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Generalized Traffic Equations

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ABSTRACT

This paper considers the problem of computing certain performance parameters of stochastic networks that have Markovian routing but are not otherwise Jackson networks. A scheme using this model for approximating address-based routing is described. The method relies on generalizing the traffic equation to special subsets of the flows in the network.

KEY WORDS: Latency, jitter, routing, path-additivity.

1. INTRODUCTION

1.1. Rationale

The Jackson network [1] is a basic model favored as a useful beginning approach to many queueing network problems. Its appeal stems from not only its product-form solution for the joint distribution of the number of customers in each queue but also from the simplicity of the structures of the model. Random (Markovian) routing is one such structure. The purpose of this paper is to study the ramifications of Markovian routing in a flow network that need not be a Jackson network, principally so that a method can be devised to compute cross-network values of functions whose values on single links are known. The main result is a simple equation for path-additive functions in a flow network in equilibrium (not necessarily a Jackson network) that has random routing. We use this equation to compute latency and jitter in telecommunications packet networks. The class of flow networks we consider includes queueing networks as well as generalizations of flow networks of the type introduced by Ford and Fulkerson [2].

Informally, a path-additive network function is one whose value on a path is the sum of its values on any partition of the path (see Section 3.2 for the definitions). For example, the length of a path in a road network is a path-additive phenomenon: the total length traveled along the path is the sum of the individual lengths of the elements of a partition of the path.

1.2. Scope

This paper concerns a directed, acyclic flow network $\mathcal{H} = (\mathcal{N}, \mathcal{L})$ having $n = |\mathcal{N}| < \infty$ nodes and random routing. A link is an ordered pair of nodes, *i. e.*, $\mathcal{L} \subset \mathcal{N} \times \mathcal{N}$. The first element of the pair is called the head and the second element is called the tail. It will sometimes be convenient to consider a node such as j as equivalent to the link (j, j). The paper restricts consideration to random routing in which a customer[1] (unit of flow) departing a node chooses the next node to which it travels according to a (conditional) probability distribution that depends only on the current node: if a customer is currently at node $i \in \mathcal{N}$, then the customer next travels to node $j \in \mathcal{N}$ with probability $r_{ij} \geq 0$. If X_m denotes the node at which a customer is found at the m^{th} step, then $r_{ij} = P\{X_{m+1} = j \mid X_m = i, X_{m-1}, \ldots, X_0\} = P\{X_{m+1} = j \mid X_m = i\}$, $m = 1, 2, \ldots$. By this definition, random routing is Markovian and homogeneous; the stochastic process $\{X_m : m = 1, 2, \ldots\}$ having state space $\mathcal{N}$ is a homogeneous Markov chain. Let $R = (r_{ij})$ denote the routing, or switching, matrix for the network. R is also the transition matrix for this Markov chain and is in general a stochastic matrix.

Without further specification, "random routing" as a phrase could include many types of probabilistic schemes for routing jobs in the network. However, in the context of flow networks and queueing networks, random routing has come to mean exclusively Markovian routing of the nature indicated in the previous paragraph. This paper conforms to this use of the terminology.

A flow network is characterized by a movement of customers, jobs, packets, etc., among nodes. It is *open* if there is at least one node at which traffic units (customers, packets, etc.)

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[1] We freely interchange the customer, packet, and job terminology throughout.

can leave the network. We additionally postulate an exogenous demand of jobs, packets, etc., requiring transportation from originating nodes to destination nodes. All jobs leave the network once they reach their destinations. That is, there is a matrix U whose entries $u_{ij} \geq 0$ represent the number of packets required to be transported through the network from node i to node j, at which point they leave the network. In an open network, the routing matrix R is substochastic because there is at least one node where jobs leave the network. In various contexts, the u_{ij} are taken to be constant (a fixed number of packets per hour), the rates of homogeneous Poisson processes (a random number of packets per hour having expected value u_{ij}), or some other characterization of a random process describing arrival streams at the origin nodes. Examples of flow networks include telecommunication networks, package delivery networks, commodity pipeline networks such as oil and natural gas pipelines, etc. In a telecommunications context, we may think of u_{ii} as the intraoffice demand at node i. The networks we shall consider in this paper are open.

