The Firefighter Problem on d-dim. Grids and Hartke's Conjecture

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Let H be an inf. d-dimensional grid (any positive integer d). Consider the following game on H (played over Steps $0, 1, 2, \ldots$):

(1) Each vtx in H is either *fireproofed*, *on-fire*, or *suseptible*, and each vertex that becomes fireproofed or on-fire stays that way through all the later steps.

(2) At Step 0, a finite subset U_0 of the vertices in H are on-fire, the rest, suseptible.

(3) For each Step l and suseptible vtx v adj in H to a vertex on-fire, either v becomes fire-proofed or on-fire during Step $l + 1$.

(4) For each Step l, only a limited number $f(l)$ of vertices can become fire-proofed each step (we choose the set), AND none of these vertices may be on fire already.

Our object is to *contain the fire*–bound the number of vertices that become on-fire.

An infinite d-dim. grid H is the graph w/

$V(H) = \{(a_1, \ldots, a_d) |\ a_j$ integers$\}$,

and two vertices are adjacent in H if they agree in $d-1$ coordinates and differ in the remaining coordinate by exactly 1. (So each vertex in H has degree $2d$.)

Prop: If $f(l) = kl^{d-2}$ and $k = k(d)$ is large enough, then for any finite set S_0 of vertices on-fire at Step 0, we can contain the fire by fire-proofing only $f(l)$ vertices each step (if we're clever enough).

Conjecture [Hartke]: If f is such that $f(l)/l^{d-2}$ vanishes for l large, then for some finite set S_0, it is impossible for us to contain the fire by fireproofing only $f(l)$ vertices each step (no matter how clever we are).

The point of this talk is to prove that Hartke's conjecture is correct!

A few observations:

Let $S_0, S_1, \ldots$ and $V_1, V_2, \ldots$ be sequences of subsets of $V(H)$ such that

$S_{i+1} = N_H(S_i) \setminus (V_1 \cup \ldots \cup V_{i+1})$, and

(I) If V_{i+1} is disjoint from S_i for each i, (iff each vtx in V_{i+1} is not on-fire by then), then one can check that indeed, no vertex in V_{i+1} is on-fire by Step $i+1$ (so we may fireproof), and that S_i is the set of vertices on-fire by Step i.

(II) Iff the S_i's are bounded, then we have contained the fire.

Definition: Let G be any infinite graph and let U_0 be any finite subset of G. Let $U_1, U_2, \ldots$ and $X_1, X_2, \ldots$ be sequences of sets of $V(G)$ such that $U_{i+1} = N_H(U_i) \setminus (X_1 \cup \ldots X_{i+1})$. We say that $X_1, X_2, \ldots$ is a *blocking sequence of* U_0 *in* G iff

(I) each X_{i+1} is disjoint from U_i, and

(II) the U_i's are bounded.

Thm 1 [Hartke's Conj true]: Let f be any function such that $f(t)/t^{d-2}$ vanishes for t large. Then there exists a finite S_0 such that there is no blocking sequence $V_1, V_2, \ldots$ of S_0 in H where $|V_l| \leq f(l)$ for each l. (i.e., for some set S_0 on-fire at Step 0, the fire is going to keep spreading no matter how clever we might be, if we get to fireproof only $f(l)$ vertices at each Step l).

And so the rest of our talk is to focus on the proof of Thm 1.