Algorithmic Problems in Epidemiology

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The first Network Approach to Epidemiology

- Made an implicit network between people and water pumps
- People who drank water from Broad St. pump died
- also proved that cholera spreads through water

"Riddle of the Cholera Outbreak" solved by Dr. John Snow

EpiSims/Simdemics[1] - a network approach to epidemiology

Main steps:

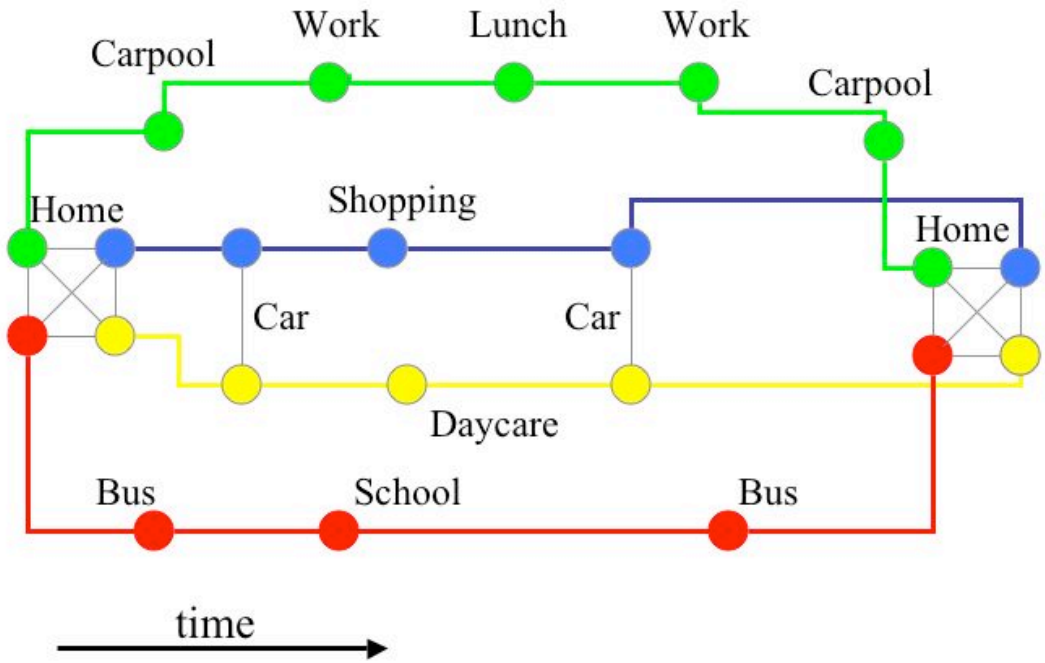
1. Construct a realistic social network
   - Not readily available
2. Develop an efficient disease simulation
   - Must scale to millions of people
   - Static analysis to guide simulation
3. Help in policy questions
   - Abstract out a simpler combinatorial model for the disease dynamics
   - whom to vaccinate/quarantine
   - where to place sensors

[1]S. Eubank et al., http://ndssl.vbi.vt.edu/

Constructing a Social Network

Main steps:

- Synthetic population from census data
- Location Information from census, land use, GIS, etc.
- Individual activities from surveys, traffic data, behavioral models, etc.

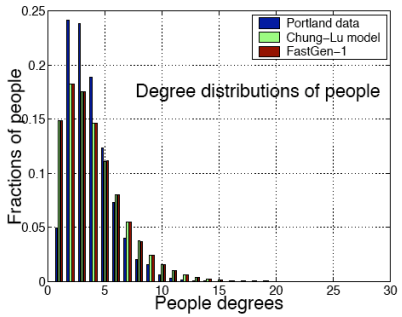


Main Difficulty: Slow and expensive

Can we construct graphs that look "like" real social networks?

