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Minimal and locally minimal games and game forms.

by

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ABSTRACT

By Shapley's (1964) theorem, a matrix game has a saddle point whenever each of its 2×2 subgames have one. In other words, all minimal saddle point free (SP-free) matrices are of size 2×2. We strengthen this result and show that all **locally** minimal SP-free matrices are of size 2×2. In other words, if A is a SP-free matrix in which a saddle point appears after deleting an arbitrary row or column, then A is of size 2×2. Furthermore, we generalize this result and characterize the minimal and locally minimal Nash equilibrium free (NE-free) bimatrix games.

Let us recall that a two-person game form is Nash-solvable if and only if it is tight (Gurvich, 1975). We show that all (locally) minimal non tight game forms are of size 2×2. In contrast, it seems difficult to characterize the locally minimal tight game forms (while all minimal ones are just trivial); we only obtain some necessary and some sufficient conditions. We also recall an example from cooperative game theory: a maximal stable effectivity function that is not self-dual and not convex.
Key words: game, game form, saddle point, Nash equilibrium, effectivity function, minimal, locally minimal.

1 Introduction

1.1 Minimal and locally minimal Boolean vectors

Let us recall that x is a *minimal true vector* of a Boolean function f if $f(x) = 1$ and $f(x') = 0$ whenever $x' < x$. Furthermore, x is a *locally minimal* true vector of f if $f(x) = 1$ and $f(x') = 0$ for every immediate predecessor of x, that is, for each x' such that $x - x'$ is a unit vector. Clearly, each minimal true vector is locally minimal and these two concepts are equivalent for monotone Boolean functions, and not only for them, as we shall see below.

If a Boolean function is given by a polynomial oracle then the local minimality of a vector can be tested in polynomial time. In contrast, verifying the minimality can be NP-hard. For instance if f is represented by a CNF, then testing minimality is as hard as satisfiability.

In this paper we consider Boolean functions related to the families of games and game forms mentioned in the Abstract. The variables of these functions are the strategies of the players, for example, the rows and columns of a matrix in case of two-person games.

In Section 5 we also consider a problem from cooperative game theory related to characterizing maximal stable effectivity functions.

1.2 Minimal and locally minimal matrix games without saddle points

A matrix (or matrix game) $A = (a(i,j))_{i\in I}^{j\in J}$ is a real valued mapping $a : I \times J \to \mathbb{R}$. Its rows I and columns J are the strategies of two players: R, the maximizer, and C, the minimizer. A situation (that is, a pair of strategies) $(i,j) \in I \times J$ is called a *saddle point* (in pure strategies) if no player can improve the result by choosing another strategy, that is, if

$$a(i,j) \geq a(k,j)\ \forall k \in I \qquad \text{and} \qquad a(i,j) \leq a(i,\ell)\ \forall \ell \in J. \tag{1.1}$$

It is well-known and easy to see that if (i,j) and (k,ℓ) are two saddle points then (i,ℓ) and (k,j) are also saddle points and $a(i,j) = a(k,j) = a(i,\ell) = a(k,\ell)$.

Furthermore, a matrix game has a saddle point if and only if its maxmin and minmax are equal, that is,

$$v_R \;=\; \max_{i\in I}\min_{j\in J} a(i,j) \;=\; \min_{j\in J}\max_{i\in I} a(i,j) \;=\; v_C. \tag{1.2}$$

It is also well-known and easy to see that $v_R \leq v_C$ holds for all real matrices.

In particular, it is easy to see that a 2×2 matrix ($|I| = |J| = 2$) has no saddle point if and only if both entries of one diagonal are strictly larger than both entries of the other diagonal, that is,

$$\begin{aligned} \min\{a(1,1), a(2,2)\} &> \max\{a(1,2), a(2,1)\} \quad \text{or} \\ \min\{a(1,2), a(2,1)\} &> \max\{a(1,1), a(2,2)\}. \end{aligned} \tag{1.3}$$

In 1964 Shapley proved that there are no other minimal saddle point free (SP-free) matrices.

Theorem 1 *([16]). A matrix has a saddle point whenever each of its 2×2 submatrices have one.*

Clearly, this condition is only sufficient but not necessary. Indeed, row i and column j uniquely define whether a situation (i, j) is a saddle point or not; the rest of the matrix is irrelevant.

Shapley's proof of Theorem 1 is short and elegant. Assume indirectly that a matrix A has no saddle point. Then $v_R < v_C$. Let us choose a number v such that $v_R < v < v_C$. Since $v < v_C$, each column in A contains an entry strictly greater than v. Let us choose a column $j \in J$ in which the number of such entries is minimal. Still, $a(i, j) > v$ for some row $i \in I$. Since $v > v_R$, each row contains an entry strictly lesser than v. In particular, $a(i, j') < v$ for some $j' \in J$. Clearly, $j' \neq j$, since $a(i, j) > v > a(i, j')$. Moreover, these inequalities and the choice of j imply that $a(i', j) \leq v < a(i', j')$ for some $i' \in I$. The obtained 2×2 submatrix $\{i, i'\} \times \{j, j'\}$ has no saddle point, by (1.3). □

We strengthen this theorem and show that not only all minimal but also all **locally** minimal SP-free matrices are of size 2×2.

Theorem 2 *If a matrix A is SP-free but a saddle point appears after deleting an arbitrary row or column of it, then A is of size 2×2.*

The proof will be given in Section 2. Next, we will extend these results further to arbitrary (not necessarily zero-sum) two-person games.

1.3 Locally minimal bimatrix games without Nash equilibria

Let again I and J be two finite sets of strategies of the players R and C, respectively. A bimatrix game (A, B) is defined as a pair of mappings $a : I \times J \to \mathbb{R}$ and $b : I \times J \to \mathbb{R}$ that specify the utility (or payoff) functions of the players R and C, respectively. Now both players can be considered as maximizers. A situation $(i, j) \in I \times J$ is called a *Nash equilibrium* if no player can improve the result by choosing another strategy, that is, if

$$a(i, j) \geq a(k, j) \; \forall k \in I \quad \text{and} \quad b(i, j) \geq b(i, \ell) \; \forall \ell \in J. \tag{1.4}$$

Clearly, Nash equilibria generalize saddle points that correspond to the zero-sum case: $a(i, j) + b(i, j) = 0$ for all $i \in I$ and $j \in J$. However, unlike SP-free games, the minimal Nash equilibria free (NE-free) bimatrix games may be larger than 2×2. Let us recall an example from [11].

Example 1 *Consider a 3×3 bimatrix game (A, B) such that*

$$\begin{array}{rcccl} b(i_1,j_1) & > & b(i_1,j_2 & \geq & b(i_1,j_3), \\ b(i_2,j_3) & > & b(i_2,j_1) & \geq & b(i_2,j_2), \\ b(i_3,j_2) & > & b(i_3,j_3) & \geq & b(i_3,j_1); \end{array}$$

$$\begin{array}{rcccl} a(i_2,j_1) & > & a(i_1,j_1) & \geq & a(i_3,j_1), \\ a(i_1,j_2) & > & a(i_3,j_2) & \geq & a(i_2,j_2), \\ a(i_3,j_3) & > & a(i_2,j_3) & \geq & a(i_1,j_3). \end{array}$$

Naturally, for situations in the same row (respectively, column) the values of b (respectively, a) are compared, since player R controls rows and has utility function a, while C controls columns and has utility function b. It is easy to see that: $b(i_1,j_1)$ is the unique maximum in the row i_1 and $a(i_1,j_1)$ is the second largest in the column j_1. Similarly, $b(i_2,j_3)$ is the unique maximum in i_2 and $a(i_2,j_3)$ is the second largest in j_3; $b(i_3,j_2)$ is the unique maximum in i_3 and $a(i_3,j_2)$ is the second largest in j_2; $a(i_2,j_1)$ is the unique maximum in j_1 and $b(i_2,j_1)$ is the second largest in i_2; $a(i_1,j_2)$ is the unique maximum in j_2 and $b(i_1,j_2)$ is the second largest in i_1; $a(i_3,j_3)$ is the unique maximum in j_3 and $b(i_3,j_3)$ is the second largest in i_3.