A key observation motivating this study is that packet routing in telecommunication networks is not completely random. It is driven by an addressing scheme such as IPv4 that supplies each packet with a destination address, and routers with routing tables that determine the progress of the packet through the network based on this address. A certain amount of randomness is present in, for example, the choice of next router, which may vary depending on the amount and location of congestion in the network, but this process is subject to rules that place more restrictions on flows than does the simple Markovian routing scheme. The model described in this paper for telecommunication networks is motivated by a need to better describe address-based routing of packets. The model arises from restricting basic Markovian routing to cause paths that cannot be encountered in practice to be excluded. In particular, we study two special subsets of packet flows: one for which packets leave the network when they reach their destination node and another for which packets do not revisit previously visited nodes. Let the subscript 0 indicate the environment (*i. e.*, the world outside the network). Then, if $r_{j0} < 1$, unrestricted Markovian routing would permit a packet reaching its destination node to possibly travel to some other node without leaving the network, so our first model of address-based routing restricts consideration to the subset of those sample paths of the routing process stimulated by the demand u_{ij} that contain node j exactly once, at the end of the path. A second model is constructed by restricting attention to that subset of sample paths of the routing process in which packets do not revisit previously visited nodes because, under nominal conditions, is it unlikely (if not impossible) for a packet to return to a router it has already traversed. Neither of these models is a completely faithful rendering of IPv4 routing, but the two approximations taken together enhance our understanding of the appropriateness of Markovian routing in modeling address-based routing.

In this paper, we consider only flow networks in equilibrium. That is, the (vector) stochastic process $\{Y_1, \ldots, Y_n\}$ is stationary, where Y_{ij} is the number of customers in the queue at the node i (in service and waiting). For this, it is sufficient that the Markovian routing process be homogeneous, and we shall assume this throughout. The networks originally studied by Ford and Fulkerson [2] possess deterministic routing, no queues at the nodes, and blocked-calls-cleared queues on the links with zero service times and number of servers equal to the link capacity. These are a special case of queueing networks because deterministic routing is certainly Markovian and the nodes may be thought of as infinite-server queues with zero service times. This paper also considers Ford-Fulkerson-type networks in which routing may be random as described above. Clearly these satisfy the equilibrium condition because the number of customers in queue at each node and link is always zero.

1.3. Background: Beyond the Jackson Network

Jackson introduced the queueing network model we now know by his name in 1957 [1] to study the flow of jobs in a manufacturing plant. The model was developed further in [3]. A property that makes the Jackson network popular as a model for many queueing networks is the so-called product-form property which demonstrates that the joint distribution of the number of customers in each queue in the network is the product of the individual one-dimensional distributions. That is, the network behaves (at least as far as the one-dimensional distributions are concerned) as though its constituent queues were mutually stochastically independent. In many instances, however, the network under study does not conform completely to the Jackson model while computation of key network performance measures remains of interest. For example, it may be desirable to relax the exponential service time distribution assumption: TCP/IP packet service times typically have a trimodal distribution. Motivation for this paper therefore includes study of how certain key performance parameters may be computed in a more general network model that posits Markovian routing but none of the other properties necessary for the Jackson network model.

1.4. Overview of Approach

We develop for path-additive functions in a flow network in equilibrium a generalization of the traffic equation and then show how this is applied to compute the expected number of links traversed by a packet in the network, the expected delay a packet encounters across the network, and the packet jitter (which is the standard deviation of packet delay). The approach is to partition a path that a packet travels into two pieces: a link in the path and everything else, and then use total probability to write an equation for the stochastic quantities of interest. We endeavor in all cases to find the most general conditions under which this operation may be performed. We consider first unrestricted Markovian routing and then the two models with path restrictions: the first in which the tail node of the path is not visited at any other time in the path, and the second in which each node in the path is visited exactly once.

