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Nash-solvable bidirected cyclic two-person game forms

by

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ABSTRACT

We consider cyclic positional games of two players. Let $\overrightarrow{G} = (V, \overrightarrow{E})$ be a directed graph (digraph) and $P : V = V_1 \cup V_2 \cup V_T$ be a partition of its vertices (positions) in three subsets: V_1 and V_2 are positions of players 1 and 2, respectively, and V_T are the terminal positions. Directed edges going from a position $j \in V_1$ (respectively, $j \in V_2$) are called the *moves* of player 1 (respectively, 2). Furthermore, $j \in V_T$ if and only if the out-degree of j is 0. Given a digraph $\overrightarrow{G} = (V, \overrightarrow{E})$, a partition $P : V = V_1 \cup V_2 \cup V_T$, and also an initial position $j_0 \in V_1 \cup V_2$, the triplet $(\overrightarrow{G}, P, j_0)$ is called a *positional cyclic game form*. Name "cyclic" is motivated as follows. A mapping x_1 (respectively, x_2) that assigns a move (j, j') to each position $j \in V_1$ (respectively, $j \in V_2$) is called a (positional) strategy of player 1 (respectively, 2). Each pair of strategies $x = (x_1, x_2)$ uniquely defines a *play*, that is, a directed path that begins in the initial position j_0 and either ends in a terminal position $j \in V_T$ or results in a simple directed cycle (dicycle) $c \in C = C(\overrightarrow{G})$. The obtained mapping $g = g(\overrightarrow{G}, P, j_0) : X_1 \times X_2 \to V_T \cup C$ is called the *normal cyclic game form* corresponding to $(\overrightarrow{G}, P, j_0)$. Utility functions of players 1 and 2 are defined by two arbitrary mappings: $u_1 : V_T \cup C \to \mathbb{R}$ and $u_2 : V_T \cup C \to \mathbb{R}$. A game form $g = g(\overrightarrow{G}, P, j_0)$ is called *Nash-solvable* if for every utility functions $u = (u_1, u_2)$ the obtained normal form game (g, u) has at least one Nash equilibrium in pure strategies. In this paper we characterize Nash-solvable cyclic game forms $g(\overrightarrow{G}, P, j_0)$ whose digraphs are *bidirected*, that is, $(j, j') \in \overrightarrow{E}$ if and only if $(j', j) \in \overrightarrow{E}$. We derive this characterization from an old general criterion: a two-person game form g is Nash-solvable if and only if it is tight.

Key words: game form, game in normal form, positional game, mean-payoff game, cyclic game, Nash equilibrium, Nash-solvable, tight

1 Introduction. Main concepts and results

1.1 Nash-solvable game forms

Let $A = \{a_1, \ldots, a_p\}$ be a set of outcomes, $I = \{1, \ldots, n\}$ be a set of players, and X_i be a set of strategies of a player $i \in I$. (In this paper we always assume that X_i is finite. In particular, we do not consider mixed strategies.)

Furthermore, let $X = \prod_{i \in I} X_i$ be the direct product of these n sets. An element of it, n-tuple $x = (x_i, i \in I) \in X$, is called a *situation*. A *game form* $g : X \to A$ is a mapping that assigns an outcome $a \in A$ to each situation $x \in X$. Typically, the mapping g is not injective, that is, the same outcome can be assigned to several distinct situations. A two-person ($I = \{1, 2\}$, $n = 2$) game form g can be represented by a matrix whose entries are the outcomes of A; see examples in Figures 1 and 2.

In general, g is given by an n-dimensional table.

A *utility function or payoff* is a mapping $u : I \times A \to \mathbb{R}$. Its value $u(i, a)$ is interpreted as the profit of player $i \in I$ in case outcome $a \in A$ is realized.

A utility function u is called *zero-sum* if $\sum_{i \in I} u(i, a) = 0$ for all $a \in A$.

A pair (g, u) is called a *game in normal form*.

Given a game (g, u), a situation $x \in X$ is called a *Nash equilibrium* if $u(i, g(x)) \geq u(i, g(x'))$ for each player $i \in I$ and for each situation $x' = (x'_j, j \in I) \in X$ that may differ from $x = (x_j, j \in I) \in X$ only in the i-th coordinate, that is, $x'_j = x_j$ whenever $j \neq i$. In other words, in a situation $x \in X$ no player $i \in I$ can make a profit if he alone changes his strategy, x'_i for x_i, while all other players apply the same strategies, $x'_j = x_j$ for all $j \neq i$.

For zero-sum two-person games Nash equilibria are called *saddle points*.

A game (g, u) is called solvable if it has a Nash equilibrium.

A game form g is called *Nash-solvable* if for each utility function u the obtained game (g, u) is solvable.

A two-person game form g is called *zero-sum-solvable* if for each zero-sum u the obtained game (g, u) is solvable (that is, it has a saddle point).

Finally, a two-person game form g is called ± 1-*solvable* if (g, u) is solvable for each zero-sum utility function u that takes only two values, $+1$ and -1.