Consequently, this game is NE-free, since no situation is simultaneously the best in its row with respect to b and in its column with respect to a. Yet, if we delete a row or column then a Nash equilibrium appears. For example, let us delete i_1. Then the situation (i_3,j_2) becomes a Nash equilibrium. Indeed, $b(i_3,j_2)$ is the largest in the row i_3 and $a(i_3,j_2)$ was the second largest in the column j_2, after $a(i_1,j_2)$ that was deleted with row i_1. Similarly, the situations (i_1,j_1), (i_2,j_3), (i_1,j_2), (i_3,j_3), (i_2,j_1) become Nash equilibria after deleting i_2,i_3,j_1,j_2,j_3, respectively. Thus, (A,B) is locally minimal NE-free bimatrix game. Moreover, it is also minimal. Indeed, one can easily verify that all 2×2 subgames of (A,B) have a Nash equilibrium and, of course, 1×2, 2×1, and 1×1 games always have it.

In general, the following criterion of the local minimality holds.

Theorem 3 *A bimatrix game (A,B) is a locally minimal NE-free game if and only if the following conditions hold*

(i) it is square, that is, $|I| = |J| = n$;

(ii) there exist two one-to-one mappings (permutations) $\sigma : I \to J$ and $\delta : J \to I$ such that their graphs, $gr(\sigma)$ and $gr(\delta)$, are disjoint in $I \times J$, or in other words, if $(i,\sigma(i)) \neq (\delta(j),j)$ for all $i \in I$ and $j \in J$;

(iii a) the entry $a(i,\sigma(i))$ is the unique maximum in row i and the second largest (though not necessarily unique) in column $\sigma(i)$;

(iii b) the entry $b(\delta(j),j)$ is the unique maximum in column j and the second largest (though not necessarily unique) in row $\delta(j)$.

Obviously, in any bimatrix game at most one such pair of permutations (σ, δ) can exist and its existence can be verified in $O(n^2)$ time. Thus, Theorem 3 provides a linear time algorithm for verifying **local** minimality. Let us remark that the definition of local minimality itself guarantees an $O(n^3)$ time recognition algorithm.

The "if" part of Theorem 3 is easy to prove. We just repeat the same arguments as in Example 1. Suppose that conditions (i, ii, iii a, iii b) hold. Then (A, B) has no Nash equilibria, that is, no situation $(i, j) \in I \times J$ is the best simultaneously in its row i (with respect to b) and in its column j ()with respect to a. Indeed, for each situation (i, j) we have: $a(i, \sigma(i)) \geq a(i, j)$ whenever $j \neq \sigma(i))$ and $b(\delta(j), j) \geq b(i, j)$ whenever $i \neq \delta(j))$. Moreover, at least one of these two inequalities must hold, since if they both fail then $(i, \sigma(i)) = (\delta(j), j) = (i, j)$ in contradiction to condition (ii).

Remark 1 *In particular, $2n$ situations $(i, \sigma(i)), i \in I$ and $(\delta(j), j), j \in J$ form so-called strict improvement cycle. It is not surprising, since such a cycle must exist in every NE-free bimatrix game.*

Let us next consider the deletion of a row $i \in I$. Consider two columns $j = \sigma(i)$ and $j' = \delta^{-1}(i)$, row $i' = \sigma^{-1}\delta^{-1}(i)$, and three situations: $(i, j) = (i, \sigma(i))$, $(i, j') = (i, \delta^{-1}(i))$, and finally, $(i', j') = (\sigma^{-1}\delta^{-1}(i), \delta^{-1}(i))$. Let us show that (i', j') is a Nash equilibrium. Indeed, $a(i', j')$ is a unique maximum in the row i' and $b(i', j')$ is the second largest in the column j', by definition of σ. Yet, the unique maximum $b(i, j')$ in the column j' is deleted with the row i. Hence, (i', j') is a Nash equilibrium in the remaining matrix.

Similarly, if we delete a column $j \in J$, the situation $(\sigma^{-1}(j), \delta^{-1}\sigma^{-1}(j))$ becomes a Nash equilibrium.

However, the "only if part" of Theorem 3 is more difficult and we postpone its proof till Section 3.

1.4 Minimal Nash equilibria free bimatrix games

Clearly, every minimal NE-free bimatrix game is locally minimal and hence, it must satisfy all conditions of Theorem 3. Yet, not vice versa. A locally minimal NE-free bimatrix game might not be minimal for two reasons. First, $I \times J$ may contain a proper NE-free subgame $I' \times J'$ disjoint from $gr(\delta) \cup gr(\sigma)$. Second, a locally minimal NE-free game $I \times J$ might be decomposed in two NE-free subgames. Let us consider the superpositions $\tau = \delta\sigma : I \to I$ and $\mu = \sigma\delta : J \to J$. A permutation $\pi : S \to S$ is called *transitive* if the orbit of each element of S is the whole set S. It is obvious and well-known that τ and μ can be transitive, or not, only simultaneously. It is also clear that if they are not transitive then $I \times J$ can be decomposed, that is, there are two partitions $I = I' \cup I''$ and $J = J' \cup J''$ such that all four sets I', I'', J', J'' are non-empty and

$$gr(\delta) \cup gr(\sigma) \subseteq (I' \times J') \cup (I'' \cup J'').$$

Obviously, in this case both subgames $I' \times J'$ and $I'' \times J''$ satisfy all conditions of Theorem 3 (in particular, they are both square, that is, $|I'| = |J'|$, $|I''| = |J''|$) and hence, they are both NE-free and locally minimal. Thus, the original game $I \times J$ is not minimal, although it is locally minimal. Summarizing, we obtain the following conditions that are necessary for minimality.

Proposition 1 *Each minimal NE-free bimatrix game is locally minimal and, in particular, it is square. Moreover, the product $\tau = \delta\sigma : I \to I$ (or equivalently, $\mu = \sigma\delta : J \to J$) of the permutations defined by Theorem 3 is transitive.* □

To verify the minimality of a locally minimal NE-free game we have to check its proper (square) subgames. Yet, many of them can be excused from inspection.

Lemma 1 *If τ and μ are transitive then a proper subgame $I' \times J' \subset I \times J$ has a Nash equilibrium whenever*

$$(gr(\delta) \cup gr(\sigma)) \cap (I' \times J') \neq \emptyset. \tag{1.5}$$

Proof . Assume indirectly that the subgame is NE-free, although (1.5) holds. Without loss of generality, suppose that $I' \times J'$ contains a situation $(i, j) = (i, \sigma(i)) \in gr(\sigma)$. By definition of σ, this situation (i, j) is a Nash equilibrium unless $I' \times J'$ contains also $(i_1, j) = (\delta\sigma(i), \sigma(i))$. In its turn, by definition of δ, the situation (i_1, j) is a Nash equilibrium unless $I' \times J'$ contains also $(i_1, j_1) = (\delta\sigma(i), \sigma\delta\sigma(i))$, etc. We conclude that $(I' \times J') \supseteq gr(\sigma) \cup gr(\delta)$. Yet, in this case $I' \times J' = I \times J$, that is, the considered subgame is not proper in contradiction to our assumptions. □

Summarizing, we obtain the following necessary and sufficient conditions for minimality.

Proposition 2 *A bimatrix game (A, B) is a minimal NE-free game if and only if there exist two permutations δ and σ satisfying all conditions of Theorem 3 and Proposition 1 and such that no square subgame $I' \times J'$ disjoint from $gr(\delta) \cup gr(\sigma)$ is locally minimal.* □

Although all subgames that are not square or intersect $gr(\sigma) \cup gr(\delta)$ can be excused, still, exponentially many subgames remain for inspection. For this reason, we conjecture that verifying minimality is NP-hard.

Remark 2 *Let us note that locally minimal (and minimal) games that have Nash equilibria (in particular, saddle points) are trivial: each player in such a game has only one strategy. This is true for k-person games, as well. Indeed, if $(j_1, \dots, j_k)$ is a Nash equilibrium then one can delete any other strategy of any player and still, the same situation will remain a Nash equilibrium in the reduced subgame.*

a_1	a_2
a_2	a_1

$g_1 =$ (left), $g_2 =$ (right):

a_1	a_1
a_2	a_3

Figure 1: Two 2×2 game forms.