2. THE EQUILIBRIUM TRAFFIC EQUATIONS

This Section establishes two basic facts about packet flow in an equilibrium flow network when routing proceeds without restrictions (in particular, without restrictions on repeated visits to previously visited nodes). Note that random routing causes the flow in the network to be a (matrix-valued) stochastic process, and so the quantities considered in this paper, as (measurable) functions of the flow, are also random quantities.

Consider a flow network with Markovian routing in which customers enter at certain nodes for transport to other nodes at which they exit. Denote by u_{ij} the number of customers per unit time requiring transport from node i to node j. The units of u_{ij} could be terabits per second, packages per day, megawatts per hour, etc., depending on the application, and the u_{ij} themselves may be constants representing deterministic demands or the expected demands in a flow network with stochastic demands. Let $U = (u_{ij})$. Note that in equilibrium, the rate of customers leaving a node must equal the rate of customers entering the node. The traffic equation expresses a relationship between the exogenous demands and the arrival rates at each node. Denote by λ_i the total arrival rate at node i, including both the exogenous arrivals and arrivals due to routing from other nodes, and let $u_i^0 = \sum_{j=1}^{n} u_{ij}$ denote the arrival rate of exogenous demand at node i. The following is a well-known result for queueing networks (*e. g.*, [4]) and is included here to show its application to more general flow networks and to further illustrate the importance of the matrix $(I-R)^{-1}$ in the work to come. The "traffic equation" and the description of R as the "switching matrix" were introduced by Beutler and Melamed [5] and Beutler, Melamed, and Ziegler [6].

Proposition 1. Suppose R is convergent (*i. e.*, $R^m \to 0$ as $m \to \infty$). Then $(I-R)^{-1}$ exists and the total arrival rate at node i is given by the i^{th} entry in the row vector $\lambda = u^0(I-R^T)^{-1}$.

Proof. Let $Z_m = \sum_{s=0}^{m} R^s$. Then $(I-R)Z_m = I - R^{m+1}, m = 1, 2, \ldots$. Using the hypothesis and passing to the limit we obtain $(I-R)\sum_{s=0}^{\infty} R^s = I$, which is enough to show [7, theorem 3.4] that $\sum_{s=0}^{\infty} R^s = (I-R)^{-1}$.

The traffic equation is obtained from observing that the total arrival rate at node i is the sum of all exogenous arrivals at node i and the packets arriving at node i that are routed from other nodes:

$$\lambda_i = u_i^0 + \sum_{j=1}^{n} \lambda_j r_{ji}, \quad i = 1, \ldots, n.$$

This may be expressed in matrix notation as $\lambda = u^0 + \lambda R^T$ (with λ and u^0 as row vectors). R^T is also convergent, so we obtain $\lambda = u^0(I-R^T)^{-1}$, which is the desired equilibrium traffic equation. ■

From the traffic equation, we can obtain an expression, a second equilibrium traffic equation, for the rate at which customers leave the network from each node. In equilibrium, the rate of customers leaving each node is equal to the rate of customers arriving at that node, and the total rate of departures from the network is equal to the total rate of arrivals to the network. However, this does not enable us to see the rate at which customers leaving each node leave the network. Let μ_i denote the total departure rate from node i (including departures to outside the network) and let c_i denote the rate of departures from node i to outside the network. Then we may write

$$c_i = \left(1 - \sum_{j=1}^{n} r_{ij}\right)\mu_i = \left(1 - \sum_{j=1}^{n} r_{ij}\right)\lambda_i = \left\langle (I-R)\mathbf{1}, \mathrm{e}_i \right\rangle \left\langle u^0 \left(I-R^T\right)^{-1}, \mathrm{e}_i \right\rangle, \quad i = 1, \ldots, n,$$

where $\mathbf{1}$ represents an n-vector of ones, $\mathbf{e}_i$ represents an n-vector with a 1 in the i^{th} place and zeros elsewhere, and $\langle \, , \rangle$ represents the ordinary Euclidean inner product. This gives an expression for the departure rate from node i to outside the network in terms of the routing matrix and the overall demand.

