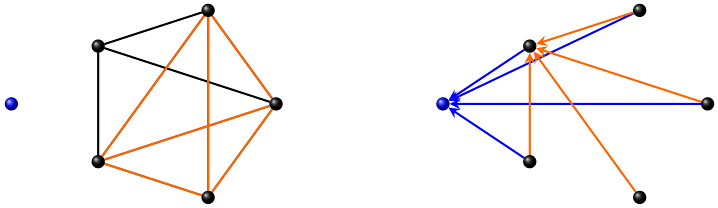

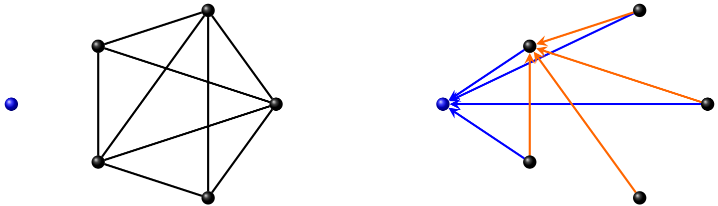

Since G is nontrivial connected and D is an acyclic digraph such that $C(D) = G \cup I_1$, $k(G) = 1$.

Characterization results

Theorem (Dutton and Brigham, 1983)

Suppose that G is a graph with n vertices. Then G is the competition graph of an acyclic digraph if and only if G has an edge clique cover $\{C_1, C_2, ..., C_n\}$ and a labeling $v_1, v_2, ... v_n$ of the vertices such that if $v_i \in C_j$, then $i > j$.

Theorem (Lundgren and Maybee, 1983)

If G is a graph with n vertices and $m < n$, then $k(G) \leq m$ if and only if G has an edge clique cover $\{C_1, C_2, ..., C_{n+m-2}\}$ and a labeling $v_1, v_2, ..., v_n$ of the vertices so that if $v_i \in C_j$, then $i \geq j - m + 1$.

- If G is a nontrivial triangle-free connected graph, then $k(G) = |E(G)| - |V(G)| + 2$. (Roberts [1978])
- If G is a chordal graph with no isolated vertex, then $k(G) = 1$. (Roberts [1978])

- The computation of the competition number of a graph is an NP-hard problem.(Opsut [1982])

- For a graph G,

$$\theta_e(G) - |V(G)| + 2 \leq k(G) \leq \theta_e(G);$$

$$k(G) \geq \min\{\theta_v(N_G[x]) \mid x \in V(G)\}.$$

(Opsut [1982])

Research on competition numbers

There are efforts to find a relationship between the competition number and a structural property of a graph.

- Graphs with bounded local vertex clique cover numbers
- Graphs with a large clique or with many holes
- Graphs with some special edge clique cover

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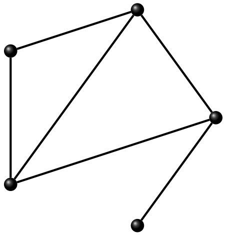
Competition numbers of line graphs

Theorem (Opsut, 1982)

For a line graph G, $k(G) \leq 2$.

The line graph $L(G)$ of a graph G is a graph such that $V(L(G)) = E(G)$ and $uv \in E(L(G))$ if and only if u and v have a common end point in G. A graph G is a line graph if $G = L(H)$ for some graph H.



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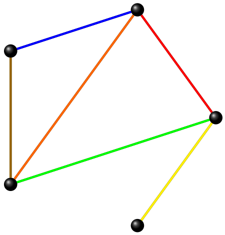

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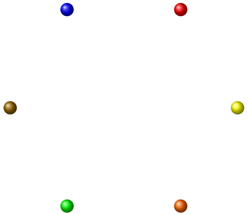

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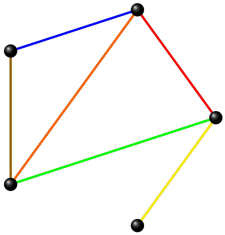

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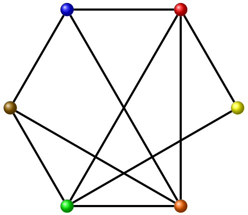

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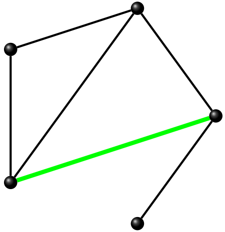

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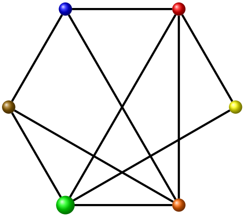

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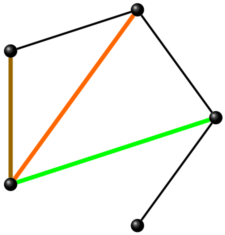

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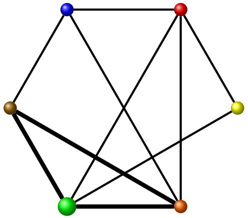

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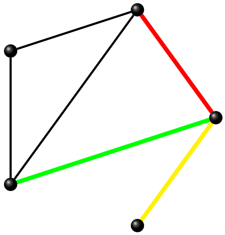

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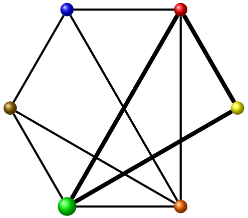

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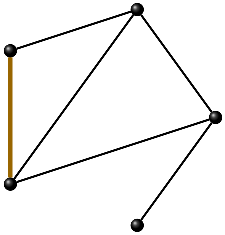

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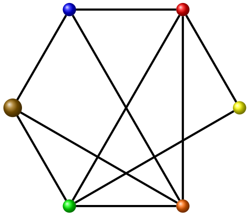

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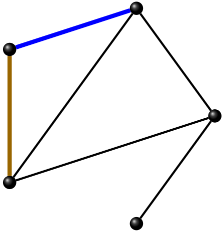

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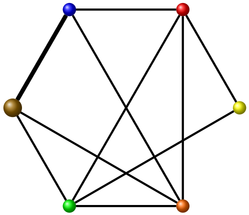

