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Inapproximability Bounds for Shortest-Path Network Interdiction Problems

by

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ABSTRACT

We consider two network interdiction problems: one where a network user tries to traverse a network from a starting vertex s to a target vertex t along the shortest path while an interdictor tries to eliminate all short s-t paths by destroying as few vertices (arcs) as possible, and one where the network user, as before, tries to traverse the network from s to t along the shortest path while the interdictor tries to destroy a fixed number of vertices (arcs) so as to cause the biggest increase in the shortest s-t path. The latter problem is known as the Most Vital Vertices (Arcs) Problem. In this paper we provide inapproximability bounds for several variants of these problems.
Keywords: approximation algorithm, most vital edges problem, network interdiction, network inhibition, shortest-path interdiction, minimal vertex cover.

1 Introduction

Network interdiction problems involve two opposing sides, a network user who operates a network in order to optimize some objective function, and the interdictor who attempts to limit the network user's achievable objective value by interdicting network vertices or arcs. We assume that the interdictor has limited resources. Several versions of this problem were considered in the literature, see e.g., [MM70, GMT71, BGV89, CS82, CMW98, FH75, GMT71, Gol78, MM70, Whi99, IW02, Woo93, Cun85].

In this paper we consider the case when the network user's objective is to traverse the network from a starting vertex s to a target vertex t along the shortest path. In the simplest example of such a problem the interdictor tries to destroy the smallest possible number of arcs in the network to prevent the network user from reaching t. This leads to the well known problem of finding a minimum s-t cut, which can be obtained by a maximum flow computation. Some other special versions, for instance when for every vertex the number of outgoing arcs which can be destroyed by the interdictor is limited, can also be efficiently solved [ZGK05].

We consider two models: one where the interdictor tries to eliminate all short s-t paths while destroying as few vertices (arcs) as possible, and one where he tries to destroy a fixed number of vertices (arcs) so as to cause the biggest increase in the shortest s-t path. The latter problem is known as the Most Vital Vertices (Arcs) Problem.

A. Bar-Noy, S. Khuller and B. Schieber [BNKS95] showed that the Most Vital Vertices Problem and the Most Vital Edges Problem are NP-hard. In this paper we strengthen this result by providing inapproximability bounds for these and several other problems.

The remainder of the paper is organized as follows. In Section 1.1 we introduce several optimization problems and state our main results. We first prove these results in Sections 2, 3, 4 and 5 for more general, restricted versions of our problems, defined in Section 1.2. The proofs are based on the NP-hardness and inapproximability of minimum vertex cover problems, presented in Section 1.3. In Section 6 we provide reductions from restricted problems to the original versions, while in Section 7 we further reduce the original problems to the special case of bipartite input graphs. Finally, in Section 8 we consider related decision problems.

1.1 Main Results

We call the length of the shortest path (dipath) from a vertex s to t in a graph (digraph) G the s-t *distance* and denote it by $d_G(s,t)$.

We consider a graph (digraph) $G = (V, E)$ with a nonnegative length associated with every edge (arc), two distinct vertices s and t, and a threshold $k \in \mathbb{Z}_+$.

A *vertex blocker of* (G, s, t, k) is a set of vertices different from s and t whose removal increases the s-t distance to at least k. We define the Minimum Vertex Blocker to Short Paths Problem (MVBP) as follows:

Minimum Vertex Blocker to Short Paths Problem (MVBP)

Input: A graph (digraph) G with a nonnegative length associated with every edge (arc), two vertices s, t and a threshold k

Output: The size $b_V(G, s, t, k)$ of the smallest vertex blocker, i.e.,

$$b_V(G, s, t, k) = min\{ \, |U| \mid d_{G[V \setminus U]}(s, t) \geq k, \; U \subseteq V \setminus \{s, t\}\}.$$

Theorem 1 *It is NP-hard to approximate the size of the smallest vertex blocker within a factor smaller than* $10\sqrt{5} - 21 \approx 1.36$*, even for bipartite graphs.*

An *edge blocker of* (G, s, t, k) is a set of edges (arcs) whose removal increases the s-t distance to at least k. We define the Minimum Edge Blocker to Short Paths Problem (MEBP) as follows:

Minimum Edge Blocker to Short Paths Problem (MEBP)

Input: A graph (digraph) G with a nonnegative length associated with every edge (arc), two vertices s, t and a threshold k

Output: The size $b_E(G, s, t, k)$ of the smallest edge blocker, i.e.,

$$b_E(G, s, t, k) = min\{ \; |F| \mid d_{(V, E \setminus F)}(s, t) \geq k, \;\; E \subseteq F\}.$$

Theorem 2 *It is NP-hard to approximate the size of the smallest edge blocker within a factor smaller than* $10\sqrt{5} - 21 \approx 1.36$*, even for bipartite graphs.*

We define the Most Vital Vertices Problem (MVVP) as follows:

Most Vital Vertices Problem (MVVP)

Input: A graph (digraph) $G = (V, E)$ with a nonnegative length associated with every edge (arc), two special vertices s, t and a threshold k

Output: The maximum $l_V(G, s, t, k)$ of s-t distances in all graphs obtained from G by removing k vertices. More precisely:

$$l_V(G, s, t, k) = max\{d_{G[V \smallsetminus U]}(s, t) \mid U \subseteq V \smallsetminus \{s, t\}, |U| = k\}.$$

Theorem 3 *It is NP-hard to approximate l_V within a factor smaller than 2, even for bipartite graphs.*

We define the Most Vital Edges Problem (MVEP) as follows:

The Most Vital Edges Problem (MVEP)

Input: A graph (digraph) $G = (V, E)$ with a nonnegative length associated with every edge (arc), two vertices s, t and a threshold k

Output: The maximum $l_E(G, s, t, k)$ of s-t distances in all graphs obtained from G by removing k edges. More precisely:

$$l_E(G, s, t, k) = max\{d_{(V, E \smallsetminus F)}(s, t) \mid F \subseteq E, |F| = k\}.$$

Theorem 4 *It is NP-hard to approximate l_E within a factor smaller than 2, even for bipartite graphs.*

In Section 8 we reformulate the above results for the case of decision problems and prove the following, more general, NP-hardness results.