The entire network's arrival-departure balance leads to the following equation:

$$\left\langle u^0, 1 \right\rangle = \left\langle c, 1 \right\rangle = \left\langle (I-R)1, u^0 \left(I-R^T\right)^{-1} \right\rangle = \left\langle 1, \left(I-R^T\right) u^0 \left(I-R^T\right)^{-1} \right\rangle,$$

leading one to inquire whether in fact $u^0 = \left(I - R^T\right) u^0 \left(I - R^T\right)^{-1}$, or, equivalently, $u^0 R^T = R u^0$. This is certainly true if routing in the network is symmetric ($R = R^T$), but need not be true otherwise.

To conclude this Section, we observe that in an open flow network, R is convergent, hence invertible. In an open network, there is at least one node where packets leave (that is, $r_{j0} > 0$ for at least one j) and this node appears in every row of R. Thus the row sums of R are strictly less than one, so the spectral radius of R is also less than one, so R is convergent

3. PACKETS LEAVING THE NETWORK UPON FIRST VISIT TO DESTINATION NODE

3.1. Introduction

The material in this Section concerns that subset of flows in the network that have the property that the terminal node in a path is visited exactly once. For brevity, we will call this the Type I restriction. This study is an attempt to better model the properties of a telecommunications network whose routing is address-based. In such a network, when a packet reaches its destination node, it leaves the network. The Type I restriction is not a perfect representation of address-based routing (for instance, arbitrary circulation in the network is permitted as long as the packet does not visit its destination node) but does provide an improvement over the unrestricted random routing model.

Definition. A path in $\mathcal{H}$ is a finite sequence of links in $\mathcal{L}$ with the property that the tail of a link in the path is the head of the previous link in the path.

This usage differs slightly from most usual notions of path, which include the nodes as part of the path also (see, *e. g.*, [8]). The links in a path need not be disjoint, *i. e.*, a path may contain a particular link more than once. Let $\mathcal{P}(\mathcal{H})$ denote the set of paths in $\mathcal{H}$ and let $\mathcal{P}_{ij}(\mathcal{H})$ denote the set of paths in $\mathcal{H}$ whose initial node is i and whose terminal node is j.

Definition. For $Z \in \mathcal{P}(\mathcal{H})$, a *partition* of Z is a finite subset $\{Z_1, \ldots, Z_m\}$ of $\mathcal{P}(\mathcal{H})$ satisfying $\bigcup_{i=1}^{m} Z_i = Z$ and $Z_i \cap Z_j = \varnothing$ for $i \neq j$.

For instance, the total (geographic) length of a path in a transportation network is path-additive. Note that this holds also for paths containing repeated links; the requirement on the elements of the partition is that they be disjoint from each other even though they may internally contain repeated links themselves.

Definition. Let a be a real-valued function on $\mathcal{P}(\mathcal{H})$, $a : \mathcal{P}(\mathcal{H}) \to \mathbf{R}$, and let $Z \in \mathcal{P}(\mathcal{H})$. a is said to be path-additive if $a(Z) = a(Z_1) + \cdots + a(Z_m)$ for every partition $\{Z_1, \ldots, Z_m\}$ of Z.

We also need to consider real-valued random processes having parameter space $\mathcal{P}(\mathcal{H})$. The above definition extends naturally to path additivity with probability one, in probability, etc.