Common approach: generative random graph models

The Social Network

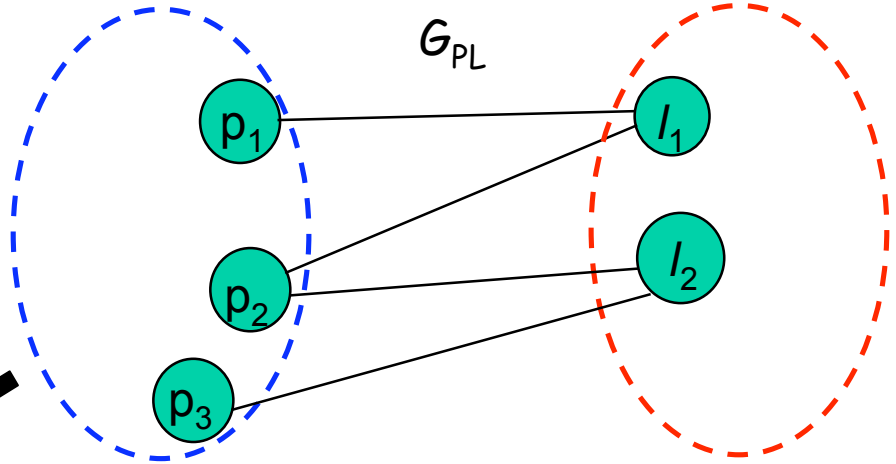


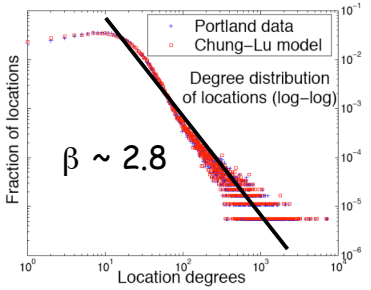
Portland: 1615860 people

people

locations

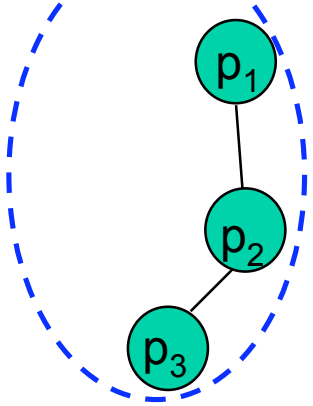
G_{PL}

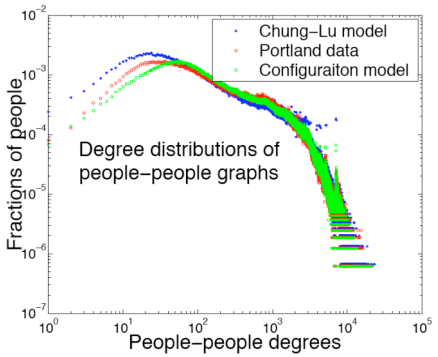




Portland: 181230 locations

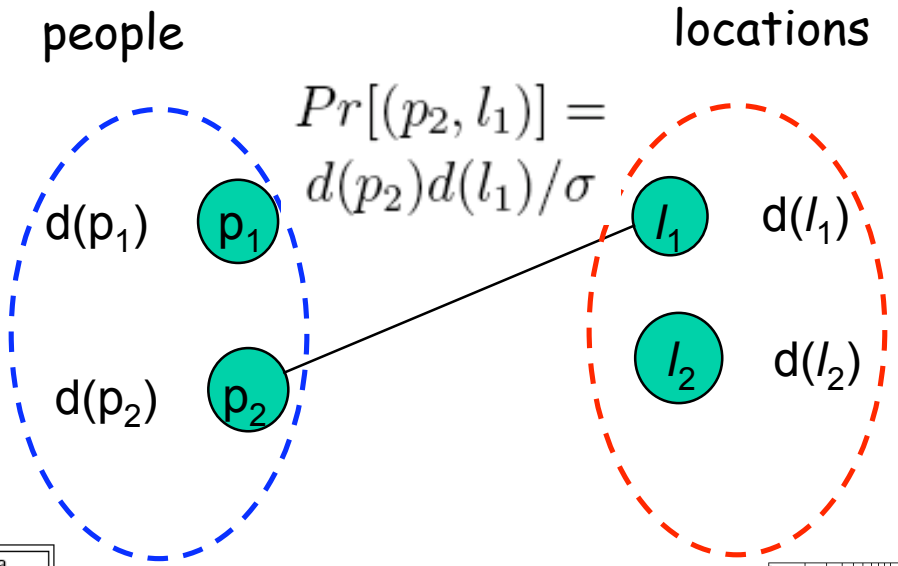
G_P



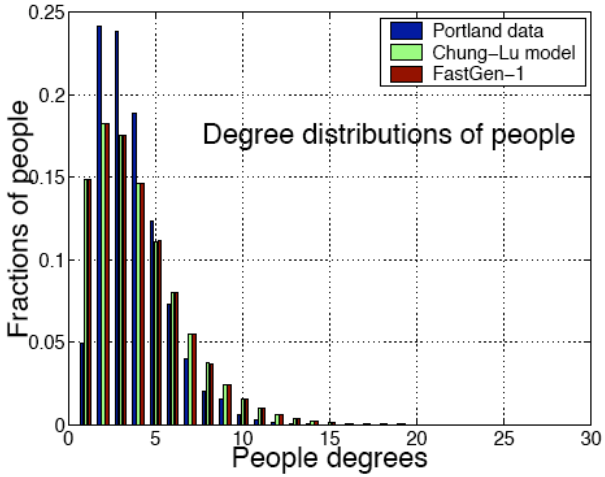


Giant component with 1615813 people

Generative Graph Models: Chung-Lu model



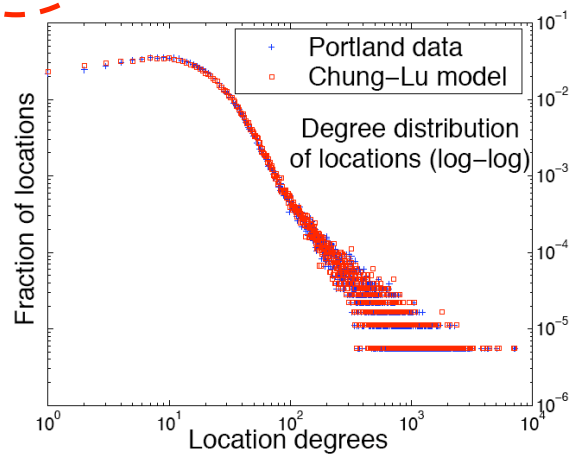
Degree distribution of people



$$\sigma = \sum_{p \in P} d(p) = \sum_{l \in L} d(l)$$

$$\max_{p \in P, l \in L} d(p)d(l) \leq \sigma$$

Degree distribution of locations

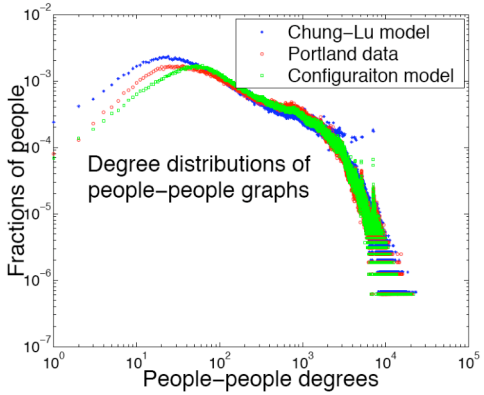


Connectivity in the people-people graph

Assume: d(p)=c, for each p∈ P

$d_0 \leq d(l) \leq d_1$, for each d(l) ∈ L

n_i: #locations with degree i ~ $c'|L|/i^{\beta}$



$$For\ p_1, p_2 \in P:$$

$$\Pr[(p_1,p_2) \notin G_P] = \Pi_{\ell \in L}(1 - \frac{d(p_1)d(p_2)d(\ell)^2}{\sigma^2})$$

$$\leq exp(-\sum_{\ell \in L} \frac{c^2 d(\ell)^2}{\sigma^2})$$

$$= exp(-\frac{1}{|P|^2}\sum_i \frac{i^2 n_i}{i^\beta})$$

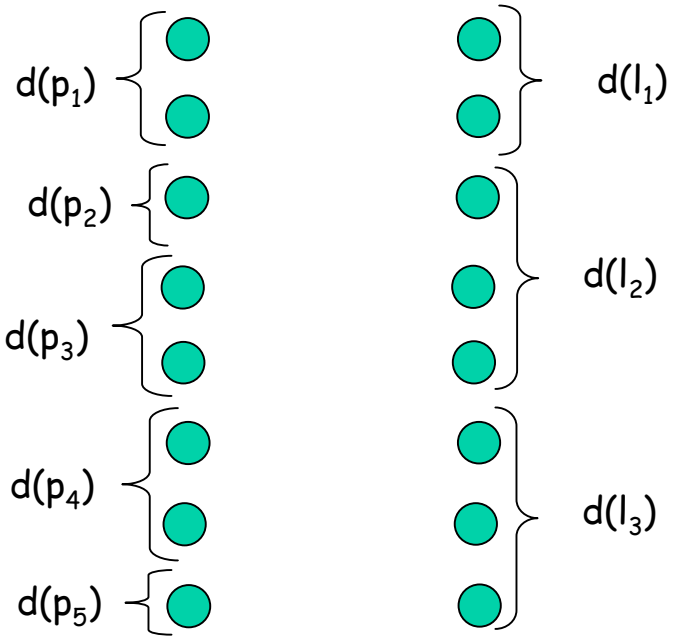
$$\simeq exp(-\frac{c'|L|}{|P|^2}\sum_{i=d_0}^{d_1} \frac{1}{i^{\beta-2}})$$

$$\simeq exp(-\frac{1}{|P|}\frac{c(\beta-2)d_0^{\beta-2}d_1^{3-\beta}}{3-\beta})$$

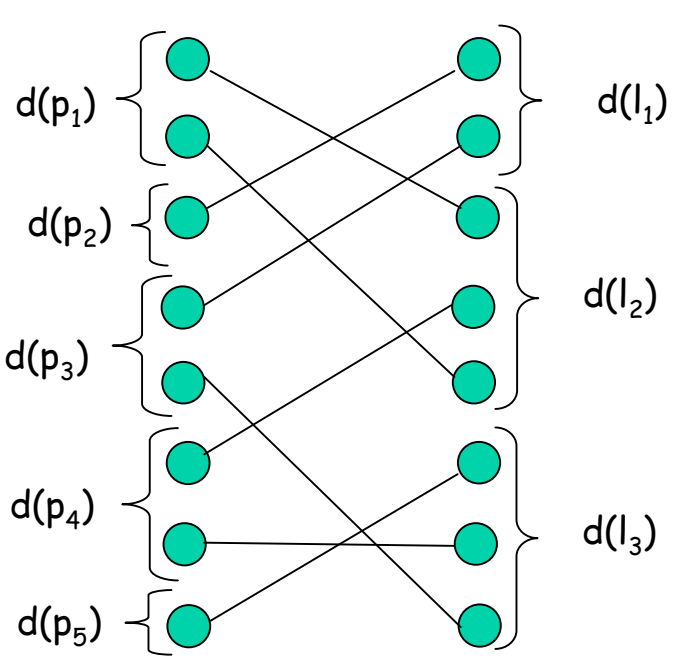
$$\ll exp(-\frac{2}{|P|}) \leq 1 - \frac{1}{|P|}$$