Theorem 5 *For every fixed $\epsilon > 0$ it is NP-hard to distinguish graphs having s-t distance d after removing k vertices (edges) from those having s-t distance less than $\frac{1}{2-\epsilon}\, d$ in all induced subgraphs obtained by removing $(\frac{34}{33} - \epsilon)k$ vertices (edges), where d and k are both parts of the input.*

Theorem 6 *For every fixed $\epsilon > 0$ it is NP-hard to distinguish graphs having a vertex (edge) blocker of size k to paths of length at most d from those having all vertex (edge) blockers of size greater than $(\frac{34}{33} - \epsilon)\, k$ to paths of length at most $\frac{1}{2-\epsilon}\, d$, where d and k are both parts of the input.*

1.2 Restricted Problems

In this section we define restricted versions of the above problems by introducing the assumption that some vertices (edges) cannot be removed. We called these vertices (edges) *fixed*. The remaining vertices (edges) are called *removable*.

We obtain *restricted*-MVBP and *restricted*-MVVP from MVBP and MVVP, respectively, by fixing some vertices (in addition to s and t). Similarly we obtain *restricted*-MEBP and *restricted*-MVEP from MEBP and MVEP, respectively, by fixing some edges.

For a graph G, two vertices s, t, a set of fixed vertices V' (or a set of fixed edges E') and a threshold k, let $b'_V(G, s, t, V', k)$, $b'_E(G, s, t, E', k)$, $l'_V(G, s, t, V', k)$ and $l'_E(G, s, t, E', k)$ denote the solutions to restricted-MVBP, restricted-MEBP, restricted-MVVP and restricted-MVEP, respectively.

Given an instance (G, s, t, V', k) of restricted-MVBP we assume that all removable vertices form a vertex blocker. Similarly given an instance (G, s, t, E', k) of restricted-MEBP we assume that all removable edges form an edge blocker.

1.3 Minimum Vertex Cover Problem

In this section we present previously known results on which the proofs of our main results are based. A *vertex cover* of an undirected graph G is a subset of vertices incident to every edge. Let $\tau(G)$ denote the size of the smallest vertex cover of G.

Deciding if G has a vertex cover of size at most k is NP-hard [GJ79], even for tripartite graphs [Pol74]. However, $\tau(G)$ can be easily approximated within a factor 2, since the vertex cover consisting of both vertices of edges belonging to the maximum matching can be computed in polynomial time and its size is at most $2\tau(G)$. Improving this simple 2-approximation algorithm has been a quite nontrivial task. The best known approximation algorithm has a factor of $2 - \Theta(\frac{1}{\sqrt{\log n}})$, where n is the number of vertices [Kar05].

On the other hand, in 1997 Håstad [Hås97] proved that it NP-hard to approximate $\tau(G)$ within a factor smaller than $\frac{7}{6} \approx 1.17$. Recently Dinur and Safra [DS05] obtained a better inapproximability factor of $10\sqrt{5} - 21 \approx 1.36$. For tripartite graphs it NP-hard to approximate $\tau(G)$ within a factor smaller than $\frac{34}{33} \approx 1.03$ [CCR99].

2 Proof of Theorem 1

In this section we prove Theorem 1 by reducing the minimum vertex cover problem to restricted-MVBP. As shown in Section 6.1 and Section 7, for each instance of restricted-MVBP we can construct an instance of MVBP with the same optimal value and a bipartite input graph. Therefore Theorem 7 below implies Theorem 1.

Theorem 7 *It is NP-hard to approximate b'_V within a factor smaller than $10\sqrt{5} - 21 \approx 1.36$.*

Proof: Let G be an undirected graph with vertices $v_1, \ldots, v_n$ (see Figure 1). We construct

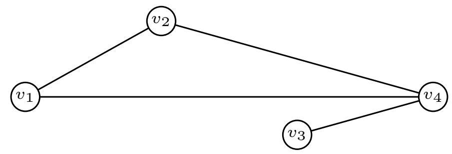

Figure 1: Graph G.

an instance of restricted-MVBP. We obtain an undirected graph H from G by adding to it a path $su_1u_2 \dots u_nt$ and connecting v_i to u_i for $i = 1, \dots, n$ (see Figure 2). Let W denote the vertex set of H. We assign length 1 to edges u_1u_2, $u_2u_3, \dots, u_{n-1}u_n$ and 0 to all other edges. Let $V' = \{u_1, \dots, u_n\}$ be the set of fixed vertices. The threshold is $n - 1$. Note that the set of all removable vertices forms a vertex blocker.

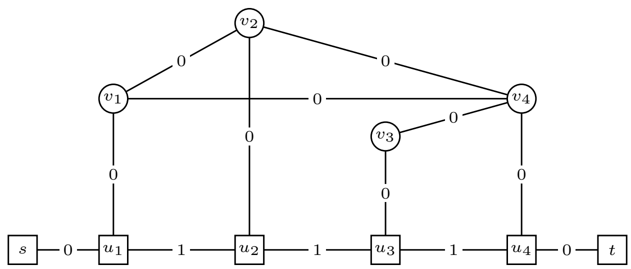

Figure 2: Graph H. Squares are fixed vertices.

Recall that $\tau(G)$ denotes the size of the smallest vertex cover of G.

Claim 1 $\tau(G) = b'_V(H, s, t, V', n - 1)$.

Proof: Let $U \subseteq \{v_1, \dots, v_n\}$ be a set of removable vertices. We show that U is a vertex cover of G if and only if U is a vertex blocker of $(H, s, t, n - 1)$.

Suppose U is a vertex cover of G. Since $V \setminus U$ is an independent set of G, there is only one s-t path, $su_1u_2 \dots u_nt$, in $H[W \setminus U]$ and the length of this path is $n - 1$ (see Figure 3).