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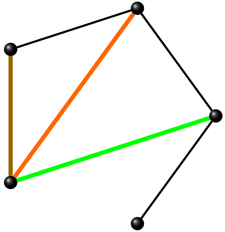
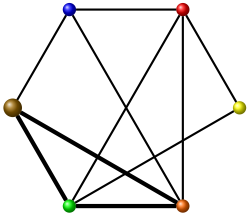


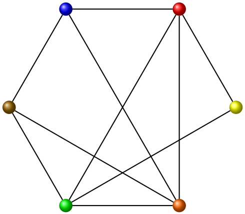
G



$L(G)$

Competition numbers of line graphs

$k(G) \leq 2$ for a line graph G



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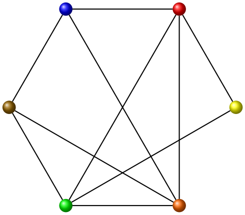

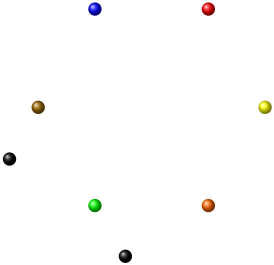

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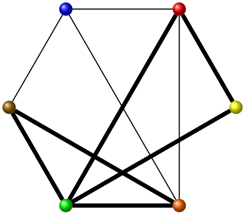

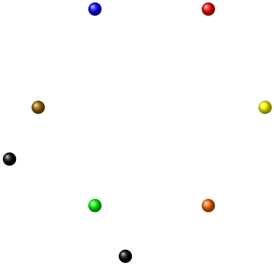

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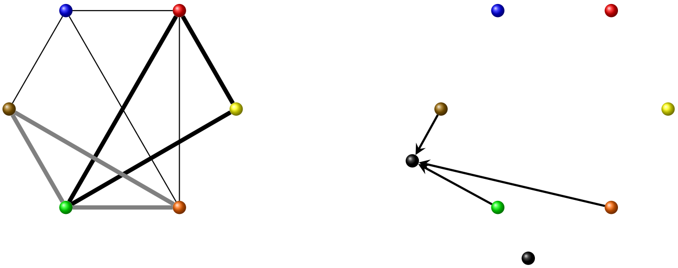

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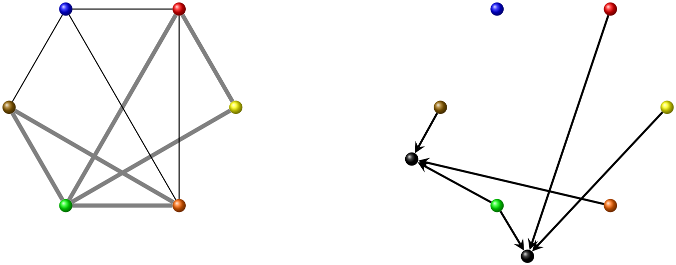

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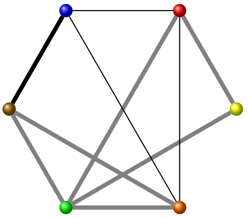

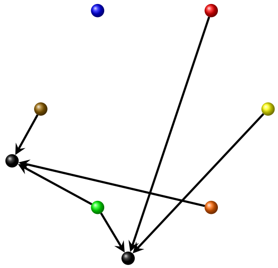

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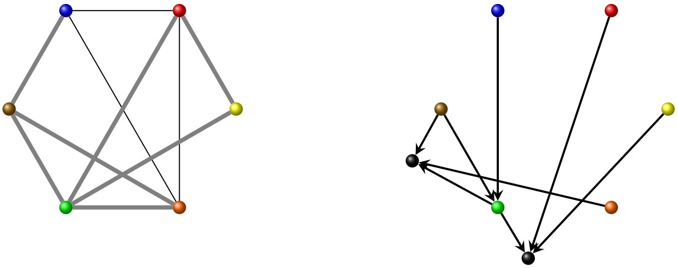

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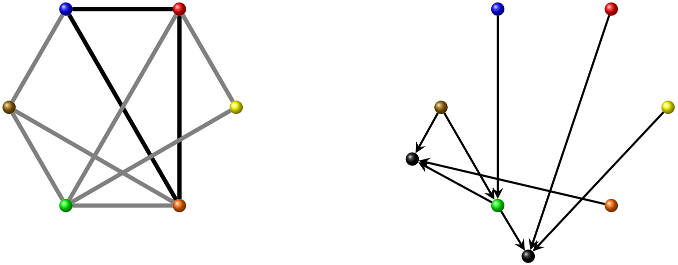

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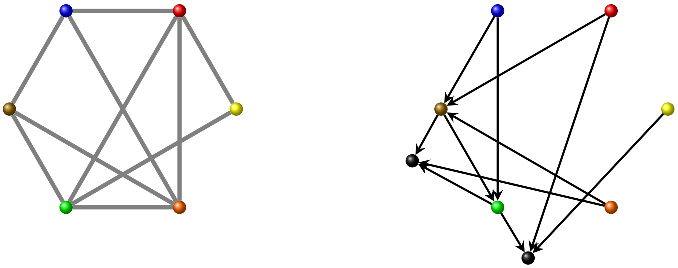

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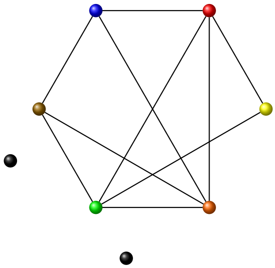

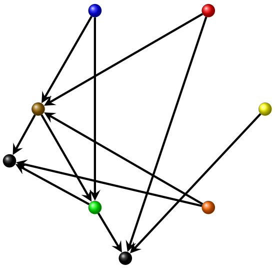

Opsut's Conjecture

Conjecture (Opsut, 1982)

For a quasi-line graph G, we have $k(G) \leq 2$, and $k(G) = 2$ if and only if $\theta_v(G[N_G[x]]) = 2$ for any vertex x.