1.5 Tight and solvable two-person game forms

Somewhat informally, a *game form* is a "game in which payoffs are not yet given". Formally, given two finite sets of strategies I and J of players R and C, respectively, and a set of outcomes $A = \{a_1, \ldots, a_p\}$, a game form is a mapping $g : I \times J \to A$. It is convenient to represent a game form by a matrix whose entries are elements of A. Consider, for example, the two 2×2 game forms given in Figure 1.

A game form g is called *Nash-solvable* if for each utility functions $u_R : A \to \mathbb{R}$ and $u_C : A \to \mathbb{R}$, the obtained bimatrix game (g, u_R, u_C) has a Nash equilibrium. Furthermore, g is called *zero-sum-solvable* if for each utility function $u : A \to \mathbb{R}$ the obtained zero-sum game, $u_R = u, u_C = -u$, has a saddle point. Finally, g is ± 1-solvable if (g, u) has a saddle point for each u that takes only two values: $+1$ and -1. We will see soon that all these properties are in fact equivalent. For example, g_2 in Figure 1 is Nash-solvable, while g_1 is not.

Given $g : I \times J \to A$, let assign a Boolean variable to each outcome $a \in A$ and denote it for simplicity by the same symbol a. Furthermore, let us assign two Disjunctive Normal Forms (DNFs) to g as follows:

$$D_R(g) = \bigvee_{j \in J} \bigwedge_{i \in I} g(i, j) \quad \text{and} \quad D_C(g) = \bigvee_{i \in I} \bigwedge_{j \in J} g(i, j).$$

Let us notice a certain similarity between these two DNFs and the maxmin and minmax. For the above two examples we get

$$D_R(g_1) = D_C(g_1) = a_1 a_2, \text{ and}$$

$$D_R(g_2) = a_1 \vee a_2 a_3$$

$$D_C(g_2) = a_1 a_2 \vee a_2 a_3.$$

Let us denote by $F_R(g)$ and $F_C(g)$ the monotone Boolean functions defined by the DNFs $D_R(g)$ and $D_C(g)$, respectively. A game form g is called *tight* if $F_R(g)$ and $F_C(g)$ are dual, $F_R^d(g) = F_C(g)$, or equivalently, $F_C^d(g) = F_R(g)$. For example, g_2 is tight, while g_1 is not, since

$$F_R(g_2)^d = (a_1 \vee a_2 a_3)^d = a_1(a_2 \vee a_3) = a_1 a_2 \vee a_1 a_3 = F_C(g_2),$$

while

$$F_R(g_1)^d = (a_1 a_2)^d = a_1 \vee a_2 \neq a_1 a_2 = F_C(g_1).$$

The above definition of tightness can be reformulated in several equivalent ways as follows. Given a game form $g : I \times J \to A$ and two more mappings $\sigma : I \to J$ and $\delta : J \to I$, let us consider two subsets of A : $[\sigma] = g(gr(\sigma))$ and $[\delta] = g(gr(\delta))$. In particular, let $\sigma_j : I \to \{j\} \in J$ and $\delta_i : J \to \{i\} \in I$ denote mappings σ and δ that take a unique value.

Proposition 3 *The following properties of a game form are equivalent:*
(i) g is tight;
(ii) $[\sigma] \cap [\delta] \neq \emptyset$ for each σ and δ;
(iii R) for each σ there is a $j \in J$ such that $[\sigma_j] \subseteq [\sigma]$;
(iii C) for each δ there is an $i \in I$ such that $[\delta_i] \subseteq [\delta]$. □

All these reformulations are well-known; see for example [7, 8].
It appears that all variants of solvability defined above are equivalent, too.

Theorem 4 *([7], see also [8]). The following properties of a game form g are equivalent:*
(i) g is Nash-solvable;
(ii) g is zero-sum-solvable;
(iii) g is ± 1-solvable;
(iv) g is tight. □

A game form g is called *solvable* if it has all these equivalent properties.

Remark 3 *The concepts of tightness and Nash-solvability can be naturally generalized for the k-person game forms, see Section 5. However, they are no longer equivalent. Already for $k = 3$, tightness is neither necessary nor sufficient for Nash-solvability; see [8].*

Given a two-person game form $g : I \times J \to A$, let us consider the problem of deciding whether it is tight, or not. This decision problem seems difficult, at least, no polynomial algorithm is known. Yet, it is very unlikely that this problem is NP-complete, since there is a quasi-polynomial algorithm by Fredman and Khachiyan [6]. Its running time is $p\ell + \ell^{o(\log \ell)}$, where $|A| = p$, $m = |I|$, $n = |J|$, and $\ell = m + n$.

1.6 Minimal and locally minimal not tight two-person game forms

Theorem 5 *All minimal and locally minimal not tight game forms are of size 2×2.*

Up to isomorphism (transposition included), there are seven 2×2 game forms:

a_1	a_2
a_2	a_1

a_1	a_2
a_3	a_1

a_1	a_2
a_3	a_4

a_1	a_1
a_2	a_2

a_1	a_1
a_2	a_3

a_1	a_1
a_1	a_2

a_1	a_1
a_1	a_1

The last four are tight, the first three are not. In other words, a 2×2 game form is tight if and only if it contains a row or column whose two entries coincide. We derive Theorem 5 from Theorems 2 and 4. Let g be a locally minimal not tight game form. Then, by definition of tightness, there is a utility function $u : A \to \mathbb{R}$ such that the zero-sum game (g, u) has no saddle point. Furthermore, since g is a locally minimal not tight game form, a saddle point appears in (g, u) after deleting a strategy $i \in I$ or $j \in J$. Hence, (g, u) is a locally minimal SP-free game and, by Theorem 2, it is of size 2×2. □

Yet, we have no characterization of minimal or locally minimal not tight k-person game forms for $k > 2$.

Somewhat surprisingly, it appears difficult to characterize locally minimal *tight* k-person game forms even for $k = 2$.

1.7 Minimal and locally minimal tight two-person game forms

At first, let us notice that all minimal tight game forms are trivial, that is, they are of size 1×1. Moreover, for each $k \geq 2$, all minimal tight k-person game forms are trivial, too.

In contrast, characterization of locally minimal tight two-person game forms seems difficult. We obtain only some necessary and some sufficient conditions for local minimality. However, Fredman and Khachiyan's algorithm [6] provides a quasi-polynomial oracle for verifying tightness of a game form g. Hence, one can also verify in quasi-polynomial time whether g is a locally minimal tight game form.

Let us recall that a game form $g : I \times J \to A$ is tight if and only if its Boolean functions $F_R(g)$ and $F_C(g)$ are dual. Furthermore, let $[i] = \{g(i,j) \mid j \in J\} \subseteq A$ and $[j] = \{g(i,j) \mid i \in I\} \subseteq A$ denote the set of outcomes in the row $i \in I$ and column $j \in J$ of g, respectively. Then

$$D_R(g) = \bigvee_{i \in I} \bigwedge_{a \in [i]} a \qquad \text{and} \qquad D_C(g) = \bigvee_{j \in J} \bigwedge_{a \in [j]} a.$$

A (monotone) DNF is called *irredundant* if its prime implicants do not contain one another. In particular, $D_R(g)$ and $D_C(g)$ are irredundant if $[i] \subseteq [i']$ for no distinct $i, i' \in I$ and $[j] \subseteq [j']$ for no distinct $j, j' \in J$, respectively. If both these conditions hold then the game form g itself will be called *irredundant*, too. Obviously, irredundancy is necessary for local minimality.

Proposition 4 *If g is a locally minimal tight game form then $D_R(g)$ and $D_C(g)$ are dual irredundant monotone DNFs.*

Proof . Indeed, $D_R(g)$ and $D_C(g)$ are dual by definition of tightness. Moreover, they are irredundant, since otherwise one can delete a row $i \in I$ or column $j \in J$ and the reduced game form remains tight. □

Given a game form $g : I \times J \to A$, row $i \in I$, column $j \in J$, and an outcome $a \in A$, let

$$k(i,a) = |\{j' \in J \mid g(i,j') = a\}| \text{ and } k(j,a) = |\{i' \in I \mid g(i',j) = a\}|$$

denote the numbers of occurrences of a in the row i and column j; if $k(i,a) = 1$ or $k(i,a) = 1$, we will say that a is a *singleton* in i or j, respectively. The following conditions are sufficient for local minimality of g.