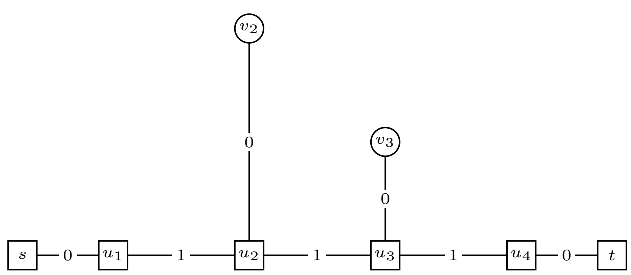

Figure 3: Graph $H[W \setminus U]$ obtained from H by removal of the vertex cover $U = \{v_1, v_4\}$ of G.

Conversely, suppose U is a vertex blocker of $(H, s, t, n-1)$. Note that for every $i < j$ there is no edge between vertices v_i and v_j in $H[W \setminus U]$, since otherwise there would exist a path $su_1 \dots u_i v_i v_j u_j \dots u_n t$ in $H[W \setminus U]$ shorter than $n-1$. Thus U is a vertex cover of G. □

Since it is NP-hard to approximate the minimum vertex cover within a factor smaller than $10\sqrt{5} - 21 \approx 1.36$ [DS05], Theorem 7 follows.

We can similarly reduce the minimum vertex cover problem to restricted-MVBP for directed graphs. Let H be a digraph obtained from G by replacing every edge $v_i v_j$, $i < j$, of G by an arc $v_i v_j$, adding to it a dipath $su_1 u_2 \dots u_n t$ and connecting v_i to u_i with two arcs $v_i u_i$ and $u_i v_i$ for $i = 1, \dots, n$ (see Figure 4).

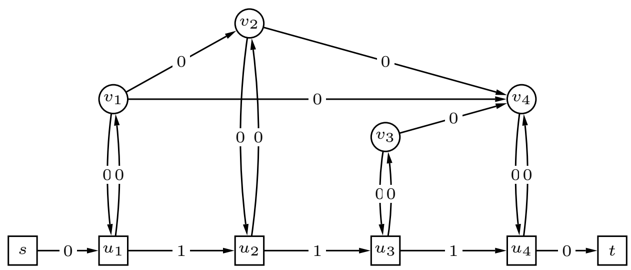

Figure 4: Digraph H. Squares are fixed vertices.

As before we assign length 1 to arcs $u_1 u_2$, $u_2 u_3, \dots, u_{n-1} u_n$ and 0 to all other arcs, vertices $u_1, \dots, u_n$ are fixed and the threshold is $n-1$. The proof that $\tau(G) = b'_V(H, s, t, V', n-1)$ is analogous. □

3 Proof of Theorem 2

In this section we prove Theorem 2 similarly to the proof of Theorem 1. We reduce the minimum vertex cover problem to restricted-MEBP. As shown in Section 6.2 and Section 7, for each instance of restricted-MEBP we can construct an instance of MEBP with the same optimal value and a bipartite input graph. Therefore Theorem 8 below implies Theorem 2.

In the proof of Theorem 8 we use a gadget first described in [BNKS95], where it was used to prove NP-hardness of the Most Vital Edges Problem.

Theorem 8 *It is NP-hard to approximate b'_E within a factor smaller than $10\sqrt{5} - 21 \approx 1.36$.*

Proof: Let G be a undirected graph with vertices $v_1, \dots, v_n$ (see Figure 1). We construct an instance of restricted-MEBP. We obtain an undirected graph H from G by

- replacing every vertex v_i of G by two vertices v'_i and v''_i connected by an edge $v'_i v''_i$ of length 1 for $i = 1, \dots, n$,

- replacing every edge v_iv_j, $i < j$, of G by $v_i''v_j'$ of length $5(j-i)-2$,
- adding to it a path $P = su_1'u_1''u_2'u_2'' \dots u_n'u_n''t$, where $u_i'u_i''$ has length 5 for $i = 1, \dots, n$ and other edges have length 0,
- adding two edges $v_i'u_i'$ and $v_i''u_i''$ of length 2 for $i = 1, \dots, n$ (see Figure 5).

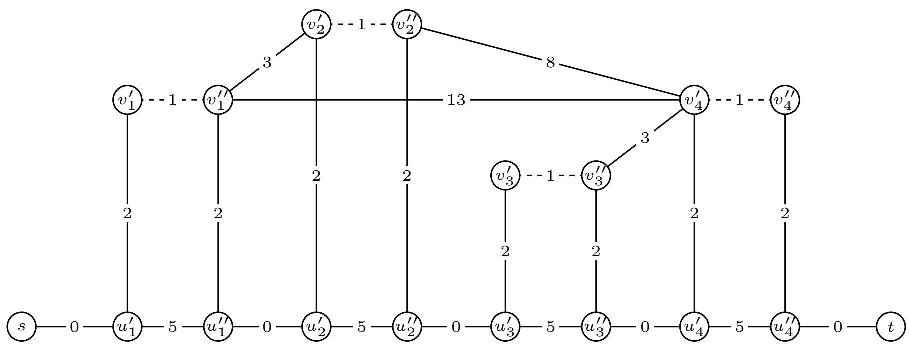

Figure 5: Graph H. Solid lines are fixed edges.

All edges except for $v_1'v_1'', \dots, v_n'v_n''$ are fixed. We denote the set of fixed edges by E'. The threshold is $5(n-1)$. Note that the set of all removable edges forms a vertex blocker. Let W and E denote the vertex set and the edge set of H, respectively.

Let $x \in \{u_i', u_i''\}$, $y \in \{u_j'', u_j''\}$, where $i \neq j$. We call the subpath of P from x to y an *x-y line.* An *x-y detour* is an x-y path D in H, where no vertices of D, apart from the first and the last, belong to P. An *i-j shortcut* is the path $v_i'v_i''v_j'v_j''$ (see Figure 6).

Let length(Q) denote the length of a path Q.

Claim 2 *If x-y detour D contains no shortcuts then* length$(D) \geq$ length(*x-y line*).

