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Process Grammar

by

Michael Leyton[1]
DIMACS
Rutgers University
New Brunswick, NJ 08854
mleyton@dimacs.rutgers.edu

[1]Permanent Member

ABSTRACT

This report gives an exposition of the Process-Grammar, published originally in the journal *Artificial Intelligence* in 1988, together with a description of some of the subsequent applications of the grammar in meteorology, biology, computer-aided design, chemical engineering, and geology. The Process-Grammar is a means of recovering the process-history of a smooth shape from its curvature extrema, and expressing that evolution in terms of transitions at those extrema. The inference of history follows from the Symmetry-Curvature Duality Theorem of Leyton (1987), which states that, to each curvature extremum, there is a differential symmetry axis leading to and terminating at that extremum; and from an inference rule that states that the symmetry axis is the record of a process. The Process-Grammar expresses the relationship between any two stages in the shape's history as an extrapolation of the processes inferred by the theorem.

1 Extraction of History from Shape

The purpose of this paper is to describe a grammar that I published in the journal Artificial Intelligence in 1988. The grammar is essentially a theorem I proved that any smooth shape evolution of a smooth 2D curve can be expressed in terms of six types of transitions at curvature extrema. These transitions constitute what I call the *Process-Grammar*.

After I published the grammar, the grammar, and the mathematics on which it is based, was applied by scientists in many disciplines: Radiology, meteorology, computer vision, chemical engineering, geology, computer-aided design, robotics, anatomy, botany, forensic science, architecture, abductive reasoning, linguistics, mechanical engineering, computer graphics, archaeology, etc.

Let us begin by understanding the purpose for which the grammar was developed: inferring history from shape; e.g., from the shapes of tumors, embryos, clouds, etc. For example, the shape shown in Fig 1 can be understood as the result of various processes such as protrusion, indentation, squashing, resistance. My book *Symmetry, Causality, Mind* (MIT Press) was essentially a 630-page rule-system for deducing the past history that formed any shape. The Process-Grammar is part of that rule-system – that related to the use of curvature extrema.

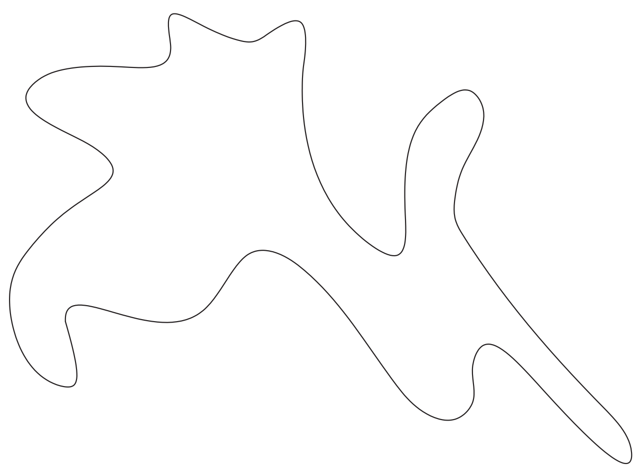

Figure 1: Shape as history.

2 The PISA Symmetry Analysis

It is first necessary to understand how symmetry can be defined in complex shape. Clearly, in a simple shape, such as an equilateral triangle, a symmetry axis is easy to define. One simply places a straight mirror across the shape such that one half is reflected onto the other. The straight line of the mirror is then defined to be a symmetry axis of the shape. However, in a complex shape, it is often impossible to place a mirror that will reflect one half of the figure onto the other. Fig 1, is an

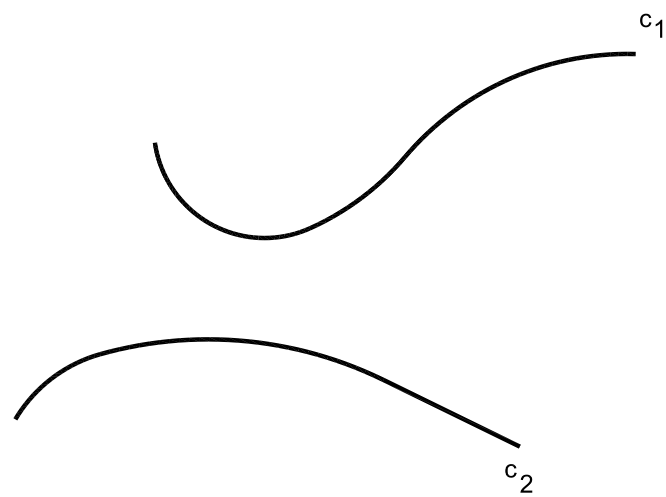

Figure 2: How can one construct a symmetry axis between these to curves?

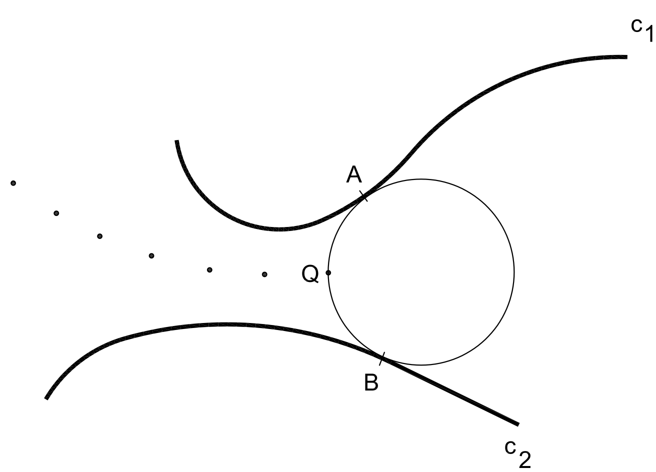

Figure 3: In the PISA system, the points Q define the symmetry axis.

example of such a shape. However, in such cases, one might still wish to regard the figure, or part of it, as symmetrical about some *curved* axis. Such a generalized axis can be constructed in the following way.

Consider Fig 2. It shows two curves c_1 and c_2, which can be understood as two sides of an object. Notice that no mirror could reflect one of these curves onto the other. The goal is to construct a symmetry axis between the two curves. One proceeds as follows: As shown in Fig 3, introduce a circle that is tangential simultaneously to the two curves. Here the two tangent points are marked as A and B.

Next, move the circle continuously along the two curves, c_1 and c_2, while always ensuring that it maintains the property of being tangential to the two curves simultaneously. To maintain this double-touching property, it might be necessary to expand or contract the circle. This initial procedure was invented by Blum in the 1960s, and he then defined the symmetry axis to be the center of the circle as it moved. However, in my book, *Symmetry, Causality, Mind*, I showed that there are serious topological problems with this definition, and it furthermore cannot infer the processes that have acted on the shape. Therefore, in contrast, I defined the axis to be the trajectory of the point Q shown in Fig 3. This is the point on the circle, half-way between the two tangent points. As the circle moves along the curves, it traces out a trajectory as indicated by the sequence of dots shown in the figure. I called this axis, *Process-Inferring Symmetry Axis*, or simply PISA. It does not have the problems associated with the Blum axis.

3 Symmetry-Curvature Duality

The Process-Grammar relies on two structural factors in a shape: symmetry and curvature. Mathematically, symmetry and curvature are two very different descriptors of shape. However, a theorem that I proposed and proved in [3] shows that there is an intimate relationship between these two descriptors. This relationship will be the basis of the entire paper. The theorem will be a crucial step in our argument:

SYMMETRY-CURVATURE DUALITY THEOREM (Leyton, 1987): *Any section of curve, that has one and only one curvature extremum, has one and only one symmetry axis. This axis is forced to terminate at the extremum itself.*

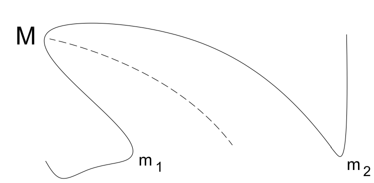

Figure 4: Illustration of the Symmetry-Curvature Duality Theorem.