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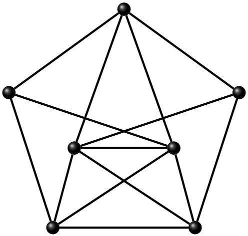

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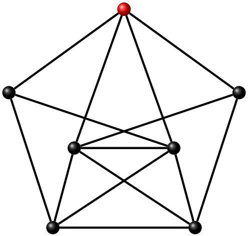

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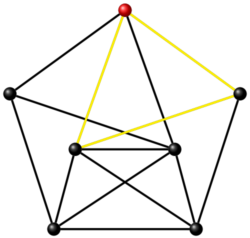

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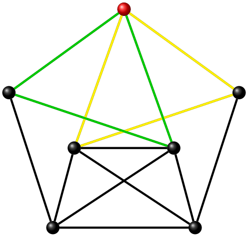

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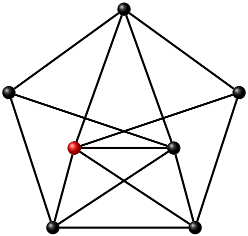

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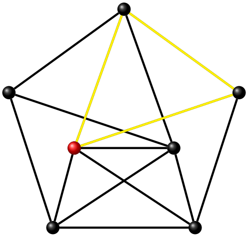

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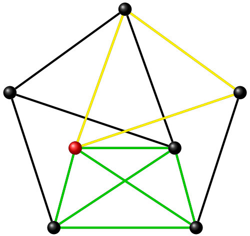

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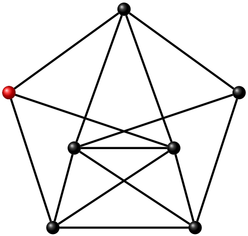

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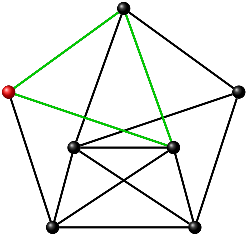

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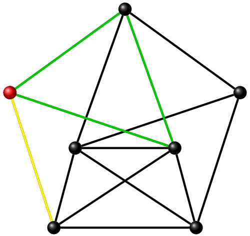

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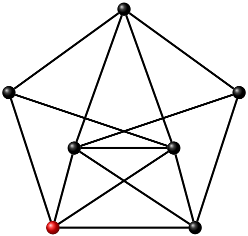

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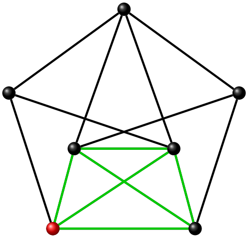

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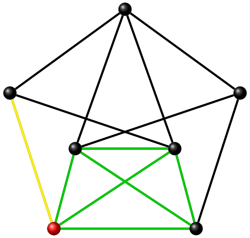


Attempts to solve Opsut's Conjecture

For a graph G and a vertex $v \in V(G)$, $\theta^*_G(v) = 1$ is and only if $\theta_v(G[N_G[v]]) = 1$; $\theta^*_G(v) = 2$ if $\theta_v(G[N_G[v]]) \neq 1$ and there exist two cliques C_1, C_2 such that $C_1 \cup C_2 = N_G[v]$ and such that for all $w \in C_1$, $\theta_v(G[N_G(w) - C_1]) \leq 1$.

(Given $x \in V(G)$, $N_G(x)$ is the set of neighbors of x in G and $N_G[x] = N_G(x) \cup \{x\}$.)

Theorem (Kim and Roberts, 1990)

For a graph G, if $\theta^*_G(v) \leq 2$ for all $v \in V(G)$, then $k(G) \leq 2$ and the equality holds if and only if $\theta^*_G(v) = 2$ for all $v \in V(G)$.

A graph G is called a critical graph if for each vertex x of G, $\theta_v(G[N_G[x]]) = 2$, and for each clique K of G, there is a vertex $x \in K$ such that $\theta_v(G[N_G(x) - K]) = 2$.

Theorem (Wang, 1992)

Opsut's conjecture is true for all non-critical graphs.

Opsut's conjecture is turned out to be true.

Theorem (McKay *et. al., 2011)*

Opsut's conjecture is true.

Theorem (Chudnovsky and Seymour, 2005)

Let G be a connected quasi-line graph. Then G is either a fuzzy circular interval graph or a composition of fuzzy linear interval strips.

Further research

- What is competition number of a graph when the vertex clique cover number of each vertex is at most d?
- What is the competition number of claw-free graph?

Research on competition numbers

There are efforts to find a relationship between the competition number and a structural property of a graph.

- Graphs with bounded local vertex clique cover numbers
- Graphs with a large clique or with many holes
- Graphs with some special edge clique cover

Competition number and the number of holes

A *hole* of a graph is an induced cycle of size at least 4.
For a graph G, we denote by $H(G)$ the set of all holes of G.

Let $\mathbb{F}_2$ be the finite field of order 2. For a graph G, let $\mathbb{F}_2^{E(G)}$ denote the set of maps from $E(G)$ to $\mathbb{F}_2$.

For a cycle C of G, we define a map $\chi_C : E(G) \to \mathbb{F}_2$ by

$$\chi_C(e) := \begin{cases} 1 & \text{if } e \in E(C); \\ 0 & \text{otherwise.} \end{cases}$$

(We may regard χ_C as a vector in $\mathbb{F}_2^{|E(G)|}$ once an edge labeling is given.)

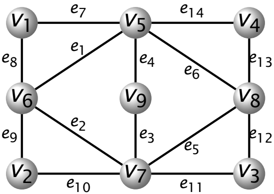
For a graph G, the *hole space* $\mathcal{H}(G)$ of G is a vector space defined by

$$\mathcal{H}(G) := \mathrm{Span}\{\chi_C \in \mathbb{F}_2^{E(G)} \mid C \in H(G)\}.$$

The *hole dimension* of a graph G is $\dim \mathcal{H}(G)$.