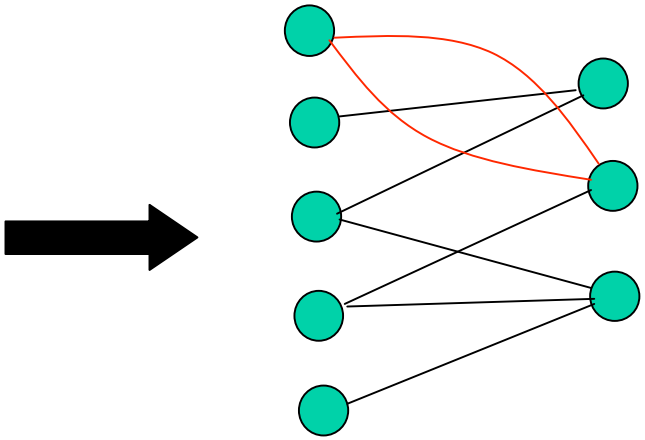
⇒ G_P of Bipartite Chung-Lu has a giant component

The Configuration Model

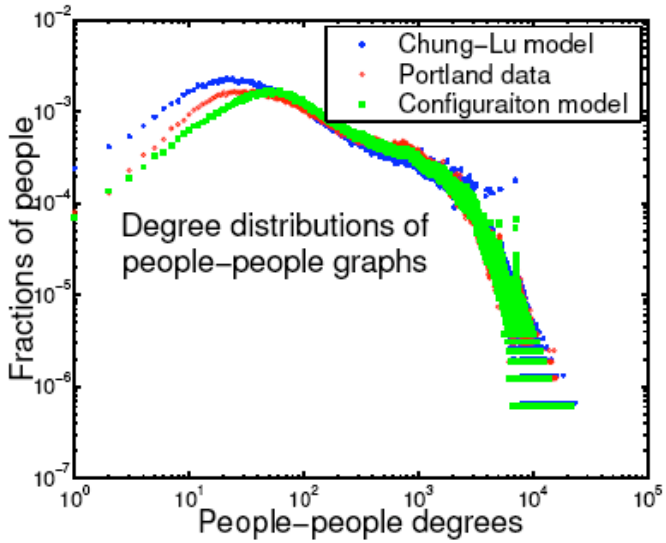


The Configuration Model

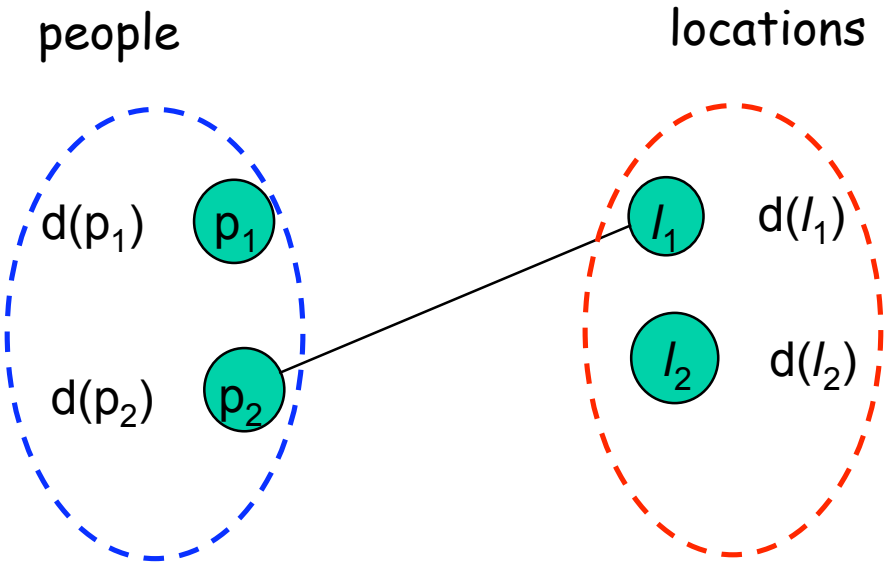




Choose a random matching



Chung-Lu vs Configuration



$$Pr[(p_2, l_1)] \propto d(p_2)d(l_1)/f(dist(p_2, l_1))$$

Candidates

$$f(dist(p_2, l_1)) = dist(p_2, l_1)^{\beta}$$

$$f(dist(p_2, l_1)) = e^{a \cdot dist(p_2, l_1)}$$

Fast Generation of CL

- Speed up (naive CL takes ~ 10 hours) random generation
- Preserve degrees
- Be "similar" to CL

Fast Generation of CL

- Speed up (naive CL takes ~ 10 hours) random generation
- Preserve degrees
- Be "similar" to CL

$$\forall p \in P, \forall \ell \in L, \Pr[X_{p,\ell} = 1] = \frac{d(p) \cdot d(\ell)}{\sigma}$$

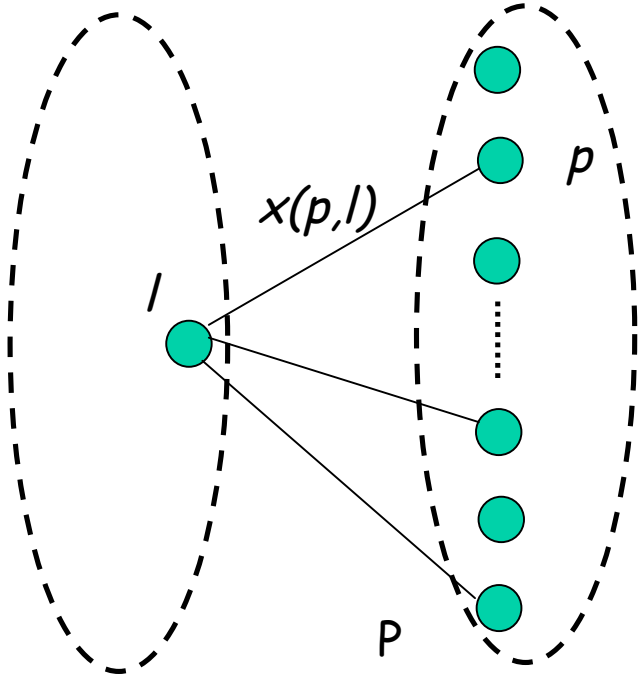
$$\forall \ell \in L, \Pr[|\{p : X_{p,\ell} = 1\}| = d(\ell)] = 1$$

$$\forall \ell \in L, P' \subseteq P :$$

Negative correlation constraints

$$\begin{cases} \Pr[\bigwedge_{p \in P'} (X_{p,\ell} = 0)] \leq \prod_{p \in P'} \Pr[X_{p,\ell} = 0], \\ \Pr[\bigwedge_{p \in P'} (X_{p,\ell} = 1)] \leq \prod_{p \in P'} \Pr[X_{p,\ell} = 1]. \end{cases}$$

Dependent Rounding[1]



Given:

Fractional $x(p,l) \in [0,1]$

Round $x(p,l)$ to $X(p,l) \in \{0,1\}$ s.t.:

- $\Pr[X(p,l)=1] = x(p,l)$
- $\forall\, l \in L$: $\Sigma_p X(p,l) \in \{\lfloor d(l) \rfloor, \lceil d(l) \rceil\}$
- $\forall\, p \in P$: $\Sigma_l X(p,l) \in \{\lfloor d(p) \rfloor, \lceil d(p) \rceil\}$
- $\forall\, P' \subset P$, $\forall\, b(p) \in \{0,1\}$