The theorem can be illustrated by looking at Fig 4. On the curve shown, there are three extrema: m_1, M, and m_2. Therefore, on the section of curve *between* extrema m_1 and m_2, there is only one extremum, M. What the theorem says is this: Because this section of curve has only one extremum, it has only one symmetry axis. This axis is forced to terminate at the extremum M. The axis is shown as the dashed line in the figure.

It is valuable to illustrate the theorem on a closed shape, for example, that shown in Fig 5. This shape has sixteen curvature extrema. Therefore, the above theorem tells us that there are sixteen unique symmetry axes associated with, and terminating at, the extrema. They are given as the dashed lines shown the figure.

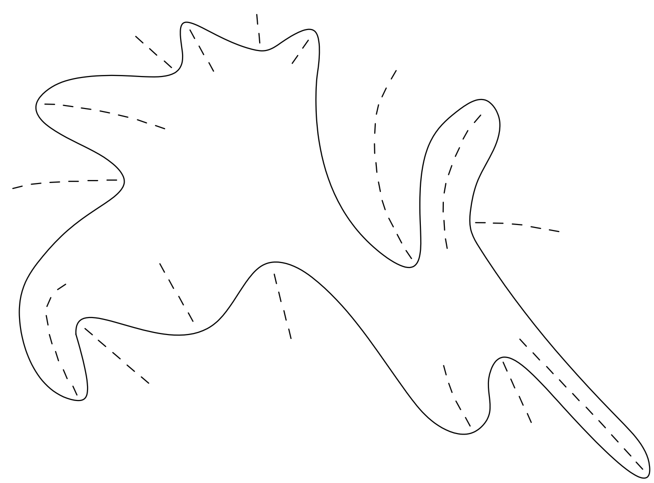

Figure 5: Sixteen extrema imply sixteen symmetry axes.

4 The Interaction Principle

The reason for involving symmetry axes is that it will be argued that they are closely related to process-histories. This proposed relationship is given by the following principle:

INTERACTION PRINCIPLE (Leyton, 1984): *Symmetry axes are the directions along which processes are hypothesized as most likely to have acted.*

The principle was advanced and extensively corroborated in Leyton [7], in several areas of perception including motion perception as well as shape perception. The argument used in Leyton [7] to justify the principle, involves the following two steps: (1) A process that acts along a symmetry axis tends to preserve the symmetry; i.e. to be structure-preserving. (2) Structure-preserving processes are perceived as the most likely processes to occur or to have occurred.

5 The Inference of Processes

We now have the tools required to understand how processes are recovered from shape. In fact, the system to be proposed consists of two inference rules that are applied successively to a shape. The rules can be illustrated by considering Fig 6.

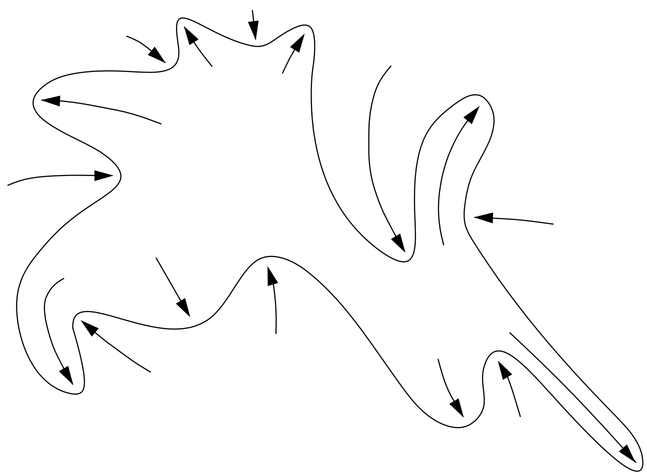

Figure 6: The processes inferred by the rules.

The first rule is the Symmetry-Curvature Duality Theorem which states that, to each curvature extremum, there is a unique symmetry axis terminating at that extremum. The second rule is the Interaction Principle, which states that each of the axes is a direction along which a process has acted. The implication is that the boundary was deformed along the axes; e.g. each protrusion was the result of pushing out along its axis, and each indentation was the result of pushing in along its axis. In fact, each axis is the trace or record of boundary-movement!

Under this analysis, processes are understood as creating the curvature extrema; e.g. the processes introduce protrusions and indentations etc., into the shape boundary. This means that, if one were to go backwards in time, undoing all the inferred processes, one would eventually remove all the extrema. Observe that there is only one closed curve without extrema: the circle. Thus the implication is that the ultimate starting shape must have been a circle, and this was deformed under various processes each of which produced an extremum.

Corroboration: To obtain extensive corroboration for the above rules, let us now apply them to a large catalogue of shapes: all shapes with up to, and including, eight curvature extrema. The catalogue provides *purely the outlines* exhibited in Fig 7, 8, and 9. Most of these outlines come from a paper by Richards, Koenderink & Hoffman [14], and the Process-Grammar was used to complete the catalogue. What I have done is taken these outlines and applied to them the above three rules for the recovery of process-history. The inference rules put the arrows on each shape, indicating how the shapes were formed over time. As the reader can see, the inferred histories accord very strongly with one's sense of how these shapes were formed.

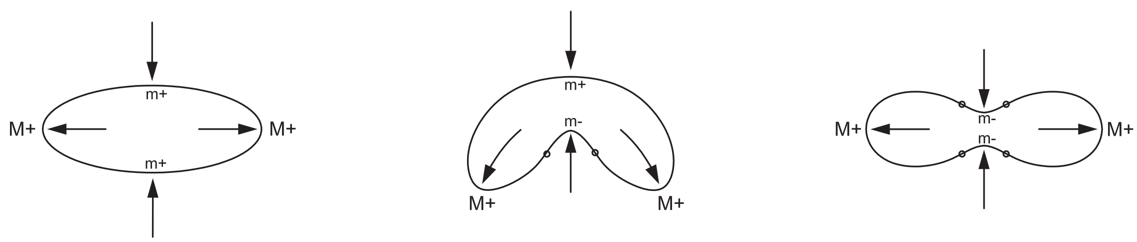

Figure 7: The inferred histories on the shapes with 4 extrema.

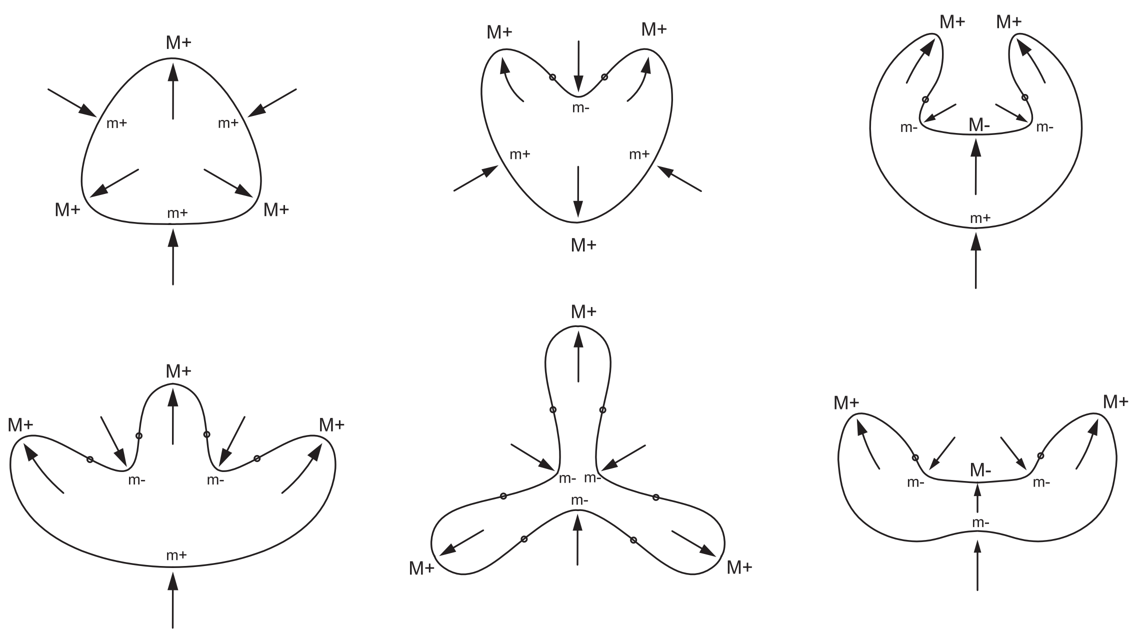

Figure 8: The inferred histories on the shapes with 6 extrema.