Proposition 5 *An irredundant tight game form $g : I \times J \to A$ is locally minimal whenever $k(i,a) \neq 1$ and $k(i,a) \neq 1$ for all $i \in I, j \in J$, and $a \in A$, or in other words, if all rows and columns contain no singletons.*

Proof . Let us delete, say, a row $i \in I$. Since DNF $D_R(g)$ is irredundant, by this, the corresponding monotone Boolean function $F_R(g)$ will be strictly reduced. On the other hand, $F_C(g)$ will remain the same, since $k(a,j) \neq 1$ for all $a \in A$ and $j \in J$. Hence, although $a = g(i,j)$ is deleted from the column j, yet, this column contains another entry $g(i',j) = a$. Thus, the reduced game form cannot be tight, since its two DNFs are no longer dual. □

Example 2 *To illustrate Proposition 5 let us consider the DNF D whose prime implicants are assigned to the lines of the Fano plane:*

$$D = a_0a_1a_4 \vee a_0a_2a_5 \vee a_0a_3a_6 \vee a_1a_2a_3 \vee a_3a_4a_5 \vee a_5a_6a_1 \vee a_2a_4a_6.$$

It is known that D is self-dual, that is $D^d = D$. Let us consider an irredundant game form g such that $D_R(g) = D_C(g) = D$. For example,

a_0	a_0	a_0	a_1	a_4	a_1	a_4
a_0	a_0	a_0	a_2	a_5	a_5	a_2
a_0	a_0	a_0	a_3	a_3	a_6	a_6
a_1	a_2	a_3	a_1	a_3	a_1	a_2
a_4	a_5	a_3	a_3	a_3	a_5	a_4
a_1	a_5	a_6	a_1	a_5	a_5	a_6
a_4	a_2	a_6	a_2	a_4	a_6	a_2

Clearly, conditions of Proposition 5 hold for g, that is, $k(i,a) \neq 1$ and $k(i,a) \neq 1$ for all $i \in I, j \in J$, and $a \in A$. Indeed, for each line ℓ of the Fano plane and a point x in it there are another two lines that intersect ℓ in x. Hence, by Proposition 5 the above 7×7 game form is a locally minimal tight one.

Less obvious sufficient conditions for local minimality will be given in Section 4. They are weaker than the conditions of Proposition 5 but still not necessary.

2 Proof of Theorems 2

Let A be a matrix game given by a mapping $a : I \times J \to \mathbb{R}$. Assume indirectly that A has no saddle point but it appears whenever we delete a row or column of A.

In particular, after we delete a row $i \in I$ a situation (i', j) becomes a saddle point. By definition, it means that the entry $a(i', j)$ becomes (though it was not) maximal in column j and it is minimal in row i'. Then we conclude that $a(i, j)$ is a *unique* maximum, while $a(i', j)$ is the second largest entry (not necessarily unique) of column j. Indeed, it is easy to see that otherwise (i', j) could not become a saddle point. Furthermore, we conclude that $m = |I| \leq |J| = n$, since for each $i \in I$ there exists a $j \in J$ such that $a(i, j)$ is a unique maximum in column j.

Similarly, after we delete a column $j \in J$ a situation (i, j') becomes a saddle point. By definition, it means that the entry $a(i, j')$ becomes (though it was not) minimal in row i and it is maximal in column j'. Then we conclude that $a(i, j)$ is a *unique* minimum, while $a(i, j')$ is the second smallest entry (not necessarily unique) of row i. Indeed, it is easy to see that otherwise (i, j') could not become a saddle point. Furthermore, we conclude that $m = |I| \geq |J| = n$, since for each $j \in J$ there exists a $i \in I$ such that $a(i, j)$ is a unique minimum in row i. Thus, $m = n$.

Furthermore, for each row $i \in I$ its unique minimum entry $a(i, j) = a(i, \sigma(i))$ is the second maximal in the column $j = \sigma(i)$. Similarly, for each column $j \in J$ its unique maximum entry $a(i, j) = a(\delta(j), j)$ is the second minimal in the row $i = \delta(j)$.

Thus, we obtain two one-to-one mappings (permutations) $\sigma : I \to J$ and $\delta : J \to I$. Let us notice that for each $i \in I$ and $j \in J$ inequality $(i, \sigma(i)) \neq (\delta(j), j)$ must hold, since otherwise the original matrix would have a saddle point. In other words, graphs of σ and δ in $I \times J$ are disjoint, $gr(\sigma) \cap gr(\delta) = \emptyset$.

Let us note that $|I \times J| = n^2$, while $|gr(\sigma) \cup gr(\delta)| = 2n$. If $n = |I| = |J| = 2$ then $gr(\sigma) \cup gr(\delta) = I \times J$, that is, these two graphs partition the matrix. In this case we obtain a locally minimal SP-free game.

However, if $m = n > 2$ then there exists a situation (i, j) such that $(i, j) \notin gr(\sigma) \cup gr(\delta)$. This case results in a contradiction, since, by definition of σ and δ, we have

$$a(i, j) > a(i, \sigma(i)) \geq a(\delta(j), \sigma(i) \geq a(\delta(j), j) > a(i, j).$$

Indeed, entry $a(i, \sigma(i))$ is a unique minimum in its row i and the second largest in its column $\sigma(i)$; respectively, $a(\delta(j), j)$ is a unique maximum in its column j and the second smallest in its row $\delta(j)$. It remains to add that

$$(\delta(j), \sigma(i)) \notin gr(\sigma) \cup gr(\delta),$$

since both $\sigma : I \to J$ and $\delta : J \to I$ are one-to-one mappings. $\square$

3 Proof of Theorem 3

The "if part" was already proven in Introduction. Let us now prove the "only if" part. Given a locally minimal NE-free bimatrix game $(A, B) = (a : I \times J \to \mathbb{R}, b : I \times J \to \mathbb{R})$, let us denote by N_i (respectively, by N^j) the set of Nash equilibria that appears after the row $i \in I$ (respectively, column $j \in J$) is deleted.

Claim 1 i. If $(i', j') \in N_i$ then $a(i, j')$ is a unique maximum in column j', while $a(i', j')$ is the second maximal entry (not necessarily unique) in this column.

Proof . Indeed, situation (i', j') was not a Nash equilibrium in the original matrix but it becomes one after row i is deleted. Hence, $a(i, j') > a(i', j')$; moreover, $a(i'', j') > a(i', j')$ for no other $i'' \in I$. In particular, $a(i'', j') = a(i, j')$ for no i'' distinct from i. Hence, $a(i, j')$ is a unique maximum in column j', while $a(i', j')$ is the second maximal entry (perhaps, not unique) in this column. □

Respectively, for N^j the following similar statement holds.

Claim 1 j. If $(i', j') \in N_j$ then $b(i', j)$ is a unique maximum in row i', while $b(i', j')$ is the second maximal entry (not necessarily unique) in this row. □

Let us denote by $R(N_i), R(N^j) \subseteq I$ and $C(N_i), C(N^j) \subseteq J$, respectively, the sets of the rows and columns of the equilibria N_i and N^j. In other words, $R(N_i)$ and $R(N^j)$ (respectively, $C(N_i)$ and $C(N^j)$) are projections of sets N_i and N^j in I (respectively, in J).

Claim 2 i. For every distinct $i, i' \in I$ we have $C(N_i) \cap C(N_{i'}) = \emptyset$.

Proof . Assume indirectly that $j \in C(N_i) \cap C(N_{i'})$. Then, by Claim 1 i, each entry $a(i, j)$ and $a(i', j)$ must be a unique maximum in j. Hence, $i = i'$ and we get a contradiction. □

Respectively, for columns we get a similar statement.

Claim 2 j. For every distinct $j, j' \in J$ we have $R(N^j) \cap R(N^{j'}) = \emptyset$. □

Furthermore, Claim 2 i (respectively, Claim 2 j) immediately implies that $m \leq n$ (respectively, $n \leq m$), and hence, $m = n$.