3.2. An Equation for Path-Additive Functions with the Type I Restriction

In a flow network with random routing, the path traveled by a packet is a random element of $\mathcal{P}(\mathcal{H})$. The following result is a basic equation for path-additive functions in a flow network with the Type I restriction.

Proposition 2. Suppose R is convergent. Let $i, j \in \mathcal{N}$ and let Z_{ij} be a random element of $\mathcal{P}_{ij}(\mathcal{H})$ satisfying the Type I restriction. Let a be a path-additive function on $\mathcal{P}(\mathcal{H})$ with probability 1 and let A be the matrix of $Ea(Z_{ij})$ for $i, j = 1, \ldots, n$. Let $s_{ij} = Ea((i, j))$ for $(i, j) \in \mathcal{L}$ be the values obtained by evaluating Ea on all the links in $\mathcal{L}$, let S be the matrix (s_{ij}), $i, j = 1, \ldots, n$, let $b_j = (1+r_{jj})^{-1}$ and let R^* be the matrix whose (i, j) entry is $r_{ij}b_j$. Finally, let S_1 be the matrix whose (i, j) entry is the (j, j) diagonal element of $S^T R^*$ (all columns of S_1 are identical). Then $A = S_1(I - R^*)^{-1}$.

Proof. Let $B_1(j)$ denote the set of nodes connecting to j over exactly one link, *i. e.*, $B_1(j) = \{k \in \mathcal{N}: (k, j) \in \mathcal{L}\}$. Note $j \notin B_1(j)$ by the Type I restriction. For any path in $\mathcal{P}_{ij}(\mathcal{H})$, the path from i to any $m \in B_1(j)$ followed by the link (m, j) is a partition because j does not appear anywhere in the path from i to m. Plainly, $P\{m \in B_1(j)\} = r_{mj}$. Then

$$\begin{aligned}
Ea\left(Z_{ij}\right) &= \sum_{k\in\mathcal{N}} Ea\left(Z_{ij} \mid k \in B_1(j)\right)P\left\{k \in B_1(j)\right\} \\
&= \sum_{k=1}^{n} E\left[\left(a\left(Z_{ik}\right)+a\left(Z_{kj}\right)\right) \mid k \in B_1(j)\right] r_{kj} - E\left[a\left(Z_{ij}\right)+a\left(Z_{jj}\right)\right] r_{jj} \\
&= \sum_{k=1}^{n} \left[Ea\left(Z_{ik}\right)+s_{kj}\right] r_{kj} - Ea\left(Z_{ij}\right) r_{jj} \\
&= \sum_{k=1}^{n} Ea\left(Z_{ik}\right) r_{kj} - Ea\left(Z_{ij}\right) r_{jj} + \sum_{k=1}^{n} s_{kj} r_{kj} \ ,
\end{aligned}$$

where we have used $Ea(Z_{jj}) = 0$ for all j. From this, we obtain

$$\left(1+r_{jj}\right) Ea\left(Z_{ij}\right) = \sum_{k=1}^{n} Ea\left(Z_{ik}\right) r_{kj} + \sum_{k=1}^{n} s_{kj} r_{kj}$$

or, in matrix notation, $A = AR^* + S_1$. R^* is also convergent because $b_j \leq 1$, so the result follows from Proposition 1. ■

The remainder of this Section provides applications of Proposition 2 to quantities of interest in telecommunication networks.

3.3. Expected Number of Links Under the Type I Restriction

The number of links in a path is clearly a path-additive function. Let Z be a random element of $\mathcal{P}(\mathcal{H})$ satisfying the type I restriction. To use this method to determine the expected number of links traveled by a packet in $\mathcal{H}$ under the Type I restriction, define c_{ij} to be $I\{(i, j) \in Z\}$. We have $Ec_{ij} = r_{ij}\, I\{(i, j) \in \mathcal{L}\} = r_{ij}w_{ij}$, where $W = (w_{ij})$ is the incidence matrix of $\mathcal{H}$. Define the matrix C_1 as indicated in Proposition 2. It follows immediately from Proposition 2 that the expected number of links traveled by a packet (satisfying the Type I restriction) whose origin node is i and whose destination node is j is given by the (i, j) entry in $C_1(I - R^*)^{-1}$.