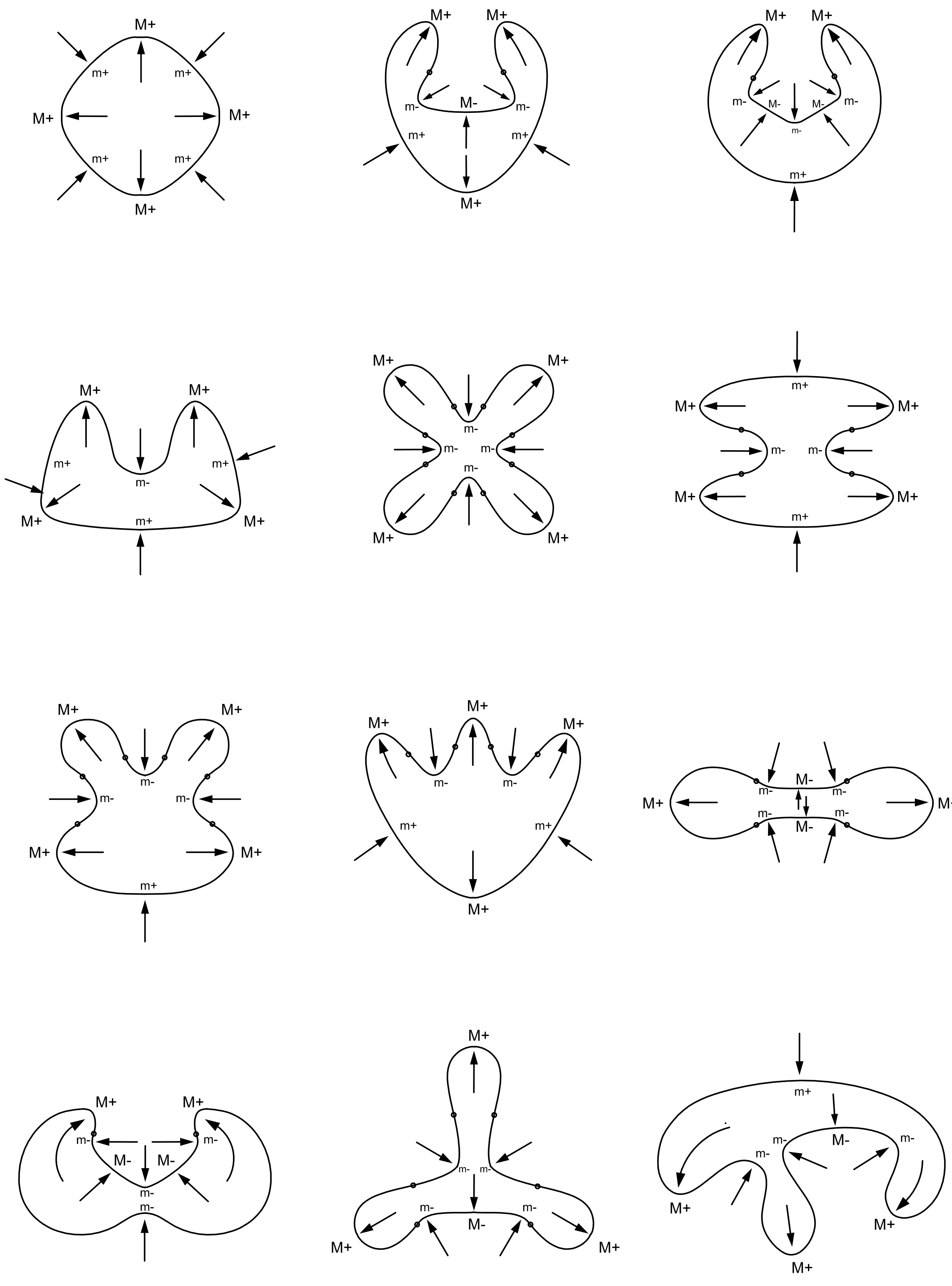

Figure 9: The inferred histories on the shapes with 8 extrema.

6 Extremum Type

Any individual outline, together with the inferred arrows, will be called a *process-diagram*. The reader should observe that, on each process-diagram in Figs 7–9, a letter-label has been placed at each extremum (the end of each arrow). There are four alternative labels, M^+, m^-, m^+, and M^-, and these correspond to the four alternative types of curvature extrema. The four types are shown in Fig 10 and are explained as follows:

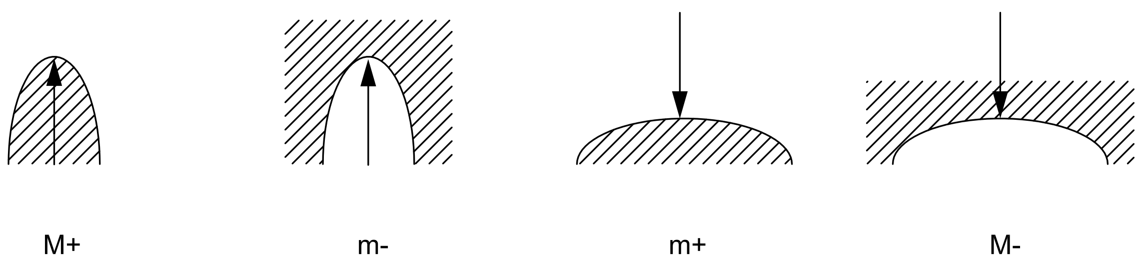

Figure 10: The four types of extrema.

First, understand the curve as the *boundary* of an object. Refer to the object side of the curve as "solid" and the other side as "empty". Now observe that the first two kinds of extrema in Fig 10 have the same shape. They are the sharpest points on their respective curves. Their difference is that they change the side on which the solid (shaded) and empty (non-shaded) occur; i.e., they are figure/ground reversals of each other. The remaining two extrema are also the same shape as each other. They are the flattest points on their respective curves. Again, they are figure/ground reversals of each other.

Now notice the following important phenomenon: The above characterizations of the four extrema types are purely structural. However, in surveying the shapes in Figs 7–9, it becomes clear that the four extrema types correspond to four English terms that people use to describe *processes*. Table 1 gives the correspondence:

EXTREMUM TYPE	$\longleftrightarrow$	**PROCESS TYPE**
M^+	$\longleftrightarrow$	protrusion
m^-	$\longleftrightarrow$	indentation
m^+	$\longleftrightarrow$	squashing
M^-	$\longleftrightarrow$	internal resistance

Table 1: Correspondence between extremum type and process type.