Claim 3. Every locally minimal NE-free bimatrix is square, that is, $n = |J| = |I| = m$. □

Claim 4. For every $i \in I$ and $j \in J$ we have $|C(N_i)| = |R(N^j)| = 1$.

Proof . Indeed, this Claim easily results from Claims 2 and 3. □

Thus, we assign a unique column $C(N_i)$ to each row $i \in I$ and unique row $R(N_j)$ to each column $j \in J$. Claims 2,3, and 4 imply that both mappings are one-to-one. Let us denote them by δ^{-1} and σ^{-1}, respectively:

$$\delta^{-1} : i \to C(N_i) \in J \text{ ; and ; } \sigma^{-1} : j \to R(N^j) \in I.$$

Such a notation appears to be consistent with Section 1.3.

Claim 5. In each row $i \in I$ the payoff $b(i,j)$ takes a unique maximum at situation $(i, \sigma(i))$.

Respectively, in each column $j \in J$ the payoff $a(i,j)$ takes a unique maximum at situation $(\delta(j), j)$.

Proof . All these claims follow from the definitions of σ, δ and Claim 1. □

Now we are ready to "transpose" Claim 2. Indeed, Claim 5 implies the following statement.

Claim 6. For every distinct $i, i' \in I$ we have $R(N_i) \cap R(N_{i'}) = \emptyset$.
Respectively, for every distinct $j, j' \in J$ we have $C(N^j) \cap C(N^{j'}) = \emptyset$. □

Furthermore, Claims 4 and 6 imply that only one Nash equilibrium appears after deleting a row or column.

Claim 7. For each row $i \in I$ and column $j \in J$ we have $|N_i)| = |N^j)| = 1$. □

From definitions of σ, δ, and Claim 1 we derive the explicit formulas:

$$N_i = \{(\sigma^{-1}\delta^{-1}(i), \delta^{-1}(i))\} \text{ and } N^j = \{(\sigma^{-1}(j), \delta^{-1}\sigma^{-1}(j))\}.$$

Finally, let us recall Claims 1 once more and derive the last statement.

Claim 8. The entry $a(i, \sigma(i))$ is the second maximal (not necessarily unique) in the column $\sigma(i)$.

Respectively, $b(\delta(j), j)$ is the second maximal (not necessarily unique) in the row $\delta(j)$. □

This concludes the proof of Theorem 3 □

4 More about locally minimal tight game forms

Let us return to sufficient conditions of Proposition 5.

Example 3 *The following game form g*

a_1	a_3	a_3	a_5	a_5	a_1
a_1	a_3	a_4	a_4	a_1	a_3
a_2	a_2	a_3	a_5	a_5	a_3
a_2	a_2	a_4	a_4	a_1	a_1

satisfies these conditions. It is easy to see that g is tight and irredundant. Indeed, its DNFs
$D_R(g) = a_1a_3a_5 \vee a_1a_3a_4 \vee a_2a_3a_5 \vee a_1a_2a_4$ *and*
$D_C(g) = a_1a_2 \vee a_2a_3 \vee a_3a_4 \vee a_4a_5 \vee a_5a_1 \vee a_1a_3$

are irredundant and dual. Furthermore, functions $k(i,a)$ and $k(j,a)$ take only values 0 and 2 for all $i \in I$, $j \in J$, and $a \in A$. Hence, conditions of Proposition 5 hold and g is a locally minimal tight game form.

However, conditions of Proposition 5 are not necessary for local minimality.

Example 4 *Let us consider the following game form g:*

a_1	a_2	a_3	a_3	a_2
a_4	a_2	a_3	a_3	a_2
a_4	a_5	a_3	a_5	a_4
a_4	a_5	a_6	a_5	a_4
a_1	a_5	a_6	a_1	a_6
a_1	a_2	a_6	a_1	a_6

It is easy to see that g is tight and irredundant, since its monotone DNFs

$D_R(g) = a_1a_2a_3 \vee a_2a_3a_4 \vee a_3a_4a_5 \vee a_4a_5a_6 \vee a_5a_6a_1 \vee a_6a_1a_2$ *and*

$D_C(g) = a_1a_4 \vee a_2a_5 \vee a_3a_6 \vee a_1a_3a_5 \vee a_2a_4a_6$

are irredundant and dual. Yet, conditions of Proposition 5 fail. Indeed, it is easy to see that each row contains a singleton (rows $i_1, i_2, i_3, i_4, i_5, i_6$ contain singletons $a_1, a_4, a_3, a_6, a_5, a_2$ in columns $j_1, j_1, j_3, j_3, j_2, j_2$, respectively). Nevertheless, g is a locally minimal tight game forms. To show it we have to verify that after deleting a row or column the reduced game form becomes not tight. Indeed, deleting a row will strictly reduce $F_R(g)$, while $F_C(g)$ remains the same, since no column contains a singleton. Hence, the reduced game form is not tight. For the same reason, it cannot remain tight after deleting one of the last two columns. Hence, we have to verify only the first three columns. Due to symmetry, all three cases are equivalent. For example, after deleting the first column we get a reduced game form g' with DNFs

$D_R(g') = a_2a_3 \vee a_2a_3 \vee a_3a_4a_5 \vee a_4a_5a_6 \vee a_5a_6a_1 \vee a_6a_1a_2$, *and*

$D_C(g) = a_2a_5 \vee a_3a_6 \vee a_1a_3a_5 \vee a_2a_4a_6$

that are not dual, and hence, the reduced game form is not tight.

To understand this example better we need some new concepts. Given a game form $g : I \times J \to A$, by definition, $g(i,j) \in [i] \cap [j]$. A situation $(i,j) \in I \times J$ is called *simple* if $[i] \cap [j] = \{g(i,j)\}$.

Remark 4 *All situations of a tight game form g are simple if and only if $D_R(g)$ and $D_C(g)$ are dual irredundant read-once DNFs.*

Remark 5 *In Example 2 all situations except seven diagonal ones are simple. In Example 3 all situations are simple, except for (i_1, j_5), (i_1, j_6), (i_2, j_6), (i_2, j_3), (i_3, j_2), (i_4, j_1); for example, $[i_4] \cap [j_1] = \{a_1, a_2\}$. If we substitute a_1 for $g(i_4, j_1) = a_2$ then the obtained game form g' will contain the singleton a_2 in the first column j_1 and last row i_4. Hence, conditions of Proposition 5 do not hold for g'. Moreover, it is easy to check that g' is tight but not* *locally minimal tight game form. In contrast, in Example 2, in which* 7 *diagonal situations* (i_ℓ, j_ℓ) *are not simple, one can substitute arbitrary outcomes from* $[i_\ell] \cap [j_\ell]$ *for* $g(i_\ell, j_\ell)$ *for* $\ell = 1, \ldots, 7$ *and still all obtained game forms satisfy conditions of Proposition 5 and hence they are all locally minimal.*

Proposition 6 *If* g *is a tight game form then for every* $i \in I$ *and* $a \in [i]$ *there exists a* $j \in J$ *such that* $g(i, j) = a$ *and* (i, j) *is simple, that is* $[i] \cap [j] = \{a\}$*. Respectively, for every* $j \in J$ *and* $a \in [j]$ *there exists an* $i \in I$ *such that* $g(i, j) = a$ *and* (i, j) *is simple.* □

This is a well-known property of dual DNFs; see for example [3].

Let us now return to Example 4. Note that $g(i_5, j_1) = a_1$ and $k(i_5, a_1) = 2$, since $g(i_5, j_4) = a_1$, too. However, the situation (i_5, j_4) is not simple: $[i_5] \cap [j_4] = \{a_1, a_5\}$. Similarly, $g(i_4, j_1) = a_4$ and $k(i_4, a_4) = 2$, since $g(i_4, j_5) = a_4$, too. However, the situation (i_4, j_5) is not simple: $[i_4] \cap [j_j] = \{a_4, a_6\}$. For this reason, after we delete column j_1 from g the obtained game form is not tight. Indeed, its DNF D_R contains implicants $a_1a_5a_6$ and $a_4a_5a_6$ generated by rows i_5 and i_4, respectively, instead of a_5a_6, a dual implicant of D_C.