Hole space

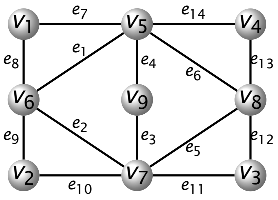
Example



A graph G

Hole space

Example



A graph G

There are exactly three holes

$C_1 := v_5 v_6 v_7 v_9 v_5$, $C_2 := v_5 v_9 v_7 v_8 v_5$
$C_3 := v_5 v_6 v_7 v_8 v_5$.

$$\chi_{C_1} = (1,1,1,1,0,0,0,0,0,0,0,0,0,0);$$
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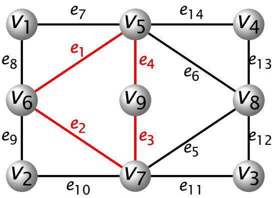

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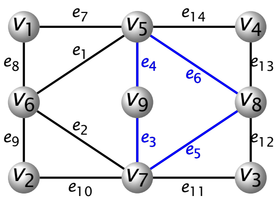

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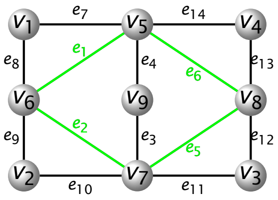

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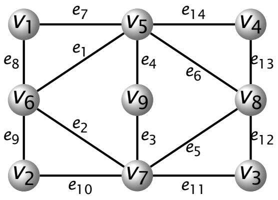

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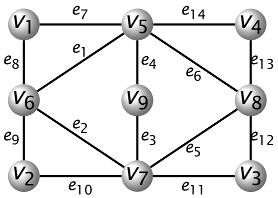

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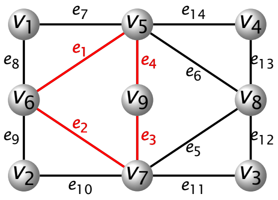
Hole space

Example



A graph G

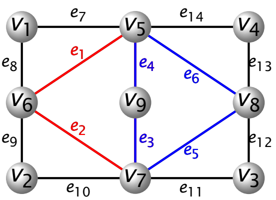
There are exactly three holes
$C_1 := v_5 v_6 v_7 v_9 v_5$, $C_2 := v_5 v_9 v_7 v_8 v_5$
$C_3 := v_5 v_6 v_7 v_8 v_5$.

$$\chi_{C_1} = (1,1,1,1,0,0,0,0,0,0,0,0,0,0);$$
$$\chi_{C_2} = (0,0,1,1,1,1,0,0,0,0,0,0,0,0);$$
$$\chi_{C_3} = (1,1,0,0,1,1,0,0,0,0,0,0,0,0).$$

$$\mathcal{H}(G) = \text{Span}\{\chi_{C_1}, \chi_{C_2}, \chi_{C_3}\}$$

Hole space

Example



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Hole space

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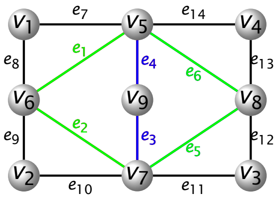

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$\mathcal{H}(G) = \text{Span}\{\chi_{C_1}, \chi_{C_2}, \chi_{C_3}\}$

$\dim \mathcal{H}(G) = 2$.

Theorem (Roberts, 1978)

If G is a chordal graph, then $k(G) \leq 1$.

Theorem (Kim, 2005)

For a graph G with exactly one hole, $k(G) \leq 2$.

Kim [2005] asked if the structure of holes in a graph is related to the competition number of a graph.

(Letting $C(D) = G$, any three vertices of a hole of G cannot have a common out-neighbor in D.)

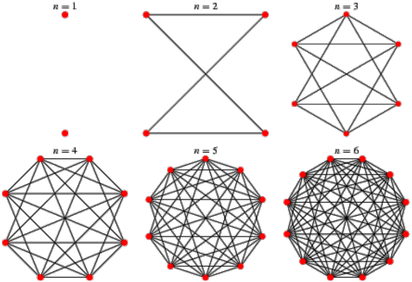
⋮

Theorem (McKay *et al.*, 2011)

For any graph G, $k(G) \leq \dim \mathcal{H}(G) + 1$.

$\dim \mathcal{H}(G) + 1$ is too large

Consider the cocktail party graph K_2^m:



Theorem (Lee *et al.*, 2011)

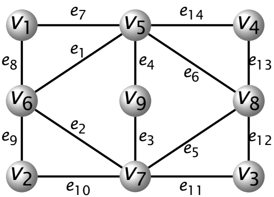
If all holes of a graph G are pairwise edge-disjoint and $\omega(G) - 1 \leq \dim \mathcal{H}(G) \leq 2\omega(G)$, then $k(G) \leq \dim \mathcal{H}(G) - \omega(G) + 3$.

Further Research

- What kind of graphs do satisfy the inequality $k(G) \leq \dim \mathcal{H}(G) - \omega(G) + 3$?
- We may define the triangle space as follows:

For a graph G, the *triangle space* $\mathcal{T}(G)$ of G is defined by

$$\mathcal{T}(G) := \text{Span}\{\chi_C \in \mathbb{F}_2^{E(G)} \mid C \text{ is a triangle of } G\}.$$



A graph G

There are exactly four triangles

$$\chi_{T_1} = (1,0,0,0,0,0,1,1,0,0,0,0,0,0);$$
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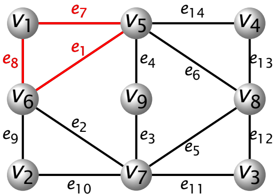

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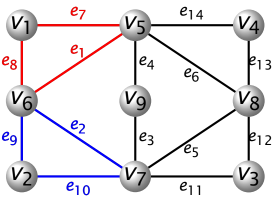

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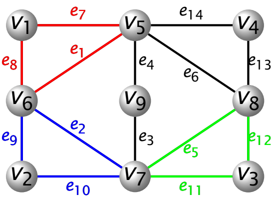

A graph G

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$$\begin{aligned}
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\end{aligned}$$

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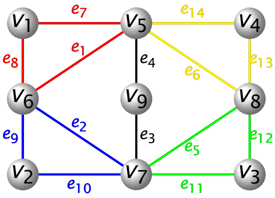

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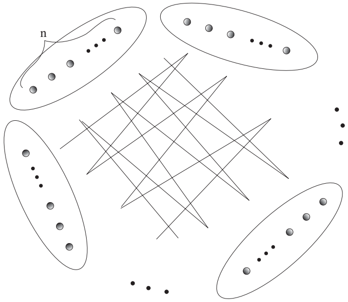
Research on competition numbers

There are efforts to find a relationship between the competition number and a structural property of a graph.