Proof: There are four possible kinds of x-y detours containing no shortcuts:

Case 1: $u_i'v_i'v_i''u_i''$, for $i = 1, \dots, n$. Then length(*x-y line*) $= 5$ and length$(detour(x, y)) =$ 5

Case 2: $u_i'v_i'v_i''v_j'u_j'$, for $i = 1, \dots, n$. Then length(*x-y line*) $= 5(j-i)$ and length$(detour(x, y)) = 5(j-i)+3$,

Case 3: $u_i''v_i''v_j'u_j'$, for $i = 1, \dots, n$. Then length(*x-y line*) $= 5(j-i-1)$ and length$(detour(x, y)) = 5(j-i)+2$.

Case 4: $u_i''v_i''v_j'v_j''u_j''$, for $i = 1, \dots, n$. Then length(*x-y line*) $= 5(j-i)$ and length$(detour(x, y)) = 5(j-i)+3$.

Thus all four kinds of x-y detours are at least as long as x-y line. □

Claim 3 *Let Q be an s-t path in H. If for every detour D contained in P* length$(D) \geq$ length(*x-y line*)*, where x and y are ends of D, then the length of Q is at least* $5(n-1)$*.*

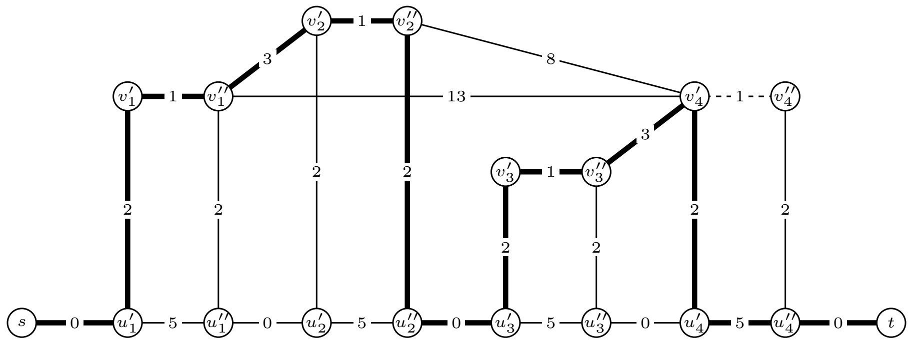

Figure 6: Thick lines are edges of the s-t path consisting of the s-u_1' line, a u_1'-u_2'' detour D_1, the u_2''-u_3' line, a u_3'-u_4' detour D_2 and the u_4'-t line. Note that the detour D_1 contains the 1-2 shortcut.

Proof: Note that Q starts with the edge su_1' and ends with the edge $u_n''t$. Thus we can decompose Q into an alternating sequence of lines and detours the s-l_1 line, an l_1-r_1 detour D_1, the r_1-l_2 line, an l_2-r_2 detour D_2, ..., an l_m-r_m detour D_m, the r_m-t line (see Figure 6). Since no x-y detour is shorter than the x-y line, we have

$$\begin{aligned}
\text{length}(Q) &= \text{length}(s\text{-}l_1 \textit{ line}) + \text{length}(D_1) + \text{length}(r_1\text{-}l_2 \textit{ line}) \\
&\quad + \text{length}(D_2) + \ldots + \text{length}(r_m\text{-}t \textit{ line}) \\
&\geq \text{length}(s\text{-}l_1 \textit{ line}) + \text{length}(l_1\text{-}r_1 \textit{ line}) + \text{length}(r_1\text{-}l_2 \textit{ line}) \\
&\quad + \text{length}(l_2\text{-}r_2 \textit{ line}) + \ldots + \text{length}(r_m\text{-}t \textit{ line}) \\
&\geq \text{length}(P) = 5(n-1).
\end{aligned}$$

□

Recall that $\tau(G)$ denotes the size of the smallest vertex cover of G.

Claim 4 $\tau(G) = b_E'(H, s, t, E', 5(n-1))$.

Proof: Let F be a set of removable edges. We show that $\{v_i \mid v_i'v_i'' \in F\}$ is a vertex cover of G if and only if F is an edge blocker of $(H, s, t, 5(n-1), E')$.

Suppose $\{v_i \mid v_iv_i' \in F\}$ is a vertex cover of G. Thus there is no shortcut in the graph $(W, E \setminus F)$. By Claim 2 all x-y detours are longer than x-y lines, which by Claim 3 implies that every s-t path has length at least $5(n-1)$.

Conversely, suppose F is an edge blocker of $(H, s, t, E', 5(n-1))$ and suppose that $v_i'v_i'', v_j'v_j'' \notin F$, for some edge v_iv_j of G. Then F does not block the path consisting of the s-u_i' line, the u_i'-u_j'' detour $u_i'v_i'v_i''v_j'v_j''u_j''$ and the u_j''-t line which has a total length $5(n-1)-1$, a contradiction. □

Since it is NP-hard to approximate the minimum vertex cover within a factor smaller than $10\sqrt{5} - 21 \approx 1.36$ [DS05], Theorem 8 follows.

Note that we can similarly reduce the Minimum Vertex Cover Problem to restricted-MEBP for directed graphs. Let H be a digraph obtained from G by

- replacing every vertex v_i of G by two vertices v_i' and v_i'' connected by an arc $v_i'v_i''$ of length 0 for $i = 1, \ldots, n$,
- replacing every edge v_iv_j, $i < j$, of G by $v_i''v_j'$ of length 0,
- adding to it a dipath $su_1u_2 \ldots u_nt$, where arcs u_1u_2, $u_2u_3, \ldots, u_{n-1}u_n$ have length 1 and all other arcs have length 0,
- adding two arcs u_iv_i' and $v_i''u_i$ of length 0 for $i = 1, \ldots, n$ (see Figure 7).

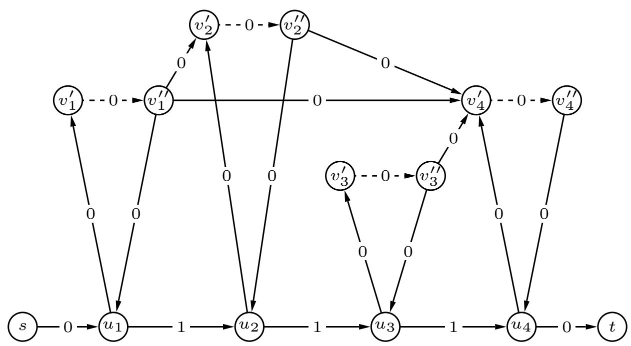

Figure 7: Digraph H. Solid lines are fixed arcs.