A situation (i, j) will be called a *hidden singleton* in i (respectively, in j) if $k(i, a) \geq 2$, (respectively, $k(j, a) \geq 2$), where $a = g(i, j)$, and the situation (i, j) is simple, while any other situations (i, j') such that $g(i, j') = a$ (respectively, (i', j)) such that $g(i', j) = a$) is not simple. A hidden singleton (i, j) in i (respectively, in j) is called *effective* if $[i] \setminus \{a\} \supset [i'] \setminus \{g(i', j)\}$ for no $i' \in I$ (respectively, if $[j] \setminus \{a\} \supset [j'] \setminus \{g(i, j')\}$ for no $j' \in J$. The sufficient conditions of Proposition 5 we can weaken as follows.

Proposition 7 *An irredundant tight game form* $g : I \times J \to A$ *is locally minimal whenever for each simple situation* (i, j) *such that* $g(i, j) = a$ *and* $k(i, a) = 1$ (*respectively,* $k(j, a) = 1$) *there exists an effective hidden singleton in* j *(respectively, in* i*).*

Proof . If we delete a row i (column j) that contains no singletons, that is, $k(j, g(i, j) \geq 2$ for each $j \in J$ (respectively, $k(i, g(i, j) \geq 2$ for each $i \in I$ then we can just repeat the arguments from the proof of Proposition 5. If i or j contains a singleton then it must contain also an effective hidden singleton and hence, the reduced game form still cannot be tight. □

Clearly, Propositions 7 provides sufficient conditions for local minimality that are weaker than in Proposition 5. However, they are still not necessary. An example is pretty complicated, so we start with two preliminary constructions.

Example 5 *Consider the following irredundant tight game form* g.

a_1	a_2	a_1	a_2
a_5	a_5	a_1	a_3
a_6	a_6	a_1	a_3
a_5	a_5	a_4	a_2
a_6	a_6	a_4	a_2
a_5	a_5	a_4	a_3
a_6	a_6	a_4	a_3

$D_R(g) = a_1a_2 \vee a_1a_3a_5 \vee a_1a_3a_6 \vee a_2a_4a_5 \vee a_2a_4a_6 \vee a_3a_4a_5 \vee a_3a_4a_6,$
$D_C(g) = a_1a_2 \vee a_1a_5a_6 \vee a_2a_5a_6 \vee a_1a_4 \vee a_2a_3; \quad F_R^d = F_C.$

If we delete the first row, the reduced game form g_1 is no longer tight. Indeed,

$D_R(g_1) = a_1a_3a_5 \vee a_1a_3a_6 \vee a_2a_4a_5 \vee a_2a_4a_6 \vee a_3a_4a_5 \vee a_3a_4a_6,$
$D_C(g_1) = a_5a_6 \vee a_5a_6 \vee a_1a_4 \vee a_2a_3; \quad F_R^d(g_1) \neq F_C(g_1).$

Let us note that the first row contains singletons in the first two columns, while all other rows do not contain singletons. Yet, the last two columns contain singletons, too (in fact, in each row, except the first one). It is easy to verify that after deleting the third or forth column the reduced game form remains tight.

Hence, we need to modify (and substantially enlarge) this example.

Example 6 *Let us consider the following irredundant tight game form g'*

a'_1	a'_1	a'_2	a'_2	a'_5	a'_5	a'_6	a'_6
a'_1	a'_1	a'_2	a'_2	a'_7	a'_8	a'_7	a'_8
a'_3	a'_4	a'_3	a'_4	a'_5	a'_5	a'_6	a'_6
a'_3	a'_4	a'_3	a'_4	a'_7	a'_8	a'_7	a'_8

$D_R(g') = a'_1a'_2a'_5a'_6 \vee a'_1a'_2a'_7a'_8 \vee a'_3a'_4a'_5a'_6 \vee a'_3a'_4a'_7a'_8,$
$D_C(g') = a'_1a'_3 \vee a'_1a'_4 \vee a'_2a'_3 \vee a'_2a'_4 \vee a'_5a'_7 \vee a'_5a'_8 \vee a'_6a'_7 \vee a'_6a'_8; \; F_R^d(g') = F_C(g').$

Notice that all situations of g' are simple, since $F_R(g')$ and $F_C(g')$ are dual read-once functions:

$$F_R(g')=(a'_1a'_2 \vee a'_3a'_4)(a'_5a'_6 \vee a'_7a'_8),$$

$$F_C(g')=(a'_1 \vee a'_2)(a'_3 \vee a'_4) \vee (a'_5 \vee a'_6)(a'_7 \vee a'_8).$$

Now let us define a game form $g'' = g + g'$, where g is from Example 5:

$a_1a_2a_1a_2$	$a_1a_2a_1a_2$	$a_1a_2a_1a_2$	$a_1a_2a_1a_2$	$a_1a_2a_1a_2$	$a_1a_2a_1a_2$	$a_1a_2a_1a_2$	$a_1a_2a_1a_2$
$a_5a_5a_1a_3$	$a_5a_5a_1a_3$	$a_5a_5a_1a_3$	$a_5a_5a_1a_3$	$a_5a_5a_1a_3$	$a_5a_5a_1a_3$	$a_5a_5a_1a_3$	$a_5a_5a_1a_3$
$a_6a_6a_1a_3$	$a_6a_6a_1a_3$	$a_6a_6a_1a_3$	$a_6a_6a_1a_3$	$a_6a_6a_1a_3$	$a_6a_6a_1a_3$	$a_6a_6a_1a_3$	$a_6a_6a_1a_3$
$a_5a_5a_4a_2$	$a_5a_5a_4a_2$	$a_5a_5a_4a_2$	$a_5a_5a_4a_2$	$a_5a_5a_4a_2$	$a_5a_5a_4a_2$	$a_5a_5a_4a_2$	$a_5a_5a_4a_2$
$a_6a_6a_4a_2$	$a_6a_6a_4a_2$	$a_6a_6a_4a_2$	$a_6a_6a_4a_2$	$a_6a_6a_4a_2$	$a_6a_6a_4a_2$	$a_6a_6a_4a_2$	$a_6a_6a_4a_2$
$a_5a_5a_4a_3$	$a_5a_5a_4a_3$	$a_5a_5a_4a_3$	$a_5a_5a_4a_3$	$a_5a_5a_4a_3$	$a_5a_5a_4a_3$	$a_5a_5a_4a_3$	$a_5a_5a_4a_3$
$a_6a_6a_4a_3$	$a_6a_6a_4a_3$	$a_6a_6a_4a_3$	$a_6a_6a_4a_3$	$a_6a_6a_4a_3$	$a_6a_6a_4a_3$	$a_6a_6a_4a_3$	$a_6a_6a_4a_3$
$a'_1a'_1a'_1a'_1$	$a'_1a'_1a'_1a'_1$	$a'_2a'_2a'_2a'_2$	$a'_2a'_2a'_2a'_2$	$a'_5a'_5a'_5a'_5$	$a'_5a'_5a'_5a'_5$	$a'_6a'_6a'_6a'_6$	$a'_6a'_6a'_6a'_6$
$a'_1a'_1a'_1a'_1$	$a'_1a'_1a'_1a'_1$	$a'_2a'_2a'_2a'_2$	$a'_2a'_2a'_2a'_2$	$a'_7a'_7a'_7a'_7$	$a'_8a'_8a'_8a'_8$	$a'_7a'_7a'_7a'_7$	$a'_8a'_8a'_8a'_8$
$a'_3a'_3a'_3a'_3$	$a'_4a'_4a'_4a'_4$	$a'_3a'_3a'_3a'_3$	$a'_4a'_4a'_4a'_4$	$a'_5a'_5a'_5a'_5$	$a'_5a'_5a'_5a'_5$	$a'_6a'_6a'_6a'_6$	$a'_6a'_6a'_6a'_6$
$a'_3a'_3a'_3a'_3$	$a'_4a'_4a'_4a'_4$	$a'_3a'_3a'_3a'_3$	$a'_4a'_4a'_4a'_4$	$a'_7a'_7a'_7a'_7$	$a'_8a'_8a'_8a'_8$	$a'_7a'_7a'_7a'_7$	$a'_8a'_8a'_8a'_8$