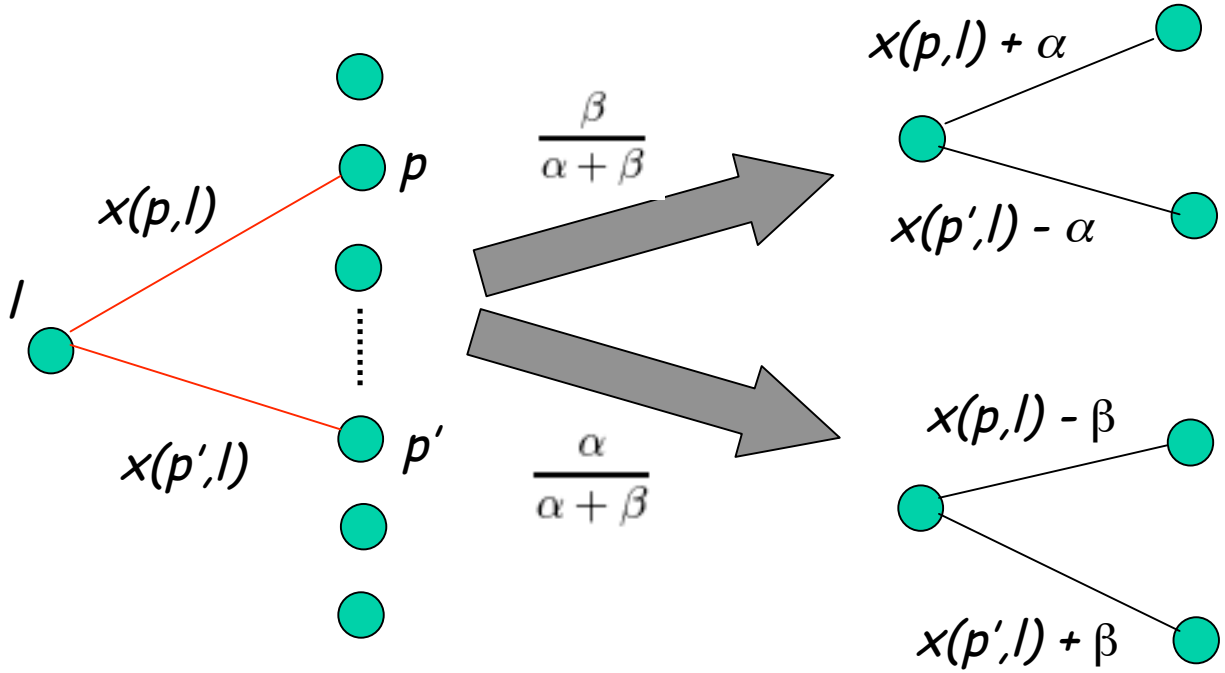
$$\Pr[\wedge_{p \in P'} X(p,l)=b(p)] \leq \Pi_{p \in P'} \Pr[X(p,l)=b(p)]$$

Time taken: *O(# fractional variables x(p,l))*

$\Rightarrow O(|P|\ |L|)$ time overall

[1] R. Gandhi, S. Khuller, S. Parthasarathy, A. Srinivasan, Dependent Rounding, FOCS 2004