It is important to understand that the entire psychological basis for peoples' use of these four process terms is explained by our inference rules. This is shown as follows: By the Interaction Principle, the inferred process is along the symmetry axis. This symmetry axis is provided by the PISA definition which I invented, given in section 2.

The fundamental fact is that PISA puts the axes on the *convex* side of the curve for the first two extrema, M^+ and m^-, and on the *concave* side of the curve for the other two extrema, m^+ and M^-. By the Interaction Principle, the inferred process arrows must be along these symmetry axes, and must therefore be as shown in Fig 10. Notice that the arrows for the first two extrema are on the convex side of the curve, and the arrows for the other two extrema are on the concave side.

As a consequence of this, we see that the first two arrows explain the *sharpening* of the curve at the extremum, and the other two explain the *flattening* at the extremum. This shows why people use the four process terms in the above table. It is entirely due to the mathematical properties of the PISA symmetry definition.

No other symmetry axis in the history of mathematics has this property. That is, all other axes would be "between" the sides of the curve; i.e., on the convex side; i.e., below the curves in all the four cases in Fig 10. In fact, the Medial Axis of Blum would not only put the axis below the curve, in the third and fourth case, but the axis would move *downwards* away from the curve, and would therefore not correspond to any meaningful process.

A detailed comparison between PISA and the other symmetry definitions is given in my book *Symmetry, Causality, Mind*, and this comparison shows the extreme inappropriateness of the other definitions of symmetry.

Finally, let us understand the meaning of the four symbols M^+, m^-, m^+, M^-. First choose the direction of traveling along the curve to be that which keeps the solid on the *left* side of the curve. Then define curvature as the rate of *anti-clockwise* rotation. Denote a curvature maximum and minimum by M and m, respectively; and denote positive and negative curvature by $+$ and $-$, respectively. Then, in the *curvature function*, the four kinds of extrema are as illustrated by the graph in Fig 11.

7 The Method to be Used

What we have done so far is to lay the ground-work of the Process-Grammar. The grammar will characterize the way shapes deform into each other in terms of the events that occur at their most important points: the curvature extrema. It turns out that there are essentially six transitions that can happen at the curvature extrema. These six transitions will constitute the grammar.

Our procedure establishing this will be as follows: Let us imagine that we have two stages in the history of the shape. For example, imagine being a doctor looking at two X-rays of a tumor taken a month apart. Observe that any doctor examines two such X-rays (e.g., on a screen), in order to assess what has happened in the intervening month. If one considers the way the doctor's thinking proceeds, one realizes that there is a basic inference rule that is being used: The doctor tries, where possible, to explain processes seen in the later shape as *extrapolations* of processes already seen in

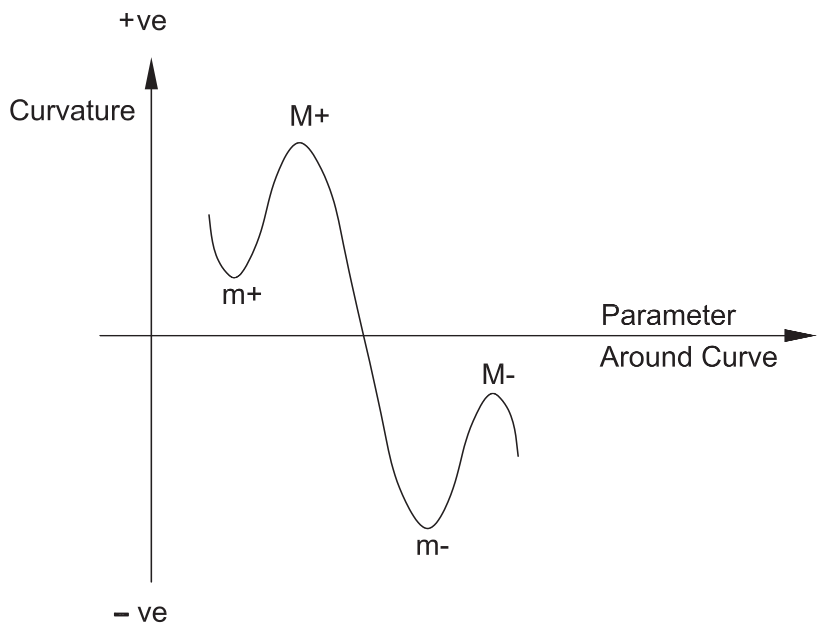

Figure 11: A curvature function showing the four kinds of extrema.

the earlier shape. That is, the doctor tries to maximize the description of *history as extrapolations*. We will show how to discover these extrapolations.

Recall that the processes we have been examining are those that move along symmetry axes, creating extrema. As a simple first cut, we can say that extrapolations have one of two forms:

> **(1) Continuation:** The process simply continues along the symmetry axis, maintaining that single axis.
>
> **(2) Bifurcation:** The process branches into two axes, i.e., creating two processes out of one.

Now recall, from section 6, that there are four types of extrema M^+, m^-, m^+, and M^-. It is necessary therefore to look at what happens when one continues the process at each of the four types, and at what happens when one branches (bifurcates) the process at each of the four types. This means that there are *eight* possible events that can occur: four continuations and four bifurcations.

8 Continuation at M^+ and m^-

Let us start by considering continuations, and then move on to bifurcations. It turns out that, when one continues a process at either of the first two extrema, M^+ or m^-, nothing significant happens, as follows:

First consider M^+. Recall from Table 1 (p8), that the M^+ extremum corresponds to a protrusion. Fig 12 shows three examples of M^+, the three protrusions. We want to understand what happens when any one of the M^+ processes is continued. For example, what happens when the protruding process at the top M^+ continues pushing the boundary further along the direction of its arrow?

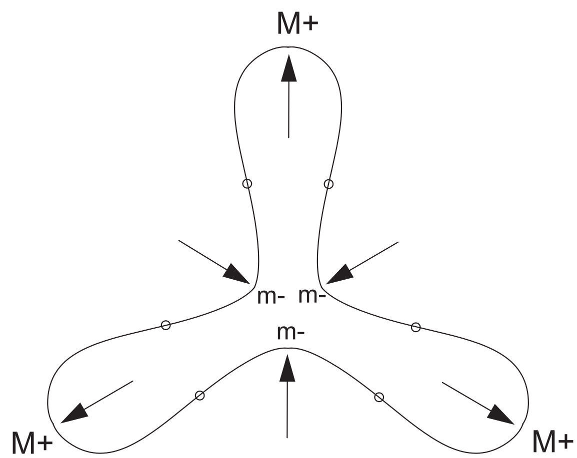

Figure 12: Continuation at M^+ and m^- do not change extremum-type.

The answer is simple: The boundary would remain a M^+ extremum, despite being extended further upwards. Intuitively, this is obvious: A protrusion remains a protrusion if it continues. Therefore, from now on, we will ignore continuation at M^+ as structurally trivial.

Now observe that exactly the same considerations apply with respect to any m^- extremum. For example, notice that the same shape, Fig 12, has three m^- extrema. Observe also that, in accord with Table 1 (p8), each of these corresponds to an *indentation*. It is clear that, if the process continues at a m^-, the boundary would remain m^-. Again, this is intuitively obvious: An indentation remains an indentation if it continues. As a consequence, we will also ignore continuation at m^- as structurally trivial.

In summary, the two cases considered in this section, continuation at M^+ and at m^-, are structurally trivial. It will now be seen that continuations at the remaining two extrema, m^+ and M^-, induce much more interesting effects on a shape.

9 Continuation at m^+

According to Table 1 (p8), a m^+ extremum is always associated with a *squashing* process. An example is shown in the top of the left shape in Fig 13. Notice therefore that the process explains the flattening at this extremum, relative to the greater bend at either end of the top.