$$
\begin{aligned}
D_R(g'') &= D_R(g) \vee D_R(g') \\
&= (a_1a_2 \vee a_1a_3a_5 \vee a_1a_3a_6 \vee a_2a_4a_5 \vee a_2a_4a_6 \vee a_3a_4a_5 \vee a_3a_4a_6) \\
&\qquad \bigvee (a'_1a'_2a'_5a'_6 \vee a'_1a'_2a'_7a'_8 \vee a'_3a'_4a'_5a'_6 \vee a'_3a'_4a'_7a'_8),
\end{aligned}
$$

$$
\begin{aligned}
F_C(g'') &= D_C(g) \wedge D_C(g') \\
&= (a_1a_2 \vee a_1a_5a_6 \vee a_2a_5a_6 \vee a_1a_4 \vee a_2a_3) \\
&\qquad \bigwedge (a'_1a'_3 \vee a'_1a'_4 \vee a'_2a'_3 \vee a'_2a'_4 \vee a'_5a'_7 \vee a'_5a'_8 \vee a'_6a'_7 \vee a'_6a'_8)
\end{aligned}
$$

$$
F_R^d(g'') = F_C(g'').
$$

Thus, game form g'' is tight. Furthermore, it is easy to see that there are no singletons in the rows, while columns have singletons and they all are in the first row. Hence, to verify local minimality of the g'' it is enough to check that it becomes not tight after deleting the first row. Clearly, the latter claim holds for g'' if and only if it holds for g and this was already verified in Example 5. Thus, g'' is a locally minimal tight game form, indeed. Yet, sufficient conditions of Propositions 5 and 6 fail for g'', since all its singletons are located in one row.

5 Maximal stable effectivity functions

Let us consider also an example from cooperative game theory.

Given a set of players (or voters) $I = \{1, \dots, n\}$ and a set of outcomes (or candidates) $A = \{a_1, \dots, a_p\}$, subsets $K \subseteq I$ are called *coalitions* and subsets $B \subseteq A$ *blocks*. An *effectivity function* (EFF) is defined as a mapping $\mathcal{E} : 2^I \times 2^A \to \{0, 1\}$. We say that coalition $K \subseteq I$ is effective (respectively, not effective) for block $B \subseteq A$ if $\mathcal{E}(K, B) = 1$ (respectively, $\mathcal{E}(K, B) = 0$. Since $2^I \times 2^A = 2^{I \cup A}$, we can say that $\mathcal{E}$ is a Boolean function whose set of variables $I \cup A$ is a mixture of the players and outcomes. The "complementary" function, $\mathcal{V}(K, B) \equiv \mathcal{E}(K, A \setminus B)$, is called *veto function*; by definition, K is effective for B if and only if K can veto $A \setminus B$. Both names are frequent in the literature [1, 13, 12, 14, 15, 4, 9, 10, 5]. An EFF is called *monotone, superadditive, subadditive,* and *convex*, respectively, if the following implications hold:

$\mathcal{E}(K, B) = 1,\ K \subseteq K' \subseteq I,\ B \subseteq B' \subseteq A \ \Rightarrow\ \mathcal{E}(K', B') = 1,$

$\mathcal{E}(K_1, B_1) = 1,\ \mathcal{E}(K_2, B_2) = 1,\ K_1 \cap K_2 = \emptyset \ \Rightarrow\ \mathcal{E}(K_1 \cup K_1, B_1 \cap B_2) = 1,$

$\mathcal{E}(K_1, B_1) = 1,\ \mathcal{E}(K_2, B_2) = 1,\ B_1 \cap B_2 = \emptyset \ \Rightarrow\ \mathcal{E}(K_1 \cap K_1, B_1 \cup B_2) = 1,$

$\mathcal{E}(K_1, B_1) = 1,\ \mathcal{E}(K_2, B_2) = 1, \Rightarrow\ \mathcal{E}(K_1 \cup K_1, B_1 \cap B_2) = 1$ or $\mathcal{E}(K_1 \cap K_1, B_1 \cup B_2) = 1.$

Monotonicity will be always assumed. The name is consistent with Boolean terminology, since an EFF $\mathcal{E}$ is monotone if and only if the corresponding Boolean function $\mathcal{E} : 2^{I \cup A} \to$

$\{0,1\}$ is monotone. We will also assume that all considered EFFs satisfy the following boundary conditions:

$\mathcal{E}(K,B) = 1$ if $K \neq \emptyset,\ B = A$ or $K = I,\ B \neq \emptyset$;

$\mathcal{E}(K,B) = 0$ if $K = \emptyset,\ B \neq A$ or $K \neq I,\ B = \emptyset$.

In particular, we assume that $\mathcal{E}(I,\emptyset) = 0$ and $\mathcal{E}(\emptyset,A) = 1$. Hence, by monotonicity, $\mathcal{E}(K,\emptyset) = 0$ and $\mathcal{E}(K,A) = 1$ for all $K \subseteq I$.

Furthermore, let X_i be a finite set of strategies of a player $i \in I$. The direct product $\prod_{i \in I} X_i$ is a set of situations. A mapping $g : X \to A$ is a *game form*. To each game form g let us assign an EFF $\mathcal{E}_g$ as follows:

$\mathcal{E}_g(K,B) = 1$ iff $\exists\, x_K = (x_i, i \in K)$ such that $g(x_K, x_{I\setminus K}) \in B\ \ \forall\, x_{I\setminus K} = (x_i, i \notin K)$.

In other words, a coalition $K \subseteq I$ is effective for a block $B \subseteq A$ if and only if the players of K have strategies that guarantee that the outcome will belong to B for any strategies of the remaining players. In 1982 Moulin and Peleg [13] proved that an EFF $\mathcal{E}$ is *playing*, i.e., $\mathcal{E} = \mathcal{E}_g$ for a game form g, if and only if $\mathcal{E}$ is monotone, superadditive, and the boundary conditions hold.

Given a utility function $u : I \times A \to \mathbb{R}$, its value $u(i,a)$ is interpreted as a profit of the player (voter) $i \in I$ in case the outcome (candidate) $a \in A$ is elected.

Given an EFF $\mathcal{E}$, utility function u, a coalition $K \subseteq I$, and outcome $a_0 \in A$, consider the set of all outcomes strictly and unanimously preferred to a_0 by all coalitionists of K, that is,

$PR(K, a_0, u) = \{a \in A \mid u(i,a) > u(i,a_0)\ \forall\, i \in K\} \subseteq A.$

We say that a coalition $K \subseteq I$ rejects an outcome $a_0 \in A$ if $\mathcal{E}(K, PR(K,a_0,u)) = 1$, that is, if K can guarantee a strictly better result than a_0 to all coalitionists.

Given $\mathcal{E}$ and u, the *core* is defined as the set of outcomes not rejected by any coalition, that is,

$C(\mathcal{E}, u) = \{a \in A \mid \mathcal{E}(K, PR(K,a,u))) = 0\ \forall\, K \subseteq I\} \subseteq A.$

This is a natural, and surely the simplest, concept of solution in cooperative game theory. Yet, the core is frequently empty, since there are too many, 2^n, coalitions.

An EFF $\mathcal{E}$ is called *stable* if the core $C(\mathcal{E}, u)$ is not empty for any u.

In 1984 Peleg proved [14] that every convex EFF is stable.

It is easy to see that convex EFFs are sub- and superadditive [14, 9, 10].

By definition, stability is anti-monotone, that is, if $\mathcal{E}$ is stable and $\mathcal{E}' \leq \mathcal{E}$ then $\mathcal{E}'$ is stable, too. Hence, maximal and locally maximal stable EFFs coincide. It is an important problem to characterize them. Indeed, such a characterization would imply also a characterization of stable EFFs. Let us remark that, given an EFF $\mathcal{E}$, it is an NP-complete problem to decide whether $\mathcal{E}$ is stable [2]. The following important family of the maximal stable EFFs is well-known.