3.4. Delay Under the Type I Restriction

3.4.1. Introduction

A primary purpose of the methods described in this paper is to enable computation of the delay experienced by a packet flowing between any two nodes in $\mathcal{N}$ when the single-link delays are given. In a telecommunications packet network, delays are attributable to queueing times at the router input and output buffers (which are in turn driven by the degree of congestion in the network), the router CPU times, and the transport times across the links in the network (under nominal conditions these are negligible compared to the buffer delays). To avoid double-counting, we define the single link delay on a link $(i, j) \in \mathcal{L}$ to be the sum of the output buffer queue time of the router at node i, the link transport delay across (i, j), and the input buffer queue time of the router at node j. Delay, so defined, is path-additive almost surely. For a complete cross-network delay (or latency) calculation, we need to add the input buffer queue time at the origin node and the output buffer queue time at the destination node.

We may also note that packet loss is the probability that packet delay is infinite (or, in some protocols, the probability that the delay exceeds a specified timeout value). Packet loss, jitter, and delay (as these terms are used in telecommunications) are all features of the packet delay distribution.

3.4.2. Expected Delay

Denote by d_{ij} the single link delay when $(i, j) \in \mathcal{L}$. The single-link delays may be taken to be deterministic or random. When they are deterministic, total delay on a path is a path-additive property and the randomness in the latency (expected delay) comes from the fact that a packet takes a random path from its origin to its destination. When the d_{ij} are random variables, they have the path-additivity property with probability 1. In particular, the expected delays Ed_{ij} have the path-additivity property as defined in Section 3.2. Let D_1 be the matrix whose (i, j) entry is the (j, j) diagonal element of $D^T R$ (all columns of D_1 are identical). It follows immediately from Proposition 2 that the expected delay encountered by a packet flowing from origin node i to destination node j in an open telecommunications network in which all packets eventually leave the network is given by the (i, j) entry in $D_1(I - R^*)^{-1}$.

3.4.3. Variance of Delay and Jitter

The variance computation proceeds similarly as long as we additionally assume that the single-link delays are uncorrelated (this property holds, *e. g.*, in Jackson networks with no overtaking [4]), so that the variance of the sum of single-link delays is the sum of the single-link delay variances. This is what's needed for almost sure path additivity. Denote by v_{ij} the variance of the single-link delay for $(i, j) \in \mathcal{L}$. Then the variance of delay along a path is the sum of the single-link delay variances over the links in the path, so variance of delay is a path-additive function. As before, let V denote the matrix of the v_{ij} and V_1 the matrix whose (i, j) entry is the (j, j) diagonal element of $V^T R$ (all rows of V_1 are identical). It follows immediately from Proposition 2 that, if all single-link delays are uncorrelated, then the variance of the delay encountered by a packet flowing from origin node i to destination node j in an open telecommunications network in which all packets eventually leave the network is given by the (i, j) entry in $V_1(I - R^*)^{-1}$.

For these variance results, we need to define the one-link delays precisely. In particular, in models where a single node (router) comprises two queues, an input buffer (before the processor) and an output buffer (serving the outgoing links), it is sufficient for the variance computation that we assume these queues are mutually uncorrelated. This added detail is not part of the standard Jackson network model[2] and would need to be separately justified on modeling grounds. This may be possible by appealing to the mixing that takes place within a router when packets with a large number of different addresses are flowing through the router (this is essentially Kleinrock's heavy-traffic independence argument).

Jitter is obtained by taking the square root of the delay variance.