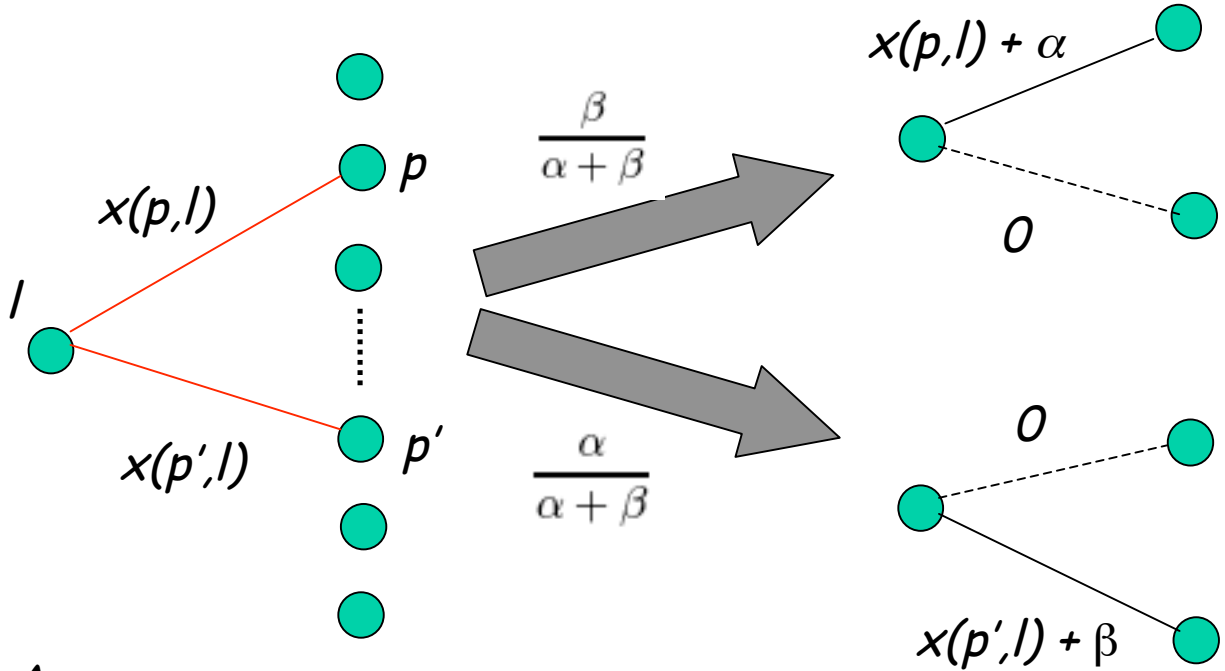
Graph generation by Dependent Rounding



α = min{ 1- x(p,l), x(p',l)}

β = min{ x(p,l), 1- x(p',l)}

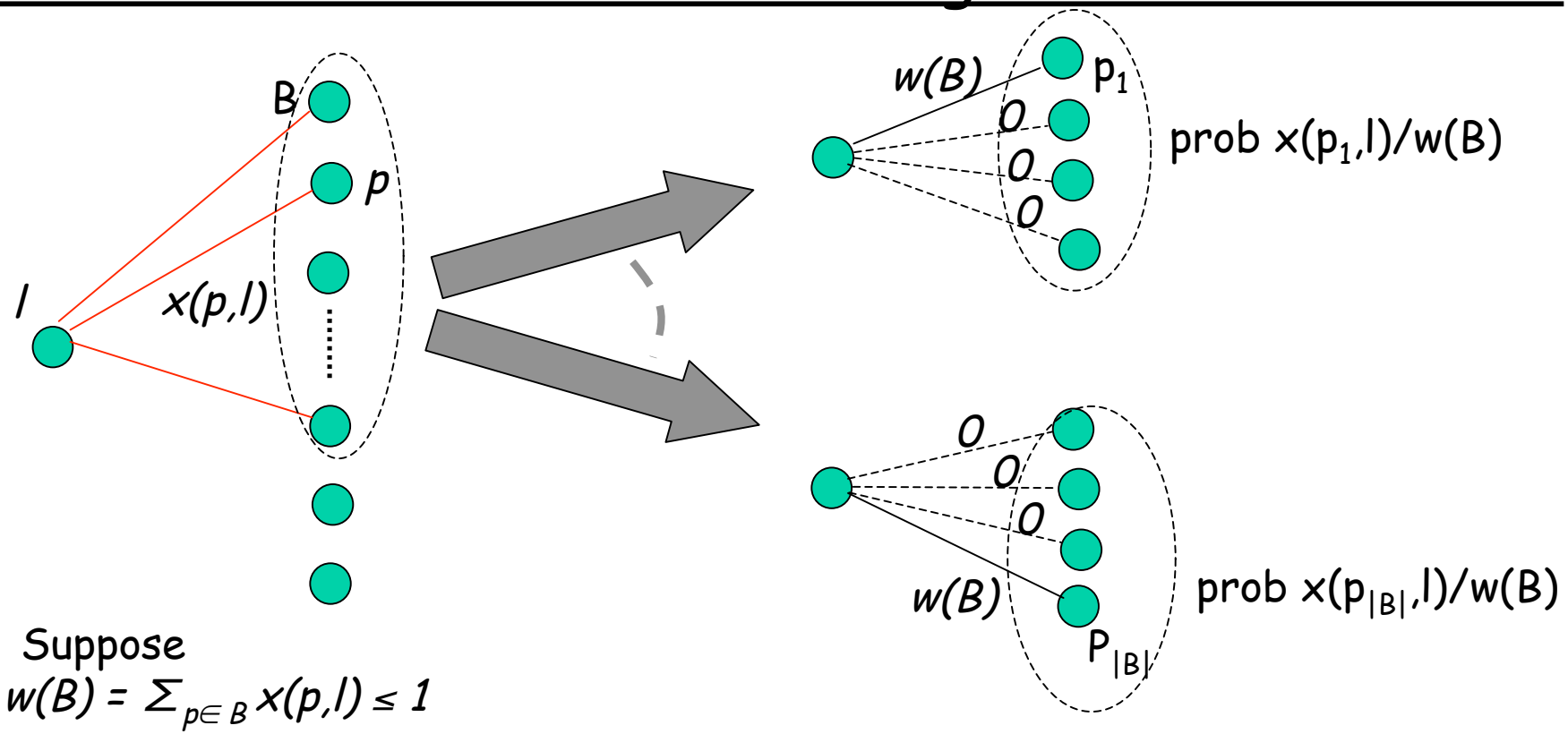
Graph generation by Dependent Rounding



Assume
$x(p,l) + x(p',l) \leq 1$

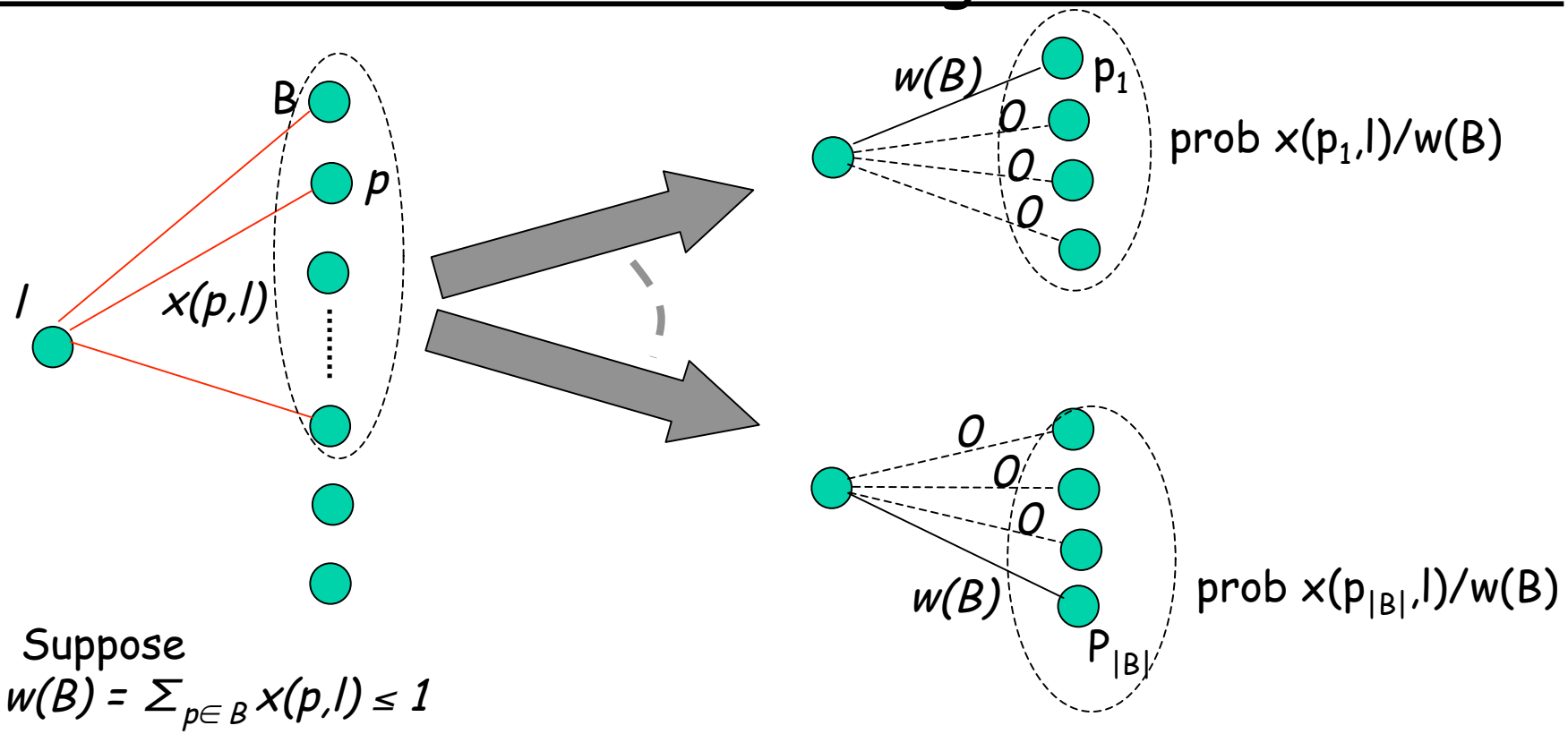
$\alpha = \min\{ 1- x(p,l), x(p',l)\} = x(p',l)$
$\beta = \min\{ x(p,l), 1- x(p',l)\} = x(p,l)$

Graph generation by Dependent Rounding



Suppose
$w(B) = \Sigma_{p \in B} x(p,l) \leq 1$

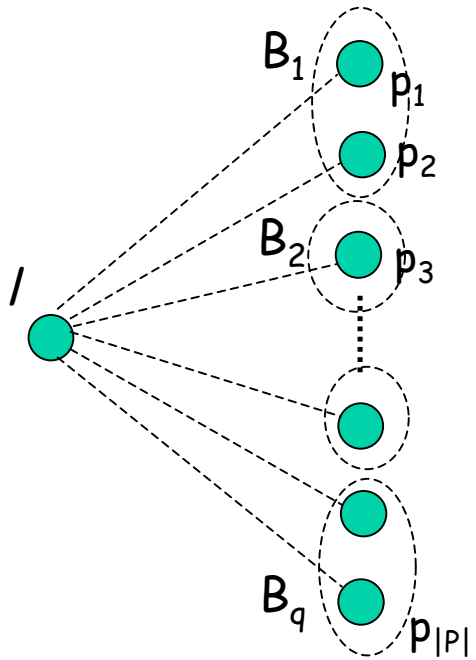
Graph generation by Dependent Rounding



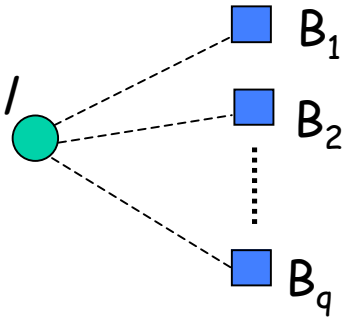
Suppose
$w(B) = \Sigma_{p \in B} x(p,l) \leq 1$

$w(B) \leq 1 \Rightarrow$ dependent rounding equivalent to sampling $p_i \in B$ with probability $x(p_i,l)/w(B)$

Fast Generation of CL



1. Partition into maximal blocks $B_1, B_2, \ldots, B_q$
   - $B_k = \{p(j_k), \ldots, p(j_{k+1})-1\}$
   - $w(B_k) = \Sigma_{j \in Bk}\, d(p_j)\, d(l)/\sigma$
   - $1 - d(p(j_{k+1}))d(l)/\sigma < w(B_k) \leq 1$
2. Reduce to smaller graph H on $\{l, B_1, B_2, \ldots, B_q\}$
3. Run Dependent rounding on H and choose subset $B' \subseteq \{B_1, B_2, \ldots, B_q\}$
4. For each block $B_i \in B'$, choose $p \in B_i$ with prob. $x(p,l) / w(B_i)$



$x(B_i, l) = \Sigma_{p \in Bi}\, x(p,l)$

Fast Generation of CL: Analysis

Observation: q <= 2(d(l) + 1)

Proof:

$\forall k$:

$$w(B_k) + w(B_{k+1}) > 1 - d(p(j_{k+1}))d(\ell)/\sigma + w(B_{k+1})$$
$$= 1 + w(B_{k+1}) - d(p(j_{k+1}))d(\ell)/\sigma$$
$$\geq 1$$

$$\Rightarrow \lfloor \frac{q}{2} \rfloor < \sum_{k=1}^{q} w(B_k)$$

$$\Rightarrow q \leq 2(d(\ell) + 1)$$

Fast Generation of CL: Running Time

$O(|P|\text{polylog})$ time overall

0. Construct prefix sums

$O(q \log |P|)$ time

1. Partition into maximal blocks $B_1, B_2, \ldots, B_q$
   - $B_k = \{p(j_k), \ldots, p(j_{k+1})-1\}$
   - $1 - d(p(j_{k+1}))d(l)/\sigma < w(B_k) \leq 1$

$O(q)$ time

2. Reduce to smaller graph H on $\{l, B_1, B_2, \ldots, B_q\}$

$O(q)$ time

3. Run Dependent rounding on H and choose subset $B' \subseteq \{B_1, B_2, \ldots, B_q\}$

$O(\log |P|)$ time

4. For each block $B_i \in B'$, choose $p \in B_i$ with prob. $x(p,l) / w(B_i)$

$\Rightarrow O(\Sigma\, q(l) \log |P|)$ time overall

$= O(\Sigma d(l) \log |P|) = O(|E| \log |P|)$

Fast Generation Properties

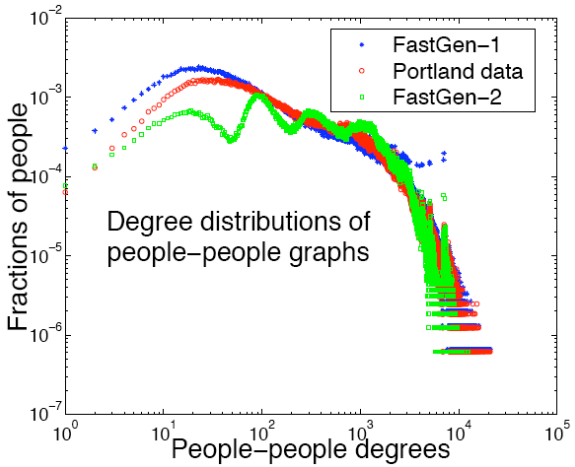
Equivalent to running Dependent Rounding on $G(\{l\}, P, \{(p,l): p \in P\})$ with fractional solution: $x(p,l) = d(p)d(l)/\sigma$

1. Partition into maximal blocks $B_1, B_2, \ldots, B_q$
   - $B_k = \{p(j_k), \ldots, p(j_{k+1})-1\}$
   - $1 - d(p(j_{k+1}))d(l)/\sigma < w(B_k) \leq 1$
2. Reduce to smaller graph H on $\{l, B_1, B_2, \ldots, B_q\}$
3. Run Dependent rounding on H and choose subset $B' \subseteq \{B_1, B_2, \ldots, B_q\}$
4. For each block $B_i \in B$, choose $p \in B_i$ with prob. $x(p,l) / w(B_i)$

$\Rightarrow$ 1. $\Pr[\text{edge } (p,l)] = x(p,l) = d(p)d(l)/\sigma$
2. Rounded-degree(l) = $\Sigma_{p \in P} x(p,l) = d(l)$
3. Negative correlation property

FastGen

	Portland data	CL	FastGen-1	FastGen-2
number of locations	181230	178746 (98.63%)	181230 (100%)	178668 (98.59%)
size of giant(locs.)	181192	178571	181088	178611
number of people	1615860	1507234 (93.28%)	1507291 (93.28%)	1615860 (100%)
size of giant(ppl.)	1615813	1507054	1507148	1615803
number of edges	6060679	6065637 (100.08%)	6060679 (100%)	6060679 (100%)
average deg. (ppl.)	3.7507	4.0227	4.0209	3.7507
time of generating a graph		$>$ 10 hours	$<$ 40 seconds	$<$ 30 seconds



	Portland data	CL	FastGen-1	FastGen-2
c.c. mean	0.6376	0.6161	0.6235	0.7021
c.c. std.	0.0167	0.0315	0.0236	0.0201
c.c l.b.	0.6286	0.6220	0.6241	0.7061
c.c. a.l.b.	0.6262	0.6220	0.6240	0.7061

A network approach to epidemiology

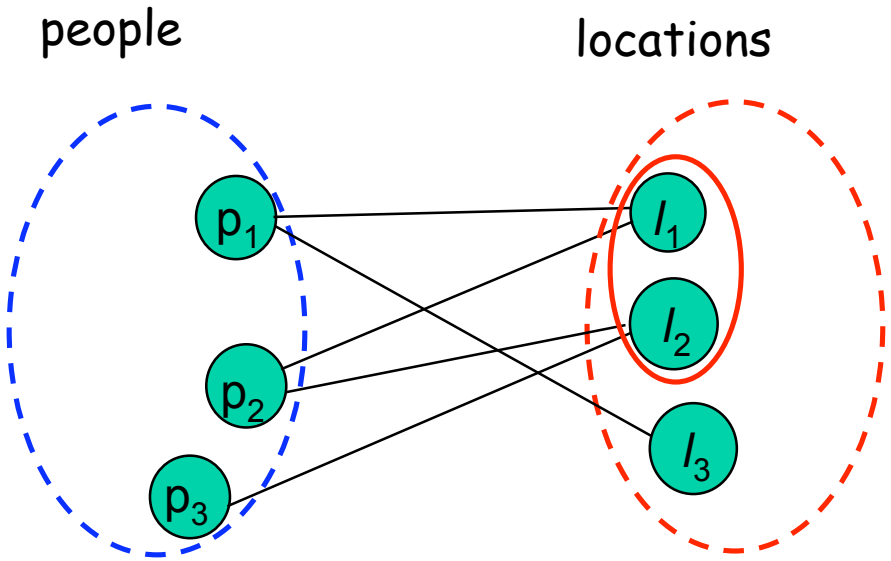
Main steps:

1. Construct a realistic social network
   - Not readily available
2. Develop an efficient disease simulation
   - Must scale to millions of people
   - Static analysis to guide simulation
3. Help in policy questions
   - Abstract out a simpler combinatorial model for the disease dynamics
   - whom to vaccinate/quarantine
   - where to place sensors

Policy Planning problems

- Disease detection
  - Fast detection + response, instead of mass vaccination
  - Sensor Location: dominating set problem
- Quarantining
  - Remove some edges so that disease is contained
- Vaccination
  - Remove some nodes so that disease is contained

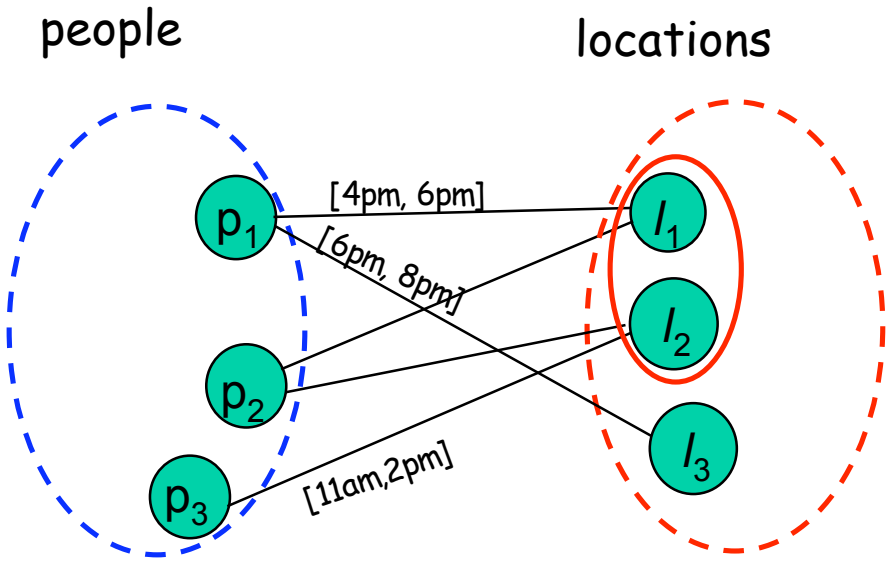
The Sensor Placement Problem



Dominating set problem: choose $L' \subseteq L$ s.t. $N(L') = P$

(1-ε)-Dominating set problem: choose $L' \subseteq L$ s.t. $|N(L')| \geq (1-\varepsilon)|P|$