- Graphs with bounded local vertex clique cover numbers
- Graphs with a large clique or with many holes
- Graphs with some special edge clique cover

Complete multipartite graphs

We denote by K_n^m the complete multipartite graph on m partite sets in which each partite set has n vertices.



Theorem (Roberts, 1978)

For an integer $n \geq 2$,

$$k(K_n^2) = n^2 - 2n + 2.$$

However, it is not easy to compute $k(K_n^m)$.

Mutually orthogonal Latin squares

For a positive integer n, a *Latin square* of order n is an $n \times n$ array L in which every cell contains an element of $[n] = \{1, 2, \ldots, n\}$ such that every row of L is a permutation of $[n]$ and every column of L is a permutation of $[n]$.
For a Latin square L, we denote the (i, j)-element of L by $L(i, j)$.

Example

$L =$

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

$L(1, 4) = 4$, $L(2, 3) = 4$, ...

Mutually orthogonal Latin squares

For two Latin squares L_1 and L_2 of order n, L_1 and L_2 are orthogonal if for any $i, j \in [n]$, there is a unique $(i^*, j^*) \in [n] \times [n]$ such that $L_1(i^*, j^*) = i$ and $L_2(i^*, j^*) = j$.

$L_1 =$

1	2	3	4	5
2	3	4	**5**	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

$L_2 =$

1	2	3	4	5
3	4	5	**1**	2
5	1	2	3	4
2	3	4	5	1
4	5	1	2	3

$(L_1, L_2) =$

(1,1)	(2,2)	(3,3)	(4,4)	(5,5)
(2,3)	(3,4)	(4,5)	(5,1)	(1,2)
(3,5)	(4,1)	(5,2)	(1,3)	(2,4)
(4,2)	(5,3)	(1,4)	(2,5)	(3,1)
(5,4)	(1,5)	(2,1)	(3,2)	(4,3)

Mutually orthogonal Latin squares

- The largest size of a family of mutually orthogonal Lain squares of order n is denoted by $L(n)$.
- It is known that $L(n) \leq n-1$ and that $L(n) = n-1$ for a prime power n.

Orthogonal Latin squares and $\theta_e(K_n^m)$

Let m and n be integers such that $2 \leq m \leq L(n) + 2$. Since $m - 2 \leq L(n)$, there exists a family $\mathcal{L} = \{L_1, L_2, \ldots, L_{m-2}\}$ of mutually orthogonal Latin squares of order n.

Consider the complete multipartite graph K_n^m, and we denote by v_j^ℓ the jth vertex in the ℓth partite set for $\ell \in [m]$ and $j \in [n]$.

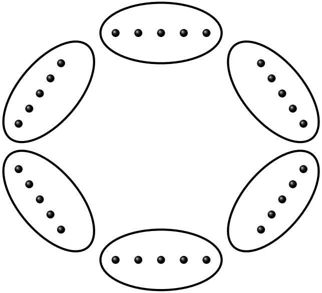
We define a set S_{ij} of vertices for $i, j \in [n]$ as follows:

$$S_{ij} = \{v_i^1, v_j^2, v_{L_1(i,j)}^3, v_{L_2(i,j)}^4, \ldots, v_{L_{m-2}(i,j)}^m\}.$$

$$\mathcal{S} := \{S_{ij} \mid i, j \in [n]\}.$$

Orthogonal Latin squares and $\theta_e(K_n^m)$

For the complete multipartite graph K_5^6, there exist 4 mutually orthogonal Latin squares of order 5.
For $i, j \in \{1, 2, \ldots, 5\}$, $S_{ij} = \{v_i^1, v_j^2, v_{L_1(i,j)}^3, v_{L_2(i,j)}^4, v_{L_3(i,j)}^5, v_{L_4(i,j)}^6\}$



$L_1 =$

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

$L_2 =$

1	2	3	4	5
3	4	5	1	2
5	1	2	3	4
2	3	4	5	1
4	5	1	2	3

$L_3 =$

1	2	3	4	5
4	5	1	2	3
1	3	4	5	1
5	1	2	3	4
3	4	5	1	2

$L_4 =$

1	2	3	4	5
5	1	2	3	4
4	5	1	2	3
3	4	5	1	2
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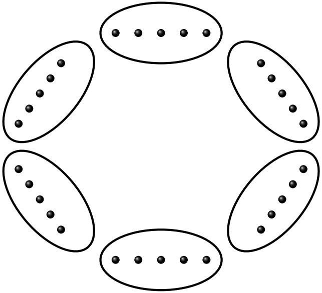
$S_{24} = \{v_2^1, v_4^2, v_5^3, v_1^4, v_2^5, v_3^6\}$



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2	3	4	5	1
3	4	5	1	2
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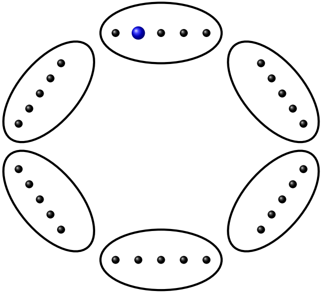

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4	5	1	**2**	3
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$L_4 =$

1	2	3	4	5
5	1	2	**3**	4
4	5	1	2	3
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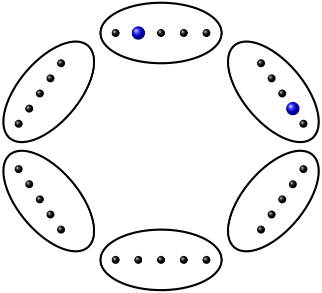

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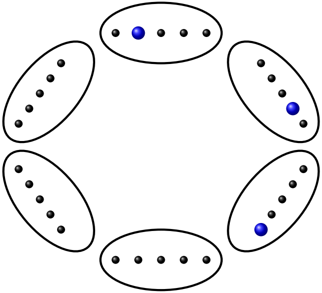

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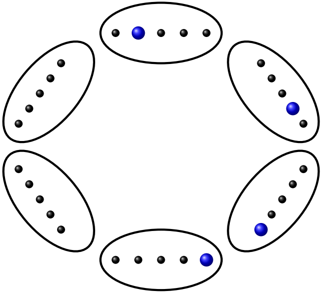