Our goal is to understand what happens when the process at this m^+ extremum is continued forward in time; i.e., the downward arrow pushes further downward. Clearly, a continuation of the process can result in the indentation shown at the top of the right shape in Fig 13.

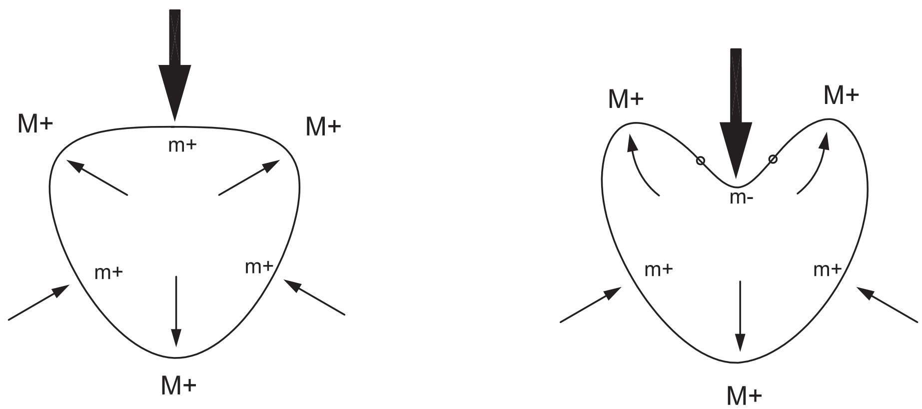

Figure 13: Continuation at m^+.

The structural change, in going from the left to the right shape, should be understood as follows: First, the m^+ at the top of the left shape changes to the m^- at the top of the right shape. Notice that the m^- extremum corresponds to an indentation, as predicted by Table 1 (p8).

An extra feature should be observed: On either side of the m^- extremum, at the top of the right shape, a small circular dot has been placed. Such a dot marks a position where the curvature is zero; i.e., the curve is, locally, completely straight. If one were driving around this curve, the dot would mark the place where the steering wheel would point straight ahead.

With these facts, one can now describe exactly what occurred in the transition from the left shape to the right shape: The m^+ extremum at the top of the left shape has changed into a m^- extremum at the top of the right shape, and two points of zero curvature, 0, have been introduced on either side of the m^-. One can therefore say that the transition from the left shape to the right shape is the replacement of m^+ (left shape) by the triple, $0m^-0$ (right shape). The transition is therefore:

$$m^+ \longrightarrow 0m^-0.$$

This transition will be labeled Cm^+ meaning *Continuation at* m^+. Thus the transition is given fully as:

$$Cm^+ \ : \ m^+ \longrightarrow 0m^-0.$$

This mathematical expression is easy to translate into English. Reading the symbols, from left to right, the expression says:

Continuation at m^+ *takes* m^+ *and replaces it by the triple* $0m^-0$.

It is worth having a simple phrase defining the transition in Fig 13. Notice that, since the extremum m^+ in the left shape is a squashing, and the extremum m^- in the right shape is an indentation, the transition can be described as:

A squashing continues till it indents.

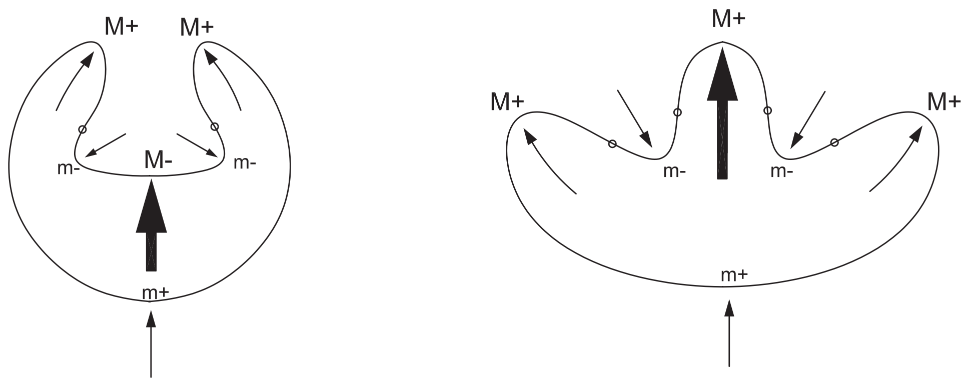

Figure 14: Continuation at M^{-}.

10 Continuation at M^{-}

We will now investigate what happens when the process at the fourth and final extremum M^{-} is continued forward in time. As an example, consider the M^{-} in the center of the bay in the left shape in Fig 14. In accord with Table 1 (p8), the process at this extremum is an *internal resistance*. In order to understand this process, let us suppose that the left shape represents an island. Initially, this island was circular. Then, there was an inflow of water at the top (creating a dip inwards). This flow increased inward, but met a ridge of mountains along the center of the island. The mountain ridge acted as a resistance to the inflow of water, and thus the bay was formed. In the center of the bay, the point labeled M^{-} is a curvature extremum. One can view it as the point on the bay with the *least* amount of bend (more rigorously it is a negative maximum).

Now return to the main issue of this section: What happens when the upward resistive arrow (terminating at the M^{-} extremum) is continued along the direction of the arrow. This could happen for example, if there is a volcano in the mountains, that erupts, sending lava down into the sea. The result would therefore be the shape shown on the right in Fig 14. In other words, a protrusion would be formed into the sea.

The structural change, in going from the left to the right shape, should be understood as follows: First, the M^{-} in the center of the bay (left shape) changes into the M^{+} at the top of the right shape, the protrusion.

An extra feature should be observed: On either side of the M^{+} extremum, at the top of the right shape, a small circular dot has been placed. Such a dot again marks a position where the curvature is zero; i.e., the curve is, locally, completely straight.

Thus we can describe what has happened in the transition from the left shape to the right shape: The M^{-} extremum in the bay of the left shape has changed into a M^{+} extremum at the top of the right shape, and two points of zero curvature, 0, have been introduced on either side of the M^{+}. In other words, the M^{-} in the left shape has been replaced by the triple, $0M^{+}0$ in the right shape. The transition is therefore:

$$M^{-} \longrightarrow 0M^{+}0.$$

This transition will be labeled CM^{-} meaning *Continuation at* M^{-}. Thus the transition is given

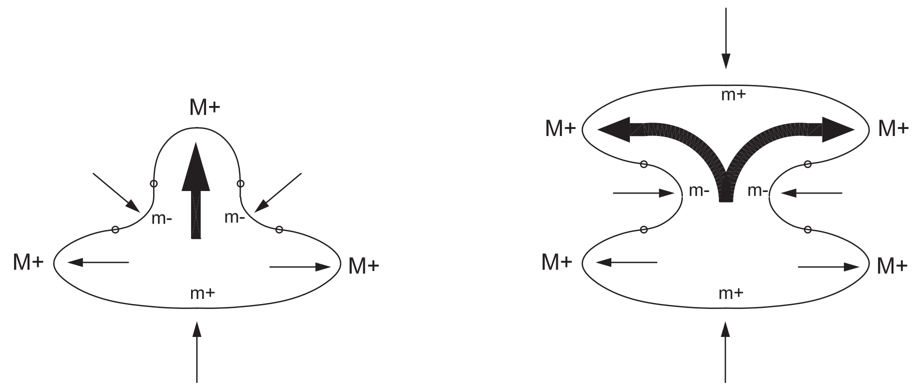

Figure 15: Bifurcation at M^+.

fully as:

$$CM^- \ : \ M^- \longrightarrow 0M^+0.$$

This mathematical expression is easy to translate into English. Reading the symbols, from left to right, the expression says:

Continuation at M^- takes M^- and replaces it by the triple $0M^+0$.