Temporal Dominating Set



I(p, l): interval during which p visits l

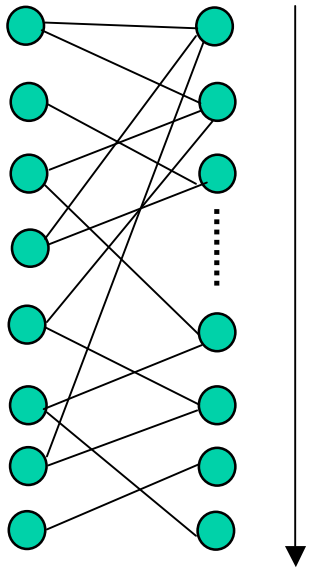
The Temporal Dominating set problem: choose $L' \subseteq L$, and time interval $[t_1, t_2]$ for each $l \in L'$ s.t. $\forall\, p \in P, \exists\, l \in L'$ s.t. $I(p,l) \cap [t_1, t_2] \neq \emptyset$

Sensor Placement: Results

- FastGreedy: choose large locations in non-increasing order of degrees sum of degrees is $(1-\varepsilon')|P|$
- FastGreedy gives (1+o(1))-approximation to optimum dominating set in Chung-Lu model
  - Same approximation in FastGeneration model
- FastGreedy works well in practice
  - 10% of the locations can dominate ~90% of people in Portland network
  - Very close to Greedy
  - Takes ~15 sec
- Temporal version hard to approximate within $\Omega(n^{\varepsilon})$

Dominating Set in the Chung-Lu model

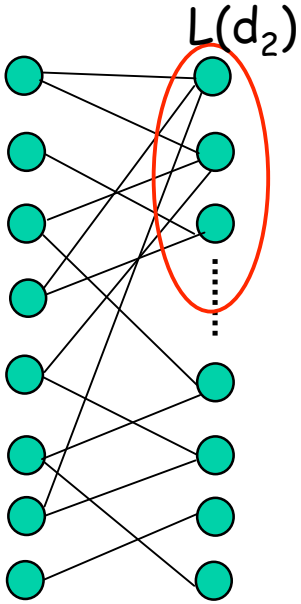
Theorem: FastGreedy is a (1+o(1))-approximation to the (1-ε)-Dominating Set in the Chung-Lu and Fast Generation models



Non-increasing
order of degrees

Dominating Set in the Chung-Lu model

Theorem: FastGreedy is a (1+o(1))-approximation to the (1-ε)-Dominating Set in the Chung-Lu and Fast Generation models



$L(d) = \{\text{locations } l \text{ with } d(l) \geq d\}$

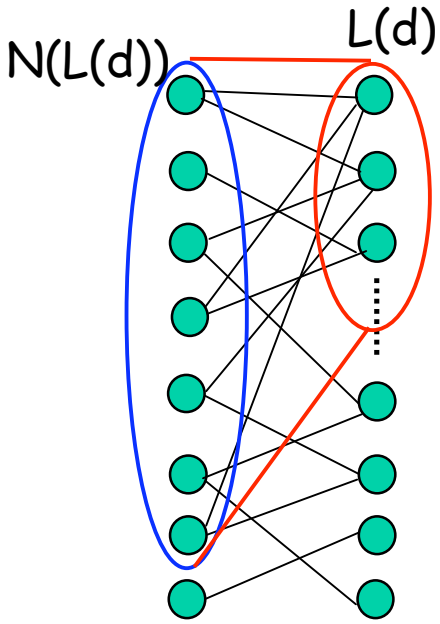
$S(L') = \Sigma_{l \in L'} d(l)$

d_2: largest degree s.t. $S(L(d_2)) \geq (1-\varepsilon')|P|$

To prove:

1. $|N(L(d_2))| \geq (1-\varepsilon)|P|$
2. Approximation ratio = $|L(d_2)|/|OPT| = 1+o(1)$

Property 1



Lemma: $S(L') \geq |P| \ln \frac{1-\delta_1}{\epsilon_1} \Rightarrow |N(L')| \geq (1-\epsilon_1-\delta_1)|P|, w.h.p.$

Proof:

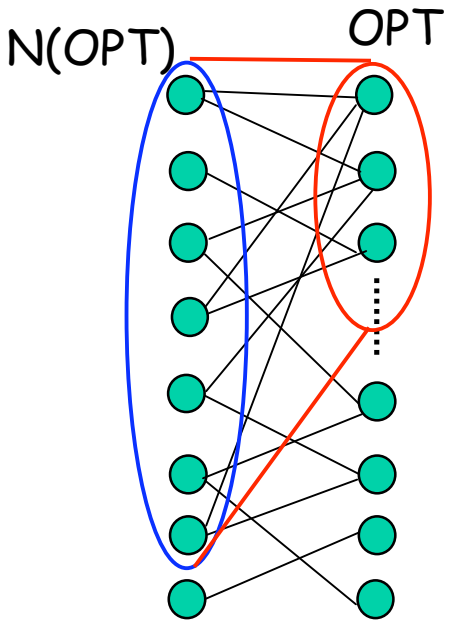
$$\forall p \in P, \Pr[p \notin N(L')] = \Pi_{\ell \in L'}(1 - \frac{d(p)d(\ell)}{\sigma})$$
$$< e^{-S(L')/|P|}$$
$$\Rightarrow E[|P \setminus N(L')|] < |P| e^{-S(L')/|P|}$$

$$S(L') \geq |P| \ln \frac{1-\delta_1}{\epsilon_1} \Rightarrow E[|P \setminus N(L')|] < |P| \frac{\epsilon_1}{1-\delta_1}$$
$$\Rightarrow E[|N(L')|] > |P|(1 - \frac{\epsilon_1}{1-\delta_1})$$
$$\Rightarrow |N(L')| > (1-\epsilon_1-\delta_1)|P| \quad w.h.p.$$

Chernoff bound, also holds for Fast Generation

⇒ Fast Greedy gives a (1-ε)-Dominating Set

Property 2



Lemma: $N(L') \geq (1-\epsilon)|P| \;\Rightarrow\; S(L') \geq |P|\frac{\ln \frac{2-\epsilon}{2\epsilon}}{1+\gamma}, \quad w.h.p.$

$\therefore$ S(OPT) is large

d_3 = largest degree s.t. $S(L(d_3)) \leq |P|\frac{\ln \frac{2-\epsilon}{2\epsilon}}{1+\gamma}$

$\Rightarrow |OPT| \geq |L(d_3)|$

Approximation factor of FastGreedy

Approximation factor $\leq \frac{|L(d_2)|}{|L(d_3)|}$

Power law $\Rightarrow$ $S(L(d)) \simeq \frac{c'|L|}{(\beta-2)d^{\beta-2}}$

$\therefore$ Approximation factor $\leq (\frac{S(L(d_2))}{S(L(d_3))})^{\frac{\beta-1}{\beta-2}}$

$\approx$ 1+o(1), w.h.p.