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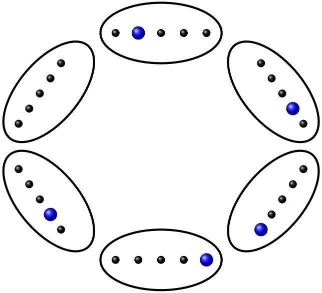

$L_1 =$

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
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Orthogonal Latin squares and $\theta_e(K_n^m)$

For the complete multipartite graph K_5^6, there exist 4 mutually orthogonal Latin squares of order 5.
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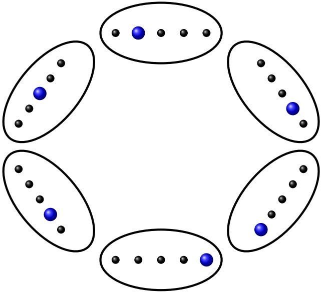

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2	3	4	**5**	1
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4	5	1	2	3
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5	1	2	3	4
2	3	4	5	1
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$L_3 =$

1	2	3	4	5
4	5	1	**2**	3
1	3	4	5	1
5	1	2	3	4
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$L_4 =$

1	2	3	4	5
5	1	2	**3**	4
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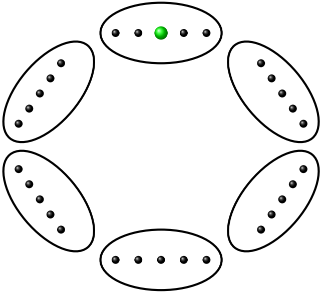

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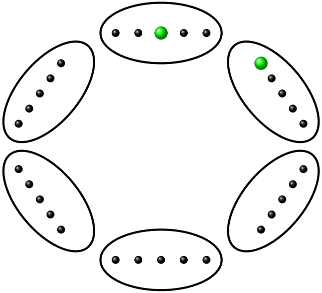

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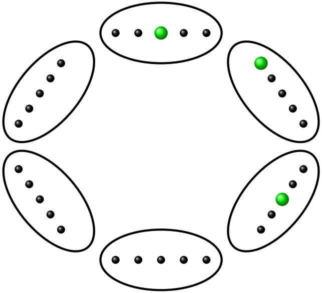

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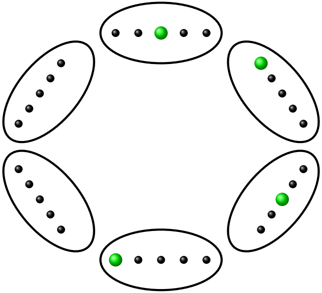

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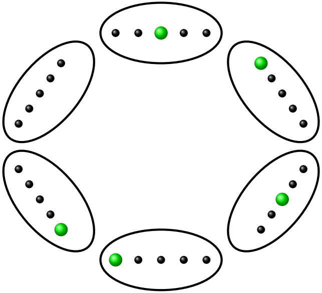

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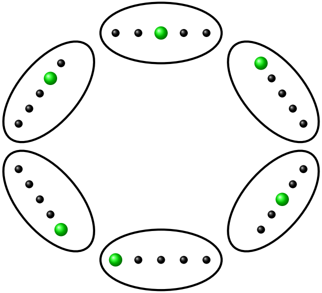

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Let m and n be positive integers such that $2 \leq m \leq L(n) + 2$. Then Park et al. (2009) showed the following are true:

(1) The family $\mathcal{S}$ is an edge clique cover of K_n^m of minimum size.

(2) $\theta_e(K_n^m) = n^2$.

Theorem (Kim and Sano, 2008)

For an integer $n \geq 2$, it holds that

$$k(K_n^3) = n^2 - 3n + 4.$$

Theorem (Park *et al.*, 2009)

For an odd integer $n \geq 5$

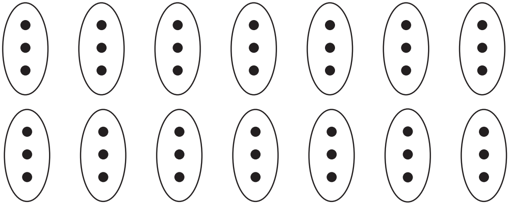
$$n^2 - 4n + 6 \leq k(K_n^4) \leq n^2 - 4n + 8.$$

Theorem (Park *et al.*, 2009)

For any integers m, n such that $3 \leq m \leq L(n) + 2$,

$$k(K_n^m) \leq n^2 - n + 1.$$

When we have many partite sets...,



(Each ellipse is a partite set, imagine that there are edges between vertices belonging to distinct ellipse.)

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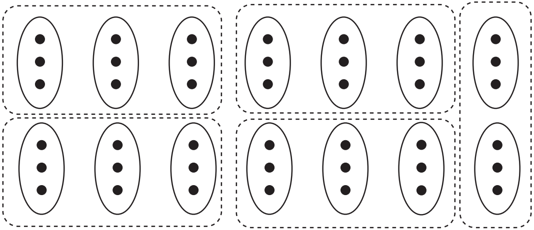


(Each ellipse is a partite set, imagine that there are edges between vertices belonging to distinct ellipse.)

Complete multipartite graphs with many partite sets

Theorem

Let n be a positive integer such that $3 \leq n \leq L(n) + 2$. For any m such that $m \geq n$,

$$k(K_n^m) \leq n^2 - n + 1.$$

- For $m \geq 2$, $k(K_2^m) = 2$.
- For $m \geq 3$, $k(K_3^m) = 4$.

Further Research

- Determine whether $k(K_n^4) = n^2 - 4n + 7$ or $k(K_n^4) = n^2 - 4n + 8$, when n is odd.
- Is it true that $k(K_n^m) \leq k(K_n^n)$ for any $m \geq n$?
- Is it true that

$$k(K_{n_1,n_2}) \geq k(K_{n_1,n_2,n_3}) \geq k(K_{n_1,n_2,n_3,n_4}) \cdots$$

for $n_1 \geq n_2 \geq n_3 \geq n_4 \cdots$?

It is interesting to see the competition number of a graph whose edge clique cover has a nice combinatorial property.

Research on Competition graphs

What kind of research topics are people working on?