As before, all edges except for $v_1'v_1'', \ldots, v_n'v_n''$ are fixed, we denote the set of fixed edges by E' and the threshold is $5(n-1)$.

Analogously to proof of Claim 1 we can show that $\tau(G) = b_E'(H, s, t, E', n-1)$, implying the theorem. □

4 Proof of Theorem 3

In this section we prove Theorem 3 by reducing the problem of deciding whether a tripartite graph has a vertex cover of size at most k, which is known to be NP-hard [Pol74], to restricted-MVVP. As shown in Section 6.3 and Section 7, for each instance of restricted-MVVP we can construct an instance of MVVP with the same optimal value and a bipartite input graph. Therefore Theorem 9 below implies Theorem 3.

Theorem 9 *It is NP-hard to approximate l_V' within a factor smaller than* 2.

Proof: We will show that a $(2-\epsilon)$-approximation algorithm, where $\epsilon > 0$, can decide whether a tripartite graph has a vertex cover of size k in polynomial time.

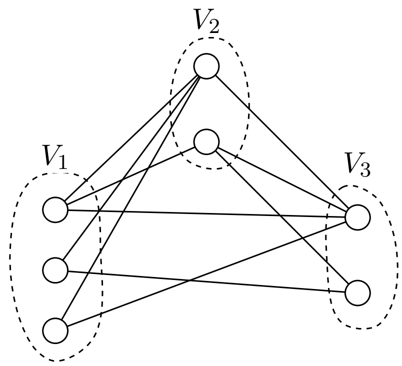

Figure 8: Tripartite graph G.

Let G be tripartite graph with vertex set $V = V_1 \cup V_2 \cup V_3$, where V_1, V_2 and V_3 are independent sets (see Figure 8). We construct an instance of restricted-MVVP. We obtain an undirected graph H from G by adding to it a path $su_1u_2u_3t$ and connecting every $v \in V_i$ to u_i, for $i = 1, 2, 3$ (see Figure 9). Let W denote the vertex set of H. We assign length 1 to edges u_1u_2, u_2u_3 and 0 to all other edges. Vertices u_1, u_2, u_3 are fixed.

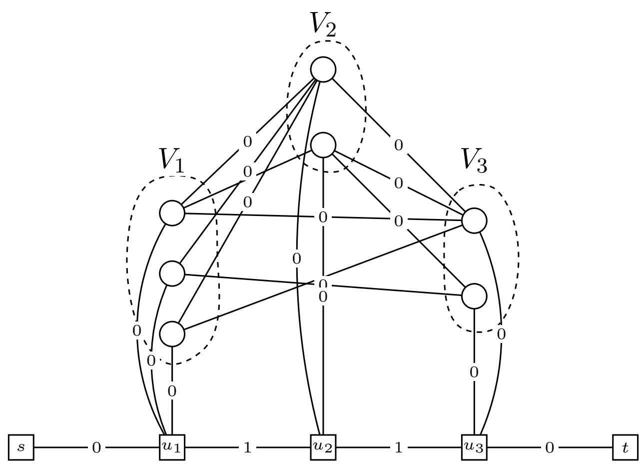

Figure 9: Graph H. Squares are fixed vertices.

Claim 5

(i) If G has a vertex cover of size at most k then $l'_V(H, s, t, V', k) = 2$.

(ii) If G does not have a vertex cover of size at most k then $l'_V(H, s, t, V', k) \leq 1$.

Proof: *(i)* Let U be a vertex cover of G such that $|U| \leq k$. Since $V \setminus U$ is an independent set in G, there is only one s-t path, $su_1u_2u_3t$, in $H[W \setminus U]$ and the length of this path is 2.

(ii) Since G has no vertex cover of size k, for every k-element subset U of removable vertices, $V \setminus U$ is not independent in G. Thus there is an edge xy in $H[W \setminus U]$ with x and y belonging to different parts of G. There are three cases:

Case 1: $x \in V_1$, $y \in V_2$. Then $su_1xyu_2u_3t$ is an s-t path of length 1.
Case 2: $x \in V_1$, $y \in V_3$. Then su_1xyu_3t is an s-t path of length 0.
Case 3: $x \in V_2$, $y \in V_3$. Then $su_1u_2xyu_3t$ is an s-t path of length 1.
Thus the s-t distance in $H[W \setminus U]$ is 0 or 1 for every k-element set U of removable vertices. □

Since a $(2-\epsilon)$-approximation algorithm, when run on H, must produce a solution smaller than 2 when $l'_V(H,s,t,V',k) \in \{0,1\}$ and a solution greater than or equal to 2 when $l'_V(H,s,t,V',k) = 2$, such an algorithm could distinguish graphs that have a vertex cover of size k from graphs that do not.

We can similarly reduce the Most Vital Vertices Problem to restricted-MVVP for directed graphs. We obtain a directed graph H from G by replacing every edge vw, where $v \in V_i$, $w \in V_j$, $i < j$, of G by an arc vw, adding to it a dipath $su_1u_2u_3t$ and two arcs vu_i and u_iv for every $v \in V_i$, for $i = 1,2,3$ (see Figure 10). We assign length 1 to arcs u_1u_2, u_2u_3 and 0 to all other arcs. Vertices u_1, u_2, u_3 are fixed. The proof of Claim 5 is essentially the same as in the undirected case. □

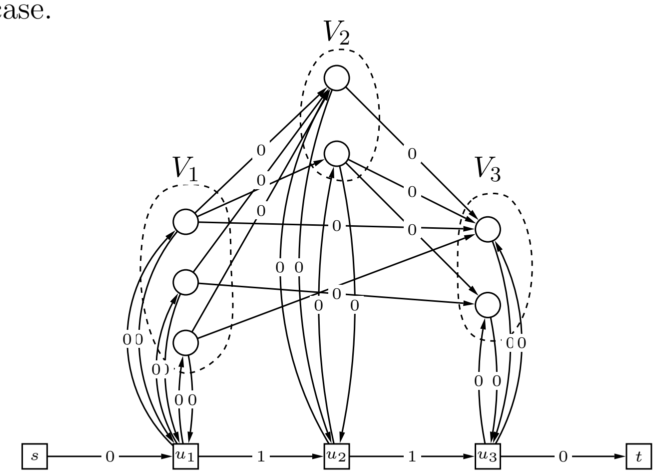

Figure 10: Digraph H. Squares are fixed vertices.