To each EFF $\mathcal{E}$ let us assign the dual EFF $\mathcal{E}^d$ defined by formula:

$\mathcal{E}(I \setminus K, A \setminus B) + \mathcal{E}^d(K,B) = 1\ \ \forall\, K \subseteq I,\ B \subseteq A.$

In other words, $\mathcal{E}^d(K, B) = 1$ if and only if $\mathcal{E}(I \setminus K, A \setminus B) = 0$. Again, the name "dual" is consistent with Boolean terminology, since two (monotone) EFFs are dual if and only if the corresponding two (monotone) Boolean functions are dual.

Respectively, an EFF $\mathcal{E}$ is called *self-dual* if $\mathcal{E}(I \setminus K, A \setminus B) + \mathcal{E}(K, B) = 1$, that is, K is effective for B if and only if $I \setminus K$ is not effective for $A \setminus B$.

Notice that a game form g is *tight* if and only if the corresponding EFF $\mathcal{E}_g$ is self-dual.

The stable self-dual EFFs form an interesting class. In 1982 Abdou [1] proved that they are both sub- and superadditive. They are also convex. Moreover, it is not difficult to see that the following three subfamilies of the self-dual EFFs coincide: stable self-dual EFFs, convex self-dual EFFs, and sub- and superadditive self-dual EFFs, see for example, [9, 10].

Furthermore, it is easy to show that stable self-dual EFFs are the maximal stable EFFs, in other words, an EFF $\mathcal{E}$ is not stable whenever $\mathcal{E} > \mathcal{E}'$, where $\mathcal{E}'$ is a self-dual EFF. Indeed, in this case $\mathcal{E}(K, B) = \mathcal{E}(I \setminus K, A \setminus B) = 1$ for some $K \subseteq I$ and $B \subseteq A$. Let us define a utility function u such that K prefers B to $A \setminus B$, while $I \setminus K$, on the contrary, prefers $A \setminus B$ to B, and both preference are unanimous and strict. Clearly, in this case the core is empty, $C(\mathcal{E}, u) = \emptyset$, since all outcomes of A are rejected: B by $I \setminus K$ and $A \setminus B$ by K. Thus, EFF $\mathcal{E}$ is not stable.

The self-dual EFFs (and corresponding veto functions) are frequently called *maximal* in the literature [1, 13, 12, 14, 15, 4, 5]. This name is logical, since, as we have just demonstrated, the self-dual stable EFFs are maximal stable EFFs. It was conjectured that there are no other maximal stable EFFs; see [12], Problem 25, and also [4]. However, this conjecture was disproved in [9], see also [5].

Example 7 *Given* 3 *players,* $I = \{1, 2, 3\}$*, and* 6 *outcomes* $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$*, let us define a monotone EFF* $\mathcal{E}$ *as follows. Each coalition that consists of* 2 *players, that is,* $\{1, 2\}, \{2, 3\}$*, or* $\{3, 1\}$*, is effective for blocks*

$\{a_1, a_3\}, \{a_3, a_5\}, \{a_5, a_1\}, \{a_2, a_4\}, \{a_4, a_6\}, \{a_6, a_2\}$,

as well as for every block that contains one of the listed. Each coalition that consists of 1 *player, that is,* $\{1\}, \{2\}$ *or* $\{3\}$*, is effective only for the total block* A*. Finally, the total coalition* I *is effective for every non-empty block.*

In is not difficult to show that EFF $\mathcal{E}$ is stable. For example, sufficient conditions of stability obtained in [2] hold. Let us prove that $\mathcal{E}$ is not majorized by a stable self-dual EFF. Assume indirectly that such an EFF exists: $\mathcal{E}'$ is stable, self-dual, and $\mathcal{E}' > \mathcal{E}$.

Let us show that, although EFF $\mathcal{E}$ is stable, but stability disappears whenever we strengthen a coalition of cardinality 2. Let us assume, without any loss of generality, that coalition $\{1, 2\}$ becomes effective for block $\{a_1, a_4\}$, that is, $\mathcal{E}'(\{1, 2\}, \{a_1, a_4\}) = 1$, while equations

$\mathcal{E}'(\{2, 3\}, \{a_3, a_5\}) = \mathcal{E}'(\{3, 1\}, \{a_2, a_6\}) = 1$

hold too. Let us consider a utility function u such that

$u(1, a_1) = u(1, a_4) > u(1, a_2) = u(1, a_6) > u(1, a_3) = u(1, a_5)$,
$u(2, a_3) = u(2, a_5) > u(2, a_1) = u(2, a_4) > u(2, a_2) = u(2, a_6)$,

$u(3,a_2) = u(3,a_6) > u(3,a_3) = u(3,a_5) > u(3,a_1) = u(3,a_4)$.

Obviously, in the game $(\mathcal{E}', u)$, coalitions $\{1,2\}$, $\{2,3\}$, and $\{3,1\}$ reject outcomes $\{a_2,a_6\}$, $\{a_1,a_4\}$, and $\{a_3,a_5\}$, respectively. Hence, $C(\mathcal{E}', u) = \emptyset$ and the obtained EFF $\mathcal{E}'$ is not stable.

Thus, to enlarge $\mathcal{E}$ it only remains to strengthen coalitions $\{1\}$, $\{2\}$, and $\{3\}$ of cardinality 1. Then, to get a (unique) self-dual EFF $\mathcal{E}'$ one should make each of these 3 coalitions effective for the following 12 blocks:

$\{a_1,a_2,a_3,a_4\}, \{a_2,a_3,a_4,a_5\}, \{a_3,a_4,a_5,a_6\}, \{a_4,a_5,a_6,a_1\}, \{a_5,a_6,a_1,a_2\}, \{a_6,a_1,a_2,a_3\},$

$\{a_1,a_2,a_4,a_5\}, \{a_2,a_3,a_5,a_6\}, \{a_3,a_4,a_6,a_1\}, \{a_4,a_5,a_1,a_2\}, \{a_5,a_6,a_2,a_3\}, \{a_6,a_1,a_3,a_4\},$

as well as for each block that contains one of the listed. Again, the obtained EFF $\mathcal{E}'$ is not stable, since, for example,

$$\begin{aligned}\mathcal{E}'(\{1\},\{a_1,a_2,a_3,a_4\}) &= \mathcal{E}'(\{3\},\{a_3,a_4,a_5,a_6\}) \\ &= \mathcal{E}'(\{2\},\{a_5,a_6,a_1,a_2\}) = 1\end{aligned}$$

and

$$\mathcal{E}'(\{1\},\{a_2,a_3,a_5,a_6\}) = \mathcal{E}'(\{2,3\},\{a_1,a_4\}) = 1.$$

Moreover, it is easy to see that $\mathcal{E}'$ cannot be stable, nor superadditive, nor convex, since if it is then the total coalition I is effective for the empty block. Thus, we got a stable EFF $\mathcal{E}$ that is not majorized by any stable self-dual EFF. Hence, there are maximal stable EFFs that are not self-dual.

Similarly, we can show that the same EFF $\mathcal{E}$ cannot be majorized by a convex EFF, either. Assume indirectly that there is a convex EFF $\mathcal{E}'$ such that $\mathcal{E}' > \mathcal{E}$. Then

$\mathcal{E}'(\{1,2\},\{a_1,a_3\}) = \mathcal{E}'(\{1,3\},\{a_4,a_6\}) = 1$

and, since $\mathcal{E}'$ is convex, we have

$\mathcal{E}'(\{1\},\{a_1,a_3,a_4,a_6\}) = 1$ or $\mathcal{E}'(\{1,2,3\},\{\emptyset\}) = 1$.

In fact, the former equation must hold, since the latter one contradicts the boundary conditions. Furthermore, due to symmetry, we get

$$\begin{aligned}\mathcal{E}'(\{1\},\{a_1,a_3,a_4,a_6\}) &= \mathcal{E}'(\{2\},\{a_2,a_4,a_5,a_1\}) \\ &= \mathcal{E}'(\{3\},\{a_3,a_5,a_6,a_2\}) = 1.\end{aligned}$$

It is easy to see that these 3 equations and convexity of $\mathcal{E}'$ contradict the boundary conditions. Alternatively, copying the above arguments, we can show that $\mathcal{E}'$ is not stable and hence, by Peleg's theorem, it is not convex either.

Thus, the considered stable EFF $\mathcal{E}$ is majorized by no convex and by no stable self-dual EFF.

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