3.4.4. Distribution of Delay

Let $F_{ij}(x) = P\{d_{ij} \le x\}$ denote the distribution of the single-link delay when $(i, j) \in \mathcal{L}$. While the distribution of delay is not a path-additive function as in Section 3.2, if we assume the single-link delays are mutually stochastically independent, then the distribution of delay along a path is the convolution of the distributions of the single-link delays for the links comprising the path. By analogy to Proposition 2, we obtain the following equation for the distribution of the cross-network delays under the Type I restriction and when the single-link delays are independent. As before, we take $a(Z_{jj}) = 0$ for all j. Then

$$\left(1 + r_{jj}\right) F_{ij}(x) = \sum_{k=1}^{n} F_{ik} * F_{kj}(x)\, r_{kj}$$

where the star indicates convolution. This constitutes a linear system of integral equations for the $F_{ij}(x)$. Further treatment of this system is beyond the scope of this paper.

4. PATHS WHERE NO NODE IS REPEATED

4.1. Introduction

The material in this Section concerns that subset of flows in the network that have the property that no node in a path is visited more than once. For brevity, we will call this the Type II restriction. Clearly, a stochastic process satisfying this condition cannot be a Markov process. However, the object of the study in this Section is that subset of flows with Markovian routing that happen to have the property that if a packet visits a node, then that node is not visited again before the packet leaves the network. Paths in such a flow cannot contain more than n nodes.

This study is a further attempt to better model the properties of a telecommunications network whose routing is address-based. In such a network, routing tables in each router determine the next node (router) to be visited based on the current node and the destination address. It is unusual for a packet to be routed back to a node it has previously visited. Therefore, the Type II restriction may help clarify our understanding of the behavior of the packet flow in such a network. The Type II restriction is not a perfect representation of address-based routing but it does provide an improvement over the unrestricted random routing model. A combination of results developed from the models with the Type I and

[2] Note that while the Jackson network model has the property that the queues in the network behave as though they were mutually stochastically independent, this only pertains to the one-dimensional distributions belonging to separate nodes. Computation of the variance requires consideration of two-dimensional distributions of the two queues internal to a single node, which the Jackson theorems do not address.

the Type II restrictions may allow a better picture of packet flow in networks with address-based routing to be discerned.

4.2. An Equation for Path-Additive Functions with the Type II Restriction

The following Proposition gives a basic equation for path-additive functions in a flow network with the Type II restriction.

Proposition 3. Suppose R is convergent. Let $i, j \in \mathcal{N}$ and let $Z_{ij} \in \mathcal{P}_{ij}(\mathcal{H})$ satisfy the Type II restriction. Let a be an almost sure path-additive function on $\mathcal{P}(\mathcal{H})$ and let A be the matrix of $Ea(Z_{ij})$ for $i, j = 1, \ldots, n$. Let $s_{ij} = Ea((i, j))$ for $(i, j) \in \mathcal{L}$ be the values obtained by evaluating Ea on all the links in $\mathcal{L}$, let S be the matrix (s_{ij}), $i, j = 1, \ldots, n$, and let R_* be the matrix whose (i, j) entry is $b_i r_{ij}$. Finally, let S_2 be the matrix whose (i, j) entry is the (j, j) diagonal element of $R_* S^T$ (all rows of S_2 are identical). Then $A = (I - R_*)^{-1} S_2$.

Proof. Let $A_1(i)$ denote the set of nodes connecting to i over exactly one link, *i. e.*, $A_1(i) = \{k \in \mathcal{N}: (i, k) \in \mathcal{L}\}$, for flows satisfying the Type II restriction. Note $i \notin A_1(i)$. For any path in $\mathcal{P}_{ij}(\mathcal{H})$, the link from i to any $m \in A_1(i)$ followed by the path from m to j is clearly a because, by the Type II restriction, $m \neq j$ and i does not appear anywhere in the path from m to j. Plainly, $P\{m \in A_1(i)\} = r_{im}$. Then