5 Proof of Theorem 4

In this section we prove Theorem 4 by reducing the problem of deciding whether a tripartite graph has a vertex cover of size at most k, which is known to be NP-hard [Pol74], to restricted-MVEP. As shown in Section 6.4 and Section 7, for each instance of restricted-MVEP we can construct an instance of MVEP with the same optimal value and a bipartite input graph. Therefore Theorem 10 below implies Theorem 4.

Theorem 10 *It is NP-hard to approximate l'_E within a factor smaller than 2.*

Proof: We will show that a $(2-\epsilon)$-approximation algorithm, where $\epsilon > 0$, can decide whether a tripartite graph has a vertex cover of size k in polynomial time.

Let G be tripartite graph with vertex set $V = V_1 \cup V_2 \cup V_3$ (see Figure 8). We next construct an instance of restricted-MVEP. We obtain an undirected graph H from G by replacing every vertex $v \in V$ by two vertices v', v'' and an edge $v'v''$, replacing every edge vw, where $v \in V_i, w \in V_j, i < j$ by $v''w'$, adding to it a path $su_1'u_1''u_2'u_2''u_3'u_3''t$ and connecting every $v' \in V_i$ to u_i' and $v'' \in V_i$ to u_i'', for $i = 1, 2, 3$ (see Figure 11). Let W denote the vertex set of H. We assign length 2 to edges $u_1'u_1''$, $u_2'u_2''$, $u_3'u_3''$, length 1 to an edge $v''u_1''$ for every $v'' \in V_1$, length 1 to edges $v'u_2'$, $v''u_2''$ for every $v', v'' \in V_2$, length 1 to an edge $v'u_3'$ for every $v' \in V_3$, and 0 to all other edges. The arcs $v'v''$ for $v \in V$ are removable.

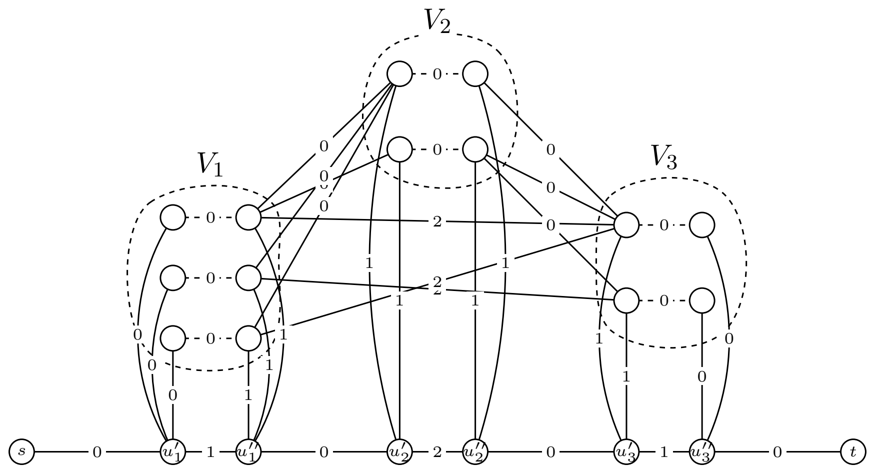

Figure 11: Graph H. Solid lines are fixed edges.

Claim 6

(i) If G has a vertex cover of size at most k then $l_E'(H, s, t, V', k) = 4$.

(ii) If G does not have a vertex cover of size at most k then $l_E'(H, s, t, V', k) \leq 2$.

Proof: *(i)* Let U be a vertex cover of G such that $|U| \leq k$. Since $V \setminus U$ is an independent set in G, there is only one s-t path, $su_1'u_1''u_2'u_2''u_3'u_3''t$, in $H[W \setminus U]$ and the length of this path is 4.

(ii) Since G has no vertex cover of size k, for every k-element subset U of removable vertices, $V \setminus U$ is not independent in G. Thus there is an edge xy in $H[W \setminus U]$ with x and y belonging to different parts of G. There are two cases:

Case 1: $x \in V_1$, $y \in V_2$ or $x \in V_2$, $y \in V_3$. Then the shortest s-x path, the edge xy and the shortest y-t path form an s-t path of length 2.

Case 2: $x \in V_1$, $y \in V_3$. Then the shortest s-x path, the edge xy and the shortest y-t path form an s-t path of length 0.

Thus the s-t distance in $H[W \setminus U]$ is 0 or 2 for every k-element set U of removable vertices. □

Since a $(2-\epsilon)$-approximation algorithm, when run on H, must produce a solution smaller than 2 when $l'_E(H,s,t,V',k) \in \{0,1\}$ and a solution greater than or equal to 2 when $l'_E(H,s,t,V',k) = 2$, such an algorithm could distinguish graphs that have a vertex cover of size k from graphs that do not.

We can similarly reduce the Most Vital Edges Problem to restricted-MVEP for directed graphs. We obtain a directed graph H from G by replacing every vertex $v \in V$ by two vertices v' and v'', replacing every arc vw, where $v \in V_i, w \in V_j, i < j$ by $v''w'$, adding to it a dipath $su_1u_2u_3t$ and two arcs u_iv' and $v''u_i$ for every $v \in V_i$, for $i = 1,2,3$ (see Figure 12). We assign length 1 to arcs u_1u_2, u_2u_3 and 0 to all other arcs. The arcs $v'v''$ for $v \in V$ are removable. The proof of Claim 6 is essentially the same as in the undirected case.