It is worth having a simple phrase defining the transition in Fig 14. Notice that, since the extremum M^- in the left shape is a resistance, and the extremum M^+ in the right shape is a protrusion, the transition can be described as:

A resistance continues till it protrudes.

Comment: We have now gone through each of the four extrema, and defined what happens when the process at the extremum is allowed to continue. The first and second extrema involved no structural change, but the second and third extrema did.

11 Bifurcation at M^+

We now turn from continuations to bifurcations (branchings) at extrema. Again, each of the four extrema will be investigated in turn.

First we examine what happens when the process at a M^+ extremum branches forward in time. As an example, consider the M^+ at the top of the left shape in Fig 15. In accord with Table 1 (p8), the process at this extremum is a *protrusion*. The effect of bifurcating is shown in the right shape. One branch goes to the left, and the other goes to the right.

The structural change, in going from the left to the right shape, should be understood as follows: First observe that the single M^+ at the top of the left shape, splits into two copies of itself, shown at the ends of the two branches in the right shape.

There is also another feature. In the center of the top of the right shape, a new extremum has been introduced, m^+. Note that the process at this extremum is a *squashing*, as predicted in Table 1 on p8. This process explains the flattening in the middle of the top, relative to the sharpening towards either end of the top. The m^+ extremum is a minimum, and is required mathematically, because the two branching extrema are maxima M, and two maxima cannot exist without a minimum in between.

With these facts, one can now describe exactly what occurred in the transition from the left shape to the right shape: The M^+ extremum at the top of the left shape has split into two copies of itself in the right shape, and a new extremum m^+ has been introduced. That is, the transition from the left shape to the right shape is the replacement of M^+ (left shape) by the triple, $M^+m^+M^+$ (right shape). The transition is therefore:

$$M^+ \longrightarrow M^+m^+M^+.$$

This transition will be labeled BM^+, meaning *Bifurcation at* M^+. Thus the transition is given fully as:

$$BM^+ \ : \ M^+ \longrightarrow M^+m^+M^+.$$

This mathematical expression is easy to translate into English. Reading the symbols, from left to right, the expression says:

Bifurcation at M^+ takes M^+ and replaces it by the triple $M^+m^+M^+$.

It will also be worth having a simple phrase to summarize the effect of the transition in Fig 15. The structure formed on the right shape has the shape of a *shield*, and therefore, the transition will be referred to thus:

Shield-formation.

12 Bifurcation at m^-

Next we examine what happens when the process at a m^- extremum branches forward in time. As an example, consider the m^- at the top of the left shape in Fig 16. In accord with Table 1 (p8), the process at this extremum is an *indentation*. The effect of bifurcating is shown in the right shape. One branch goes to the left, and the other goes to the right. That is, a *bay* has been formed! Thus one can regard the transition from the left shape to the right one as the stage preceding Fig 14 on p13.

The structural change, in going from the left to the right shape in Fig 16, should be understood as follows: First observe that the single m^- at the top of the left shape, splits into two copies of itself, shown at the ends of the two branches in the right shape.

There is also another feature. In the center of the top of the right shape, a new extremum has been introduced, M^-. Note that the process at this extremum is a *resistance*, as predicted in Table

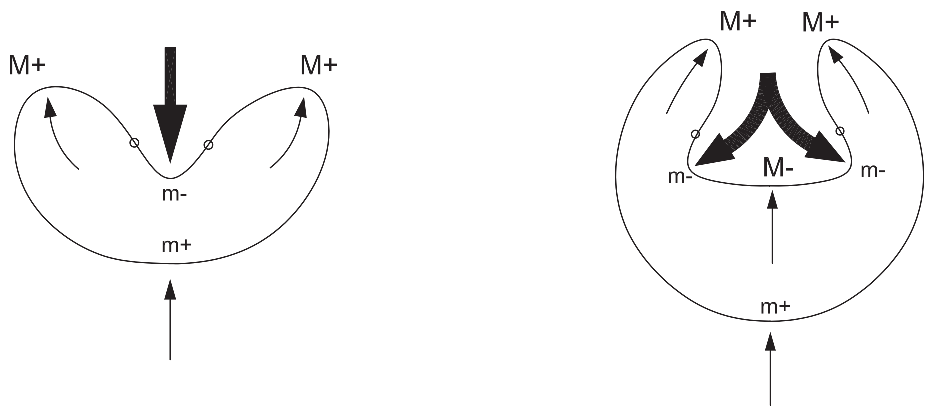

Figure 16: Bifurcation at m^-.

1 on p8. This process explains the flattening in the middle of the bay, relative to the sharpening towards either end of the bay.

With these facts, one can now describe exactly what occurred in the transition from the left shape to the right shape: The m^- extremum at the top of the left shape has been replaced by the triple, $m^-M^-m^-$ in the right shape. The transition is therefore:

$$m^- \longrightarrow m^-M^-m^-.$$

This transition will be labeled Bm^- meaning *Bifurcation at* m^-. Thus the transition is given fully as:

$$Bm^- \ : \ m^- \longrightarrow m^-M^-m^-.$$

This mathematical expression is easy to translate into English. Reading the symbols, from left to right, the expression says:

Bifurcation at m^- takes m^- and replaces it by the triple $m^-M^-m^-$.

It will also be worth having a simple phrase to summarize the effect of the transition in Fig 16. The obvious phrase is this:

Bay-formation.

13 The Bifurcation Format

The previous two sections established the first two bifurcations: those at M^+ and m^-. The next two sections will describe the remaining two bifurcations. However, before giving these, it is worth observing that the first two bifurcations allow us to see that bifurcations have the same format as each other, which is shown as follows:

$$E \ \longrightarrow \ EeE.$$

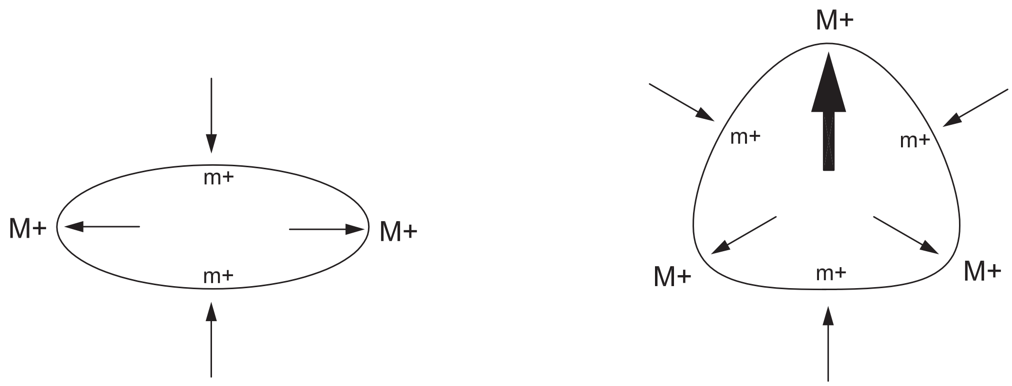

Figure 17: Bifurcation at m^+.

An extremum E is sent to two copies of itself, and a new extremum e is introduced between the two copies. The new extremum e is determined completely from E as follows: Extremum e must be the opposite type from E; that is, it much change a Maximum (M) into a minimum (m), and vice versa. Furthermore, extremum e must have the same sign as E, that is, "+" or "-".

14 Bifurcation at m^+

Next we examine what happens when the process at a m^+ extremum bifurcates, forward in time. As an example, consider the m^+ at the top of the left shape in Fig 17. In accord with Table 1 (p8), the process at this extremum is a *squashing*.

The effect of bifurcation is that m^+ splits into two copies of itself – the two copies shown on either side of the right shape. One should imagine the two copies as *sliding* over the surface till they reached their current positions.