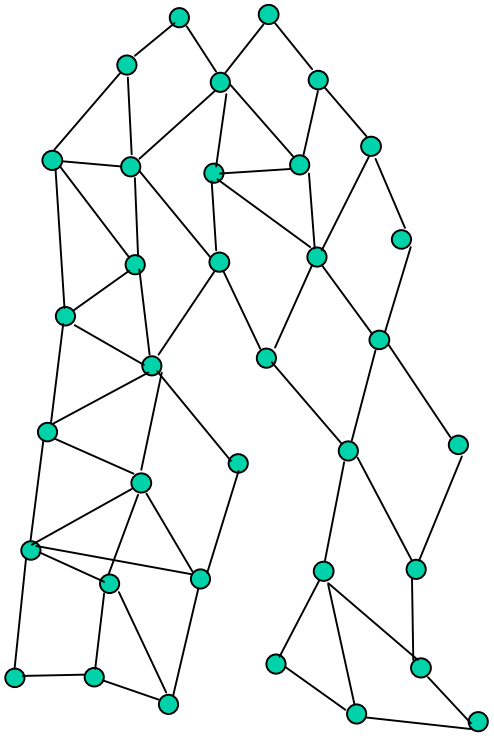
Policy Planning problems

- Disease detection
  - Fast detection + response, instead of mass vaccination
  - Sensor Location: dominating set problem
- Quarantining
  - Remove some edges so that disease is contained
- Vaccination
  - Remove some nodes so that disease is contained

Policy problems: whom to vaccinate

Given

- a social network G(P,E)
- initial infected set A
- budget B on # vaccinations



Policy problems: whom to vaccinate

Given
- a social network G(P,E)
- initial infected set A
- budget B on # vaccinations

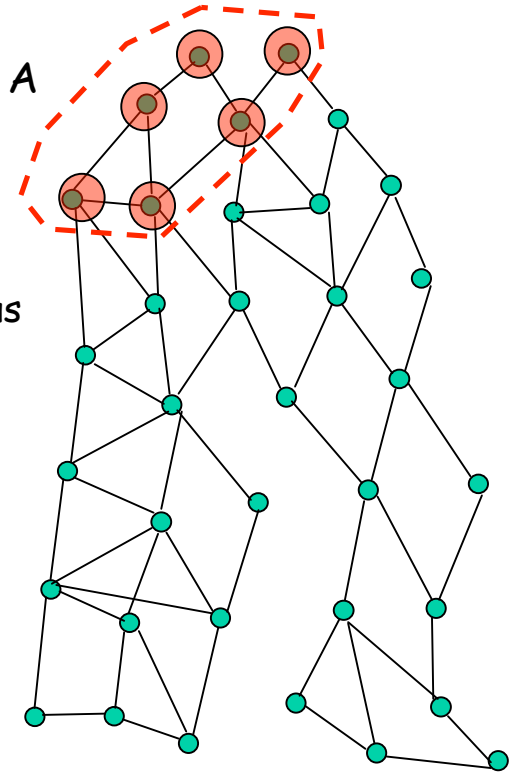
Goal

choose $S \subseteq P$ to vaccinate so that $|S| \leq B$, and # nodes reachable from A in G[P\S] is minimized

B=4

Assume: highly infectious disease

⇒ any node with a path to A gets infected

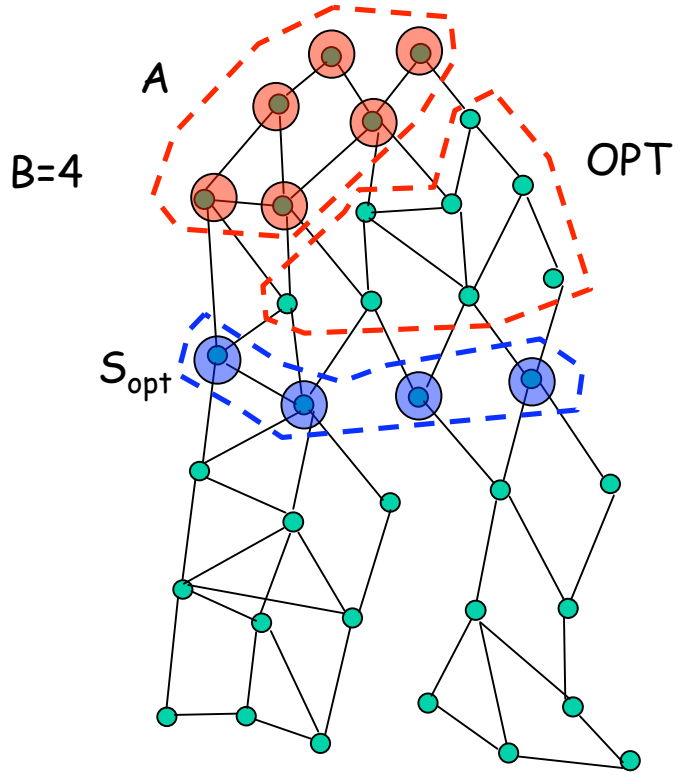


Policy problems: whom to vaccinate

Given
- a social network G(P,E)
- initial infected set *A*
- budget B on # vaccinations

Goal
choose $S \subseteq P$ to vaccinate so that $|S| \leq B$, and # nodes reachable from *A* in G[P\S] is minimized

Result
Bicriteria approximation: Vaccinate $(1+\varepsilon)B$ nodes so that at most $(1+1/\varepsilon)$ OPT infected people[1,2]



[1]S. Eubank, V.S. Anil Kumar, M. Marathe, A. Srinivasan and N. Wang, AMS-DIMACS special volume, 2005
[2]A. Hyrapetyan, D. Kempe, M. Pal and Z. Svitkina, ESA 2005

The Vaccination Problem

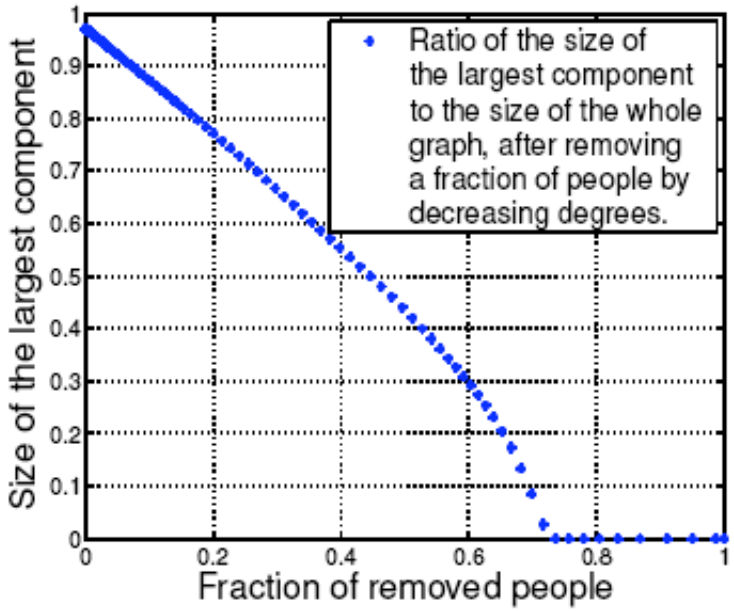
In reality: initial infected set is not known

Goal: Choose $S \subseteq P$, s.t. every component in $G[P \setminus S]$ is small

ρ-separator problem: polylog approximations known, but algorithms are not scalable

The "High Degree" Vaccination Policy

Vaccinate people of high degrees



Giant component remains after deleting a large fraction of nodes

Also true for G_P in the Chung-Lu and FastGen models

Open Problems

- Social Networks
  - Models for temporal and demographic aspects
  - Distance function
- Other network design problems for policy planning
  - Remove nodes/edges so that expansion is reduced
  - Remove nodes/edges so that average distance increases
  - Faster algorithms on large graphs
- More realistic disease models (SIR, SIS)
  - Dynamics on social networks
  - Policy planning problems
- Problems on temporal graphs

Thank You