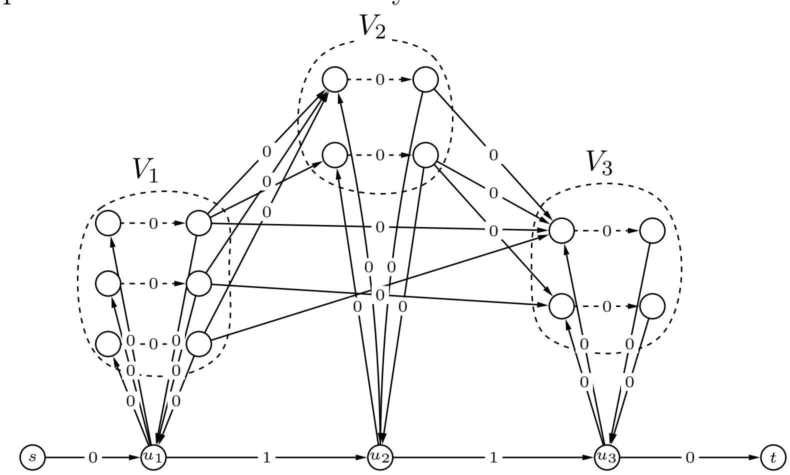

Figure 12: Digraph H. Solid lines are fixed arcs.

□

6 Reduction from Restricted to Original Problems

In this section for each instance of a restricted problem we construct in polynomial time an instance of the original problem with the same optimal value.

For an undirected graph we define the operation of *splitting a vertex x into n copies* as follows: we replace x by vertices $x^1, \ldots, x^n$ and each edge xy of length l by edges $x^1y, \ldots, x^ny$ of length l (see Figure 13). We call vertices $x^1, \ldots, x^n$ *split vertices of x*.

Analogously, for a directed graph we define the operation of *splitting a vertex x into n copies* as follows: we replace x by vertices $x^1, \ldots, x^n$, each arc xy of length l by edges

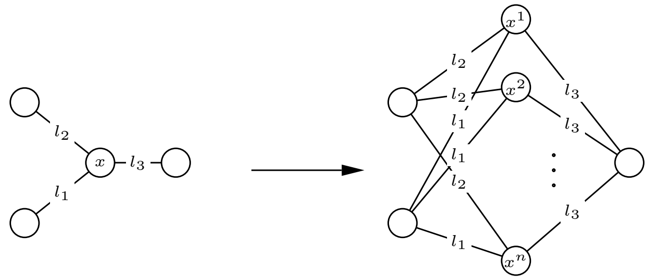

Figure 13: Operation of splitting x into n copies.

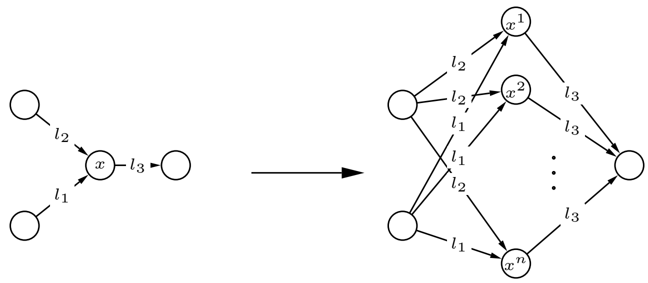

Figure 14: Operation of splitting x into n copies in directed graphs.

$x^1y, \ldots, x^ny$ of length l and each arc yx of length l by edges $yx^1, \ldots, yx^n$ of length l (see Figure 14).

For a graph (digraph) we define the operation of *splitting an edge (arc) xy into n copies* as follows: we add vertices $z^1, \ldots, z^n$, then replace the edge (arc) xy of length l by edges (arcs) $xz^1, \ldots, xz^n$ of length l and edges (arcs) $z^1y, \ldots, z^ny$ of length 0 (see Figure 15). We call vertices $z^1, \ldots, z^n$ *division vertices of xy.*

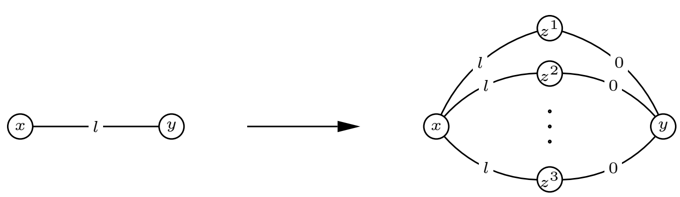

Figure 15: Operation of splitting xy into n copies.

6.1 Reduction from Restricted-MVBP to MVBP

For each instance of restricted-MVBP we construct in polynomial time an instance of MVBP with the same size of the minimum vertex blocker.

Let (G, s, t, V', k) be an instance of restricted-MVBP. Recall that we assume that all removable vertices form a vertex blocker. Let n be the number of vertices of G. We obtain a graph H from G by consecutively splitting every fixed vertex $x \in V'$ into n copies. Let W denote the vertex set of H.

Observation 1 *Let U be a subset of removable vertices. U is a vertex blocker of (H, s, t, k) if and only if U is a vertex blocker of (G, s, t, V', k).*

Claim 7 *Let U be a minimum vertex blocker of (H, s, t, k). If U contains a split vertex y of some fixed vertex x, then U contains all split vertices of x.*

Proof: Since $y \in U$, there is an s-t path in in $H[(W \setminus U) \cup y]$ through y which is shorter than k. Suppose there is a split vertex z of x such that $z \notin U$. Since the neighborhoods of y and z are the same we can replace y by z in this path and obtain a path of the same length in $H[(W \setminus U)]$, a contradiction with U being a vertex blocker. Thus all split vertices of x belong to U. □

Proposition 1 *$b'_V(G, s, t, V', k) = b_V(H, s, t, k)$.*

Proof: By Observation 1 every vertex blocker of (G, s, t, V', k) is a vertex blocker of (H, s, t, k). Thus $b'_V(G, s, t, V', k) \geq b_V(H, s, t, k)$.