The other crucial event is the introduction of a new extremum M^+ in the top of the right shape. This is in accord with the bifurcation format described in the previous section. Notice that the upward process here conforms to Table 1 on p8, which says that a M^+ extremum always corresponds to a protrusion.

Thus the transition from the left shape to the right shape is the replacement of the m^+ extremum at the top of the left shape by the triple $m^+M^+m^+$ in the right shape. The transition is therefore:

$$m^+ \longrightarrow m^+M^+m^+.$$

This transition will be labeled Bm^+ meaning ***Bifurcation at*** m^+. Thus the transition is given fully as:

$$Bm^+ \ : \ m^+ \longrightarrow m^+M^+m^+.$$

This mathematical expression is easy to translate into English. Reading the symbols, from left to right, the expression says:

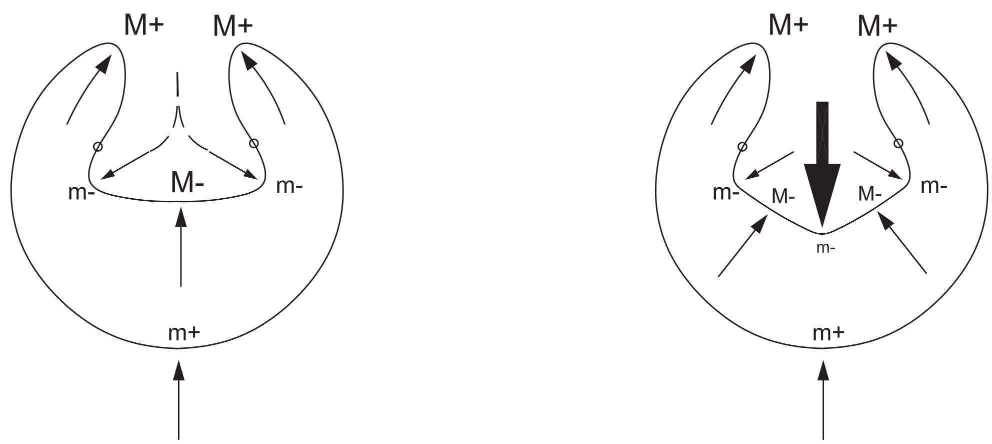

Figure 18: Bifurcation at M^{-}.

Bifurcation at m^{+} takes m^{+} and replaces it by the triple $m^{+}M^{+}m^{+}$.

It will also be worth having a simple phrase to summarize the effect of the transition, as follows: Notice that the main effect in Fig 17 is that the initial squashing process is pushed to either side by the breaking-through of an upward protrusion. Thus the transition can be summarized by the following phrase:

Breaking-through of a protrusion.

15 Bifurcation at M^{-}

Now we establish the final bifurcation. We examine what happens when the process at a M^{-} extremum bifurcates, forward in time. As an example, consider the M^{-} in the center of the bay, in left shape, in Fig 18. In accord with Table 1 (p8), the process at this extremum is an *internal resistance*.

The effect of bifurcation is that M^{-} splits into two copies of itself – the two copies shown at the two sides of the *deepened bay* in the right shape. One should imagine the two copies as *sliding* over the surface till they reached their current positions.

The other crucial event is the introduction of a new extremum m^{-} in the bottom of the right shape. This is in accord with the bifurcation format described in section 13. Notice that the downward process here conforms to Table 1 on p8, which says that a m^{-} extremum always corresponds to an indentation.

Thus the transition from the left shape to the right shape is the replacement of the M^{-} extremum in the middle of the left shape by the triple $M^{-}m^{-}M^{-}$ in the right shape. The transition is therefore:

$$M^{-} \longrightarrow M^{-}m^{-}M^{-}.$$

This transition will be labeled BM^- meaning *Bifurcation at* M^-. Thus the transition is given fully as:

$$BM^- \,:\, M^- \longrightarrow M^- m^- M^-.$$

This mathematical expression is easy to translate into English. Reading the symbols, from left to right, the expression says:

Bifurcation at M^- *takes* M^- *and replaces it by the triple* $M^- m^- M^-$.

It is also worth having a simple phrase to summarize the effect of the transition, as follows: Notice that the main effect in Fig 18 is that the initial resistance process is pushed to either side by the breaking-through of a downward indentation. Thus the transition can be summarized by the following phrase:

Breaking-through of an indentation.

16 The Process-Grammar

Having completed the bifurcations, let us now put together the entire system that has been developed in sections 7 to 15. Our concern has been to describe shape evolution by what happens at the most significant points on the shape: the curvature extrema. We have seen that the evolution of any smooth shape can be decomposed into into *six types of phase-transition* defined at the extrema involved. These phase-transitions are given as follows:

PROCESS GRAMMAR

Cm^+ :	m^+	$\longrightarrow$	$0m^-0$	(squashing continues till it indents)
CM^-:	M^-	$\longrightarrow$	$0M^+0$	(resistance continues till it protrudes)
BM^+:	M^+	$\longrightarrow$	$M^+m^+M^+$	(sheild-formation)
Bm^- :	m^-	$\longrightarrow$	$m^-M^-m^-$	(bay-formation)
Bm^+ :	m^+	$\longrightarrow$	$m^+M^+m^+$	(breaking-through of a protrusion)
BM^-:	M^-	$\longrightarrow$	$M^-m^-M^-$	(breaking-through of an indentation)

Note that the first two transitions are the two continuations, as indicated by the letter C at the beginning of the first two lines; and the last four transitions are the bifurcations, as indicated by the letter B at the beginning of the remaining lines.

17 Scientific Applications of the Process-Grammar

After I published the Process-Grammar in the journal *Artificial Intelligence* in 1988, scientists applied it and the Symmetry-Curvature Duality Theorem to many disciplines.

It is worth considering a number of applications here, to illustrate various concepts of the theory. In meteorology, Evangelos Milios [12] used the Process-Grammar to analyze and monitor high-altitude satellite imagery in order to detect weather patterns. This allowed the identification of the forces involved; i.e., the forces go along the arrows. It then becomes possible to make substantial predictions concerning the future evolution of storms. This work was done in relation to the Canadian Weather Service.

It is worth also considering the applications by Steve Shemlon, in biology. Shemlon [15] developed a continuous model of the grammar using an elastic string equation. For example, Fig 19 shows the backward time-evolution, provided by the equation. It follows the laws of the Process-Grammar. Notice how the shape goes back to a circle, as predicted in section 5. Fig 20 shows the corresponding tracks of the curvature extrema in that evolution. In this figure, one can see that the rules of the Process-Grammar mark the evolution stages. Shemlon applied this technique to analyze neuronal growth models, dental radiographs, electron micrographs and magnetic resonance imagery.

Let us now turn to an application of the Process-Grammar in Computer-Aided Design. Jean-Philippe Pernot et al [13] have developed a CAD toolbox based on the Process-Grammar for the use in design software for the automotive-aerospace industries. Their method is as follows: They begin by defining a limiting line for a feature as well as a target line. For example, the first surface in Fig 21 has a feature, a bump, with a limiting line given by its oval boundary on the surface, and its target line given by the ridge line along the top of the bump. The Process-Grammar is then used to manipulate the limiting line of the feature. Thus, applying the first operation Cm^+ of the grammar to the left-hand squashing process m^+ in the surface, this squashing continues till it indents in the second surface shown in Fig 21. With this tool-box, the designer is given considerable control over the surface to produce a large variety of free-form features.

Now let us look at an application of the Process-Grammar to chemical engineering by John Peter Lee [2]. Here the grammar was used to model molecular dynamics – in particular, the dynamical interactions within mixtures of solvent and solute particles. Fig 22 represents the data shape, in *velocity space*, of a single solute molecule as it interacts with other molecules.