1. Characterize competition graphs or, study the properties or structures of competition graphs of some special digraphs.
2. Compute the competition number of interesting graph families.
3. Introduce a concept of variants of competition graphs and then study related topics corresponding to above.

Variants of competition graphs

There are many various variations of competition graphs that have been introduced and whose properties have been studied:

- common enemy graphs (introduced by Lundgren and Maybee)
- competition-common enemy graphs (introduced by Scott)
- niche graphs (introduced by Cable *et. al.*)
- phylogeny graphs (introduced by Roberts and Sheng)
- competition hypergraphs (introduced by Sonntag and Teichert)
- p-competition graphs (introduce by Kim *et. al.*)
- tolerance competition graphs (introduced by Brigham *et. al.*)
- m-step competition graphs (introduced by Cho *et. al.*)

⋮

Competition common enemy graph (Sccot [1987])

Given a digraph D, the competition common enemy graph $CCE(D)$ of D is a graph such that
(a) $V(CCE(D)) = V(D)$;
(b) $uv \in E(CCE(D))$
$\Leftrightarrow$ u and v have both a common prey and a common enemy.

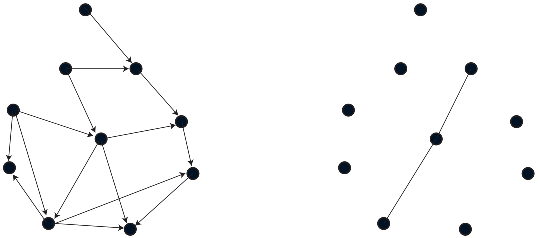


Figure: A digraph D and its competition common enemy graph $CCE(D)$

m-step competition graph (Cho *et. al.* [2000])

A vertex y is called an m-step prey of a vertex x in a digraph D if there is a directed path of length m from x to y.

(that is, there exist $z_1, z_2 \ldots, z_{m-1} \in V(D)$ such that $(x, z_1), (z_1, z_2), \ldots, (z_{m-1}, y)$ belong to $A(D)$.)

Given a digraph D, the m-step competition graph $C^m(D)$ of D is a graph such that
(a) $V(C^m(D)) = V(D)$;
(b) $uv \in E(C^m(D)) \Leftrightarrow u$ and v have an m-step common prey.

Note that $C^1(D) = C(D)$.

m-step competition graph

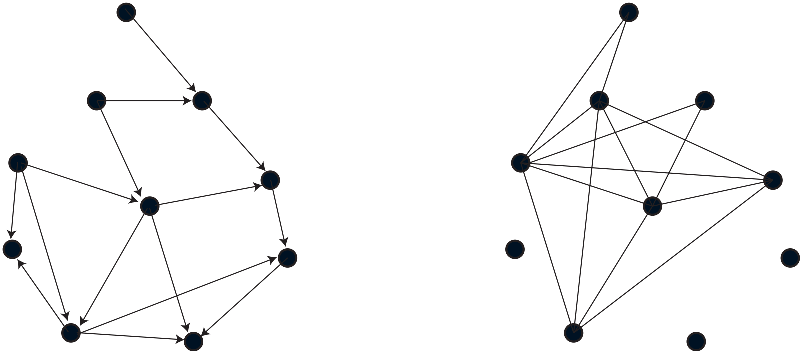
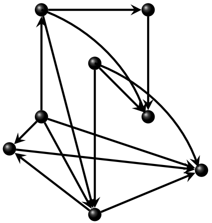


Figure: A digraph D and its 2-step competition graph $C^2(D)$

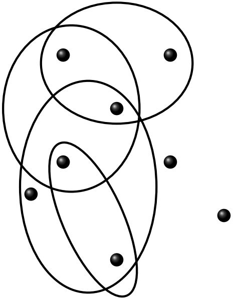
Competition hypergraphs

We assume that all hypergraphs have no loops and no multiple hyperedges.

Given a digraph D, the competition hypergraph $\mathcal{CH}(D)$ of D is a hypergraph such that $V(\mathcal{CH}(D)) = V(D)$ and $E(\mathcal{CH}(D)) = \{N_D^-(v) \mid v \in V(D), |N_D^-(v)| \geq 2\}$.



D



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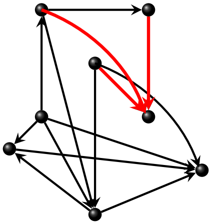

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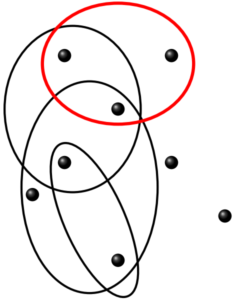

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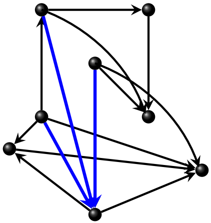

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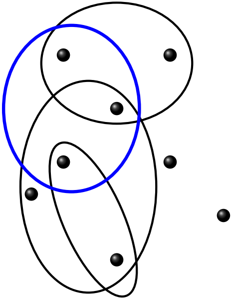

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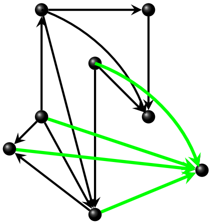

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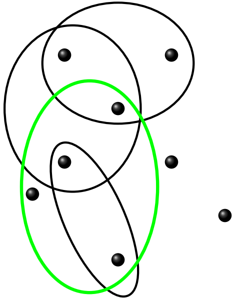

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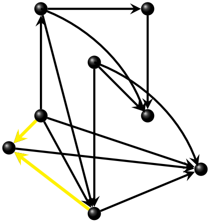

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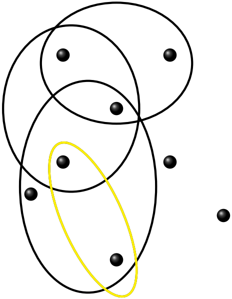
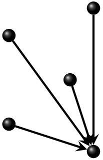
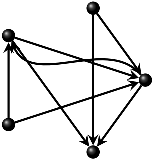