Suppose $b'_V(G, s, t, V', k) > b_V(H, s, t, k)$. Let U be a minimum vertex blocker of (H, s, t, k). Since by our assumption all removable vertices form a vertex blocker of (G, s, t, V', k), we obtain $|U| < n$. Thus by Claim 7 U cannot contain split vertices. By Observation 1 U is a vertex blocker of (G, s, t, V', k), a contradiction. □

6.2 Reduction from restricted-MEBP to MEBP

For each instance of restricted-MEBP we construct in polynomial time an instance of MEBP with the same size of the minimum edge blocker.

Let (G, s, t, E', k) be an instance of restricted-MEBP. Recall that we assume that all removable edges (arcs) form a edge blocker. Let m be the number of edges (arcs) of G. We obtain a graph H from G by consecutively splitting every fixed edge (arcs) $xy \in E'$ into m copies.

Similarly to Proposition 1 we can show that the minimum edge blockers of (G, s, t, E', k) and (H, s, t, k) have the same size.

Proposition 2 *$b'_E(G, s, t, E', k) = b_V(H, s, t, k)$.*

6.3 Reduction from restricted-MVVP to MVVP

For each instance of restricted-MVVP we construct in polynomial time an instance of MVVP with the same optimal value.

Let (G, s, t, V', k) be an instance of restricted-MVVP. We construct an instance (H, s, t, k) as in Section 6.1. Similarly to Proposition 1 we can show that the the maximum of s-t distances in all graphs obtained from G by removing k vertices and the maximum of s-t distances in all graphs obtained from H by removing k vertices are equal.

Proposition 3 *$l'_V(G, s, t, V', k) = l_V(H, s, t, k)$.*

6.4 Reduction from restricted-MVEP to MVEP

For each instance of restricted-MVEP we construct in polynomial time an instance of MVEP with the same optimal value.

Let (G, s, t, V', k) be an instance of restricted-MVEP. We construct an instance (H, s, t, k) as in Section 6.2. Similarly to Proposition 1 we can show that the the maximum of s-t distances in all graphs obtained from G by removing k edges and the maximum of s-t distances in all graphs obtained from H by removing k edges are equal.

Proposition 4 $l'_E(G, s, t, E', k) = l_E(H, s, t, k)$.

7 Reduction to Bipartite Graphs

In this section for each instance of an original problem we construct in polynomial time an instance with a bipartite input graph and the same optimal value.

Let $G = (V, E)$ be a graph (digraph). We construct a graph (digraph) H by splitting every edge of G into 1 copy, where the operation of edge splitting was defined in Section 6. Let W be the set of vertices newly added division vertices. Note that the graph H is bipartite, since every edge of H has one endpoint in V and the other in W. Analogously to Proposition 1, we can prove that $b_E(G, s, t, k) = b_E(H, s, t, k)$ and $l_E(G, s, t, k) = l_E(H, s, t, k)$.

We next obtain a graph (digraph) H' from H by splitting every vertex of W into $|V|$ copies, where the operation of vertex splitting was defined in Section 6. Note that H' is still bipartite, and we can prove that $b_V(G, s, t, k) = b_V(H', s, t, k)$ and $l_V(G, s, t, k) = l_V(H', s, t, k)$.

8 Decision Problems

Using the well known connection between optimization and decision problems (see Chapter 29 in [Vaz01]) we can restate Theorems 1, 2, 3 and 4 as follows:

Proposition 5 (Theorem 1' and Theorem 2') *It is NP-hard to distinguish instances of MVBP having a vertex (edge) blocker of size d to paths of length at most k from those having all vertex (edge) blockers of size greater than $1.36\,d$ to paths of length at most k, where d is also a part of the input.* □

Proposition 6 (Theorem 3' and Theorem 4') *For every fixed $\epsilon > 0$ it is NP-hard to distinguish instances of MVVP having s-t distance d after removing some k vertices (edges) from those having s-t distance less than $\frac{1}{2-\epsilon}\,d$ in all induced subgraphs obtained by removing k vertices (edges), where k is also a part of the input.* □

Note that Theorem 5 is the strengthening of Theorems 3' and 4'. Similarly Theorem 6 can be viewed a two-sided generalization of Theorems 1' and 2', although the corresponding factor is worse.

8.1 Proof of Theorem 5

As shown in [CCR99], it is NP-hard to approximate the size of the smallest vertex cover in tripartite graphs within a factor smaller than $\frac{34}{33}$. This can be restated as follows: for every fixed $\epsilon > 0$ it is NP-hard to distinguish tripartite graphs having a vertex cover of size k from those having all vertex covers of size greater than $(\frac{34}{33} - \epsilon)k$, where k is a part of the input.

The claim below immediately follows from Claims 5 and 6.

Claim 8 *Let G be a tripartite graph, let H_V and H_E be the graphs constructed from G in Sections 4 and 5, respectively, and let $\epsilon > 0$.*

(i) If G has a vertex cover of size k then $l'_V(H_V, s, t, V', k) \geq 2$ and $l'_E(H_E, s, t, E', k) \geq 2$.

(ii) If all vertex covers of G have size larger than $(\frac{34}{33} - \epsilon)k$ then $l'_V(H_V, s, t, V', (\frac{34}{33} - \epsilon)k) \leq 1$ and $l'_E(H_E, s, t, E', (\frac{34}{33} - \epsilon)k) \leq 1$.

Theorem 5 follows from Claim 8 and the inapproximability result stated in the beginning of this subsection.

8.2 Proof of Theorem 6

Claim 9 *Let G be a graph and let s, t be two vertices of G.*

(i) $b_V(G, s, t, b) \leq a$ if and only if $l_V(G, s, t, a) \geq b$.

(ii) $b_E(G, s, t, b) \leq a$ if and only if $l_E(G, s, t, a) \geq b$.

Proof: Both expressions $b_V(G, s, t, b) \leq a$ and $l_V(G, s, t, a) \geq b$ are equivalent to the existence of a vertex blocker of size a to s-t paths of length at most b in G. The proof of *(ii)* is identical. $\square$

We can obtain Theorem 6 by applying Claim 9 to Theorem 5.

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