The initial data shape is given by a sphere (in velocity space). This is deformed by the successively incoming data in such a way that, at any time, one can use my curvature inference rules on the current shape, in order to infer the *history of the data*. In other words, one does not have to keep the preceding data – one can use the rules to *infer* it. Incidently, the lines in Fig 22 correspond to the axes associated with curvature extrema as predicted by the rules.

Also, we will mention the application of the Process-Grammar in geology to the formation of volcanic islands. This work was done by T. W. Larsen [1] and Brian Mayoh [11]. The results, in their computer simulations, look remarkably similar to the diagrams given earlier for the Process-Grammar operations, demonstrating again the general validity of these operations.

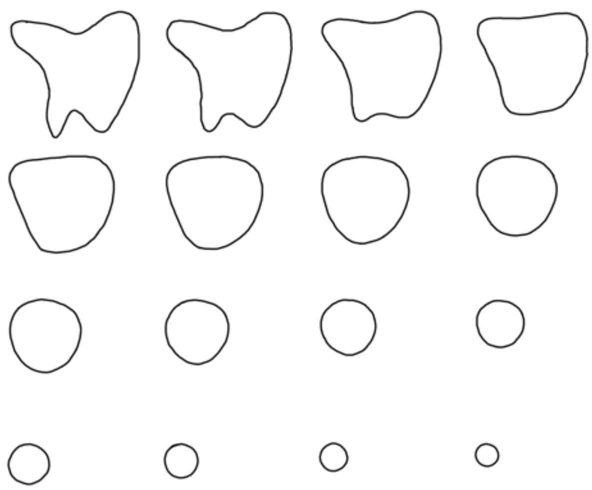

Figure 19: Continuous realization of the Process-Grammar for biological applications, by Steven Shemlon [15] using an elastic string equation.

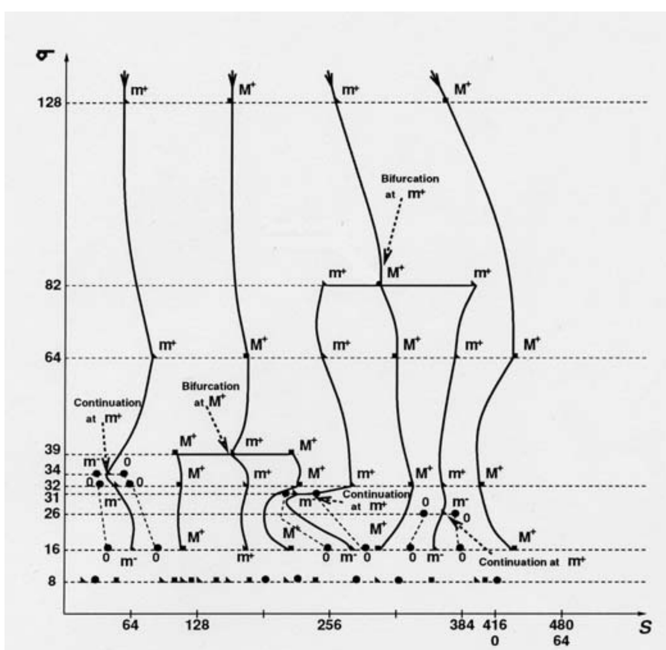

Figure 20: Shemlon's use of the Process-Grammar to label the transitions in the above biological example [15].

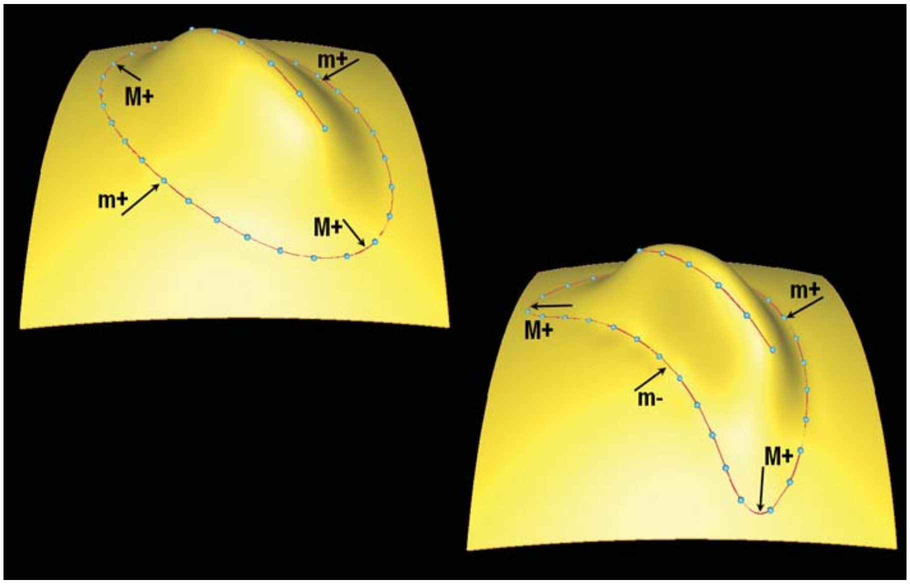

Figure 21: Application of the Process-Grammar to computer-aided design by Jean-Philippe Pernot et al [13].

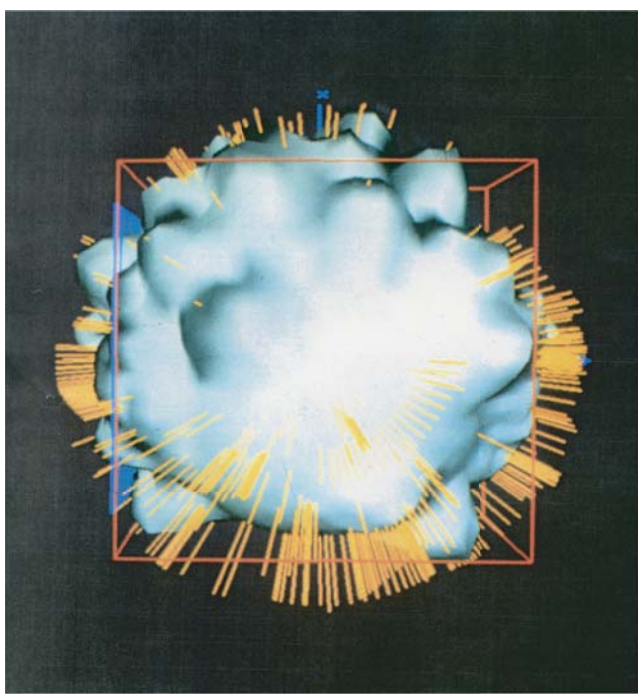

Figure 22: Application of the Process-Grammar in molecular dynamics, by J.P. Lee [2].

18 Artistic Applications of the Process-Grammar

The previous section reviewed some of the applications of the Process-Grammar in scientific disciplines. However, my book *Symmetry, Causality, Mind* (MIT Press), applied the grammar extensively to reveal the compositions of paintings. In fact, a principal argument of my books is this:

> **Artworks are structured by the rules for memory storage. That is, the rules of *aesthetics* are the rules for memory storage (Leyton, [7] [8] [10]).**

For example, my book *The Structure of Paintings* (Springer-Verlag) has demonstrated this by detailed and lengthy analyses of paintings by Picasso, Modigliani, Gauguin, Holbein, Ingres, Balthus, Raphael, Cézanne, De Kooning, etc.

In Figure 23, the rules summarized in this paper for the extraction of history from curvature extrema, are applied to Picasso's Still Life. The reader can see that this gives considerable insight into the composition of the painting.

Figure 23: Curvature extrema and their inferred processes in Picasso's Still Life.

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