$$\begin{aligned}
Ea\left(Z_{ij}\right) &= \sum_{k \in \mathcal{N}} Ea\left(Z_{ij} \mid k \in A_1(i)\right) P\left\{k \in A_1(i)\right\} \\
&= \sum_{k=1}^{n} E\left[\left(a\left(Z_{ik}\right) + a\left(Z_{kj}\right)\right) \mid k \in A_1(i)\right] r_{ik} - E\left[a\left(Z_{ii}\right) + a\left(Z_{ij}\right)\right] r_{ii} \\
&= \sum_{k=1}^{n} r_{ik}\left[s_{ik} + Ea\left(Z_{kj}\right)\right] - Ea\left(Z_{ij}\right) r_{ii} \\
&= \sum_{k=1}^{n} r_{ik} s_{ik} + \sum_{k=1}^{n} r_{ik} Ea\left(Z_{kj}\right) - Ea\left(Z_{ij}\right) r_{ii},
\end{aligned}$$

from which we obtain

$$\left(1 + r_{ii}\right) Ea\left(Z_{ij}\right) = \sum_{k=1}^{n} r_{ik} s_{ik} + \sum_{k=1}^{n} r_{ik} Ea\left(Z_{kj}\right),$$

or, in matrix notation, $A = R_* A + S_2$. As before, R_* is convergent, so the result follows. ■

The remainder of this Section provides applications of Proposition 3 to quantities of interest in telecommunication networks.

4.3. Expected Number of Links Under the Type II Restriction

Define, analogously to C_1 of Section 3.3, the matrix C_2 as indicated in Proposition 3. It follows immediately that the expected number of links traveled from origin i to destination j by a packet under the Type II restriction is given by the i, j entry in $(I - R_*)^{-1} C_2$.

4.4. Delay Under the Type II Restriction

4.4.1. Expected Delay

Let D_2 be the matrix whose (i, j) entry is the (j, j) diagonal element of $R_* D^T$ (all columns of D_2 are identical). It follows immediately from Proposition 3 that the expected delay for a packet traveling from origin node i to destination node j under the Type II restriction is given by the i, j entry in $(I - R_*)^{-1}D_2$.

4.4.2. Variance of Delay and Jitter

Under the same uncorrelatedness conditions as in Section 3.4.3, and letting V_2 denote the matrix whose (i, j) entry is the (j, j) diagonal element of R_*V^T (all columns of V_2 are identical), it follows immediately from Proposition 3 that the variance of the delay for a packet traveling from origin node i to destination node j under the Type II restriction is given by the i, j entry in $(I - R_*)^{-1}V_2$. Jitter is obtained as usual by taking the square root of the variance.

4.4.3. Distribution of Delay

By analogy to Proposition 3 and Section 3.4.4, we obtain the following equation for the distribution of the cross-network delays under the Type II restriction and when the single-link delays are independent.

$$(1+r_{ii})F_{ij}(x) = \sum_{k=1}^{n} r_{ik}F_{ik} * F_{kj}(x) \ ,$$

again a linear system of integral equations for the $F_{ij}(x)$.

5. CONCLUSION

In this paper we have formalized a procedure for computing the values of a path-additive function in a flow network in equilibrium with two kinds of restricted Markovian routing when the individual link values of this function are given. A general equation for these values is presented and applied to important quantities of interest (latency and jitter) in telecommunications networks. An approach to obtaining the distribution of cross-network delays is described. The two types of routing restrictions are an attempt to make a better model (than is afforded by unrestricted random routing) of the flow of packets in an address-based routing scheme such as is common in IP-based telecommunication networks, package delivery networks, and the like.

The Jackson network is commonly used as a model for a packet telecommunications network. In a Jackson network, routing is random (unrestricted Markovian). However, it is not necessary for a flow network with random routing to be a Jackson network. This note explores the breadth of application of random routing concepts to other kinds of flow networks. The results given here also enhance our understanding of flow networks with address-based routing.

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