$\mathcal{CH}(D)$

Competition hypergraphs

Competition graphs vs Competition hypergraphs





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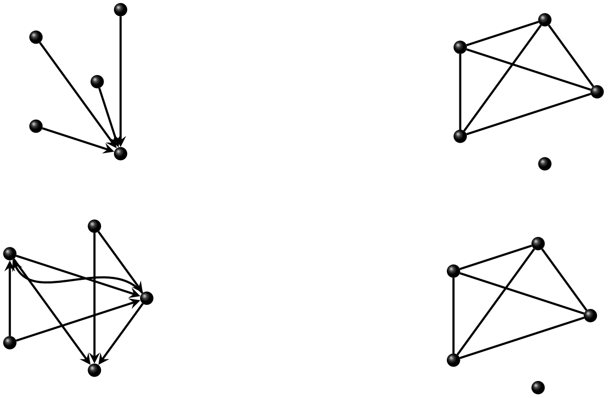
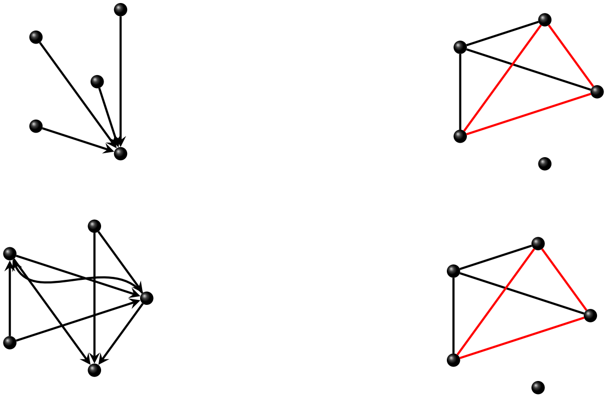


Competition hypergraphs

Competition graphs vs Competition hypergraphs



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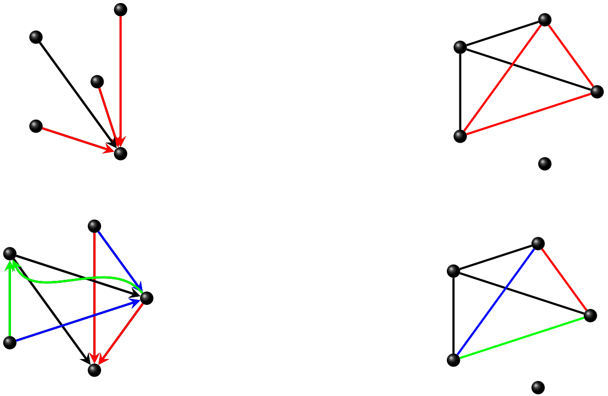

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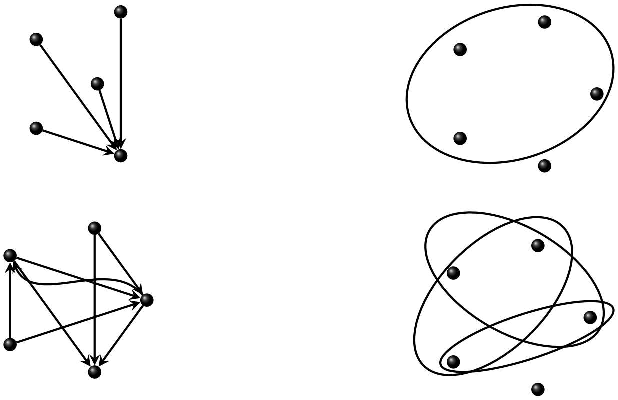


Hypercompetition numbers

Any hypergraph can be made into the competition hypergraph of some acyclic digraph if we are allowed to add sufficiently many isolated vertices.

Hypercompetition number

The hypercompetition number $hk(\mathcal{H})$ of a hypergraph $\mathcal{H}$ is defined to be the smallest number k such that $\mathcal{H} \cup I_k$ is the competition hypergraph of some acyclic digraph.

A hypercompetition number version of Opsut's bound

Opsut (1982) showed that for a graph G,

- $k(G) \geq \theta_e(G) - |V(G)| + 2$,
- $k(G) \geq \min_{x \in V(G)} \theta_v(N_G[x])$.

Park and Sano (2011) showed that for a hypergraph $\mathcal{H}$,

- $hk(\mathcal{H}) \geq |E(\mathcal{H})| - |V(\mathcal{H})| + \min_{e \in E(\mathcal{H})} |e|$,
- $hk(\mathcal{H}) \geq \min_{v \in V(\mathcal{H})} \deg_{\mathcal{H}}(v)$.

A hypercompetition number version of the Opsut's Conjecture

[Opsut's Conjecture (Opsut, 1982)]
For a quasi-line graph G, $k(G) \leq 2$, and the equality holds if and only if $\theta_v(G[N_G[x]]) = 2$ for any vertex x.

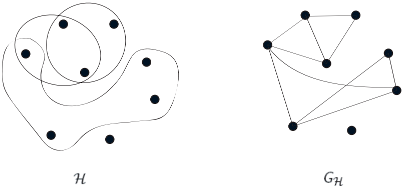
Theorem (Park and Kim, 2011)

For a hypergraph $\mathcal{H}$ with degree at most two, $hk(\mathcal{H}) \leq 2$, and the equality holds if and only if $\deg_{\mathcal{H}}(x) = 2$ for any vertex x.

Remarks

For a hypergraph $\mathcal{H}$, let $G_{\mathcal{H}}$ be the graph defined by

$$V(G) = V(\mathcal{H}), \quad E(G) = \{uv \mid \{u, v\} \subset e \text{ for some } e \in E(\mathcal{H})\}.$$



Let $\mathcal{G}$ be the set of graphs G such that $G = G_{\mathcal{H}}$ for some hypergraph $\mathcal{H}$ with degree at most two.

$\mathcal{G} \subset$ (the family of quasi-line graphs)

Research on Competition graphs

What kind of research topics are people working on?

1. Characterize competition graphs or, study the properties or structures of competition graphs of some special digraphs.
2. Compute the competition number of interesting graph families.
   - Graphs with bounded local vertex clique cover numbers
   - Graphs with a large clique or with many holes
   - Graphs with some special edge clique cover
3. Introduce a concept of variants of competition graphs and then study related topics corresponding to above.

Thank you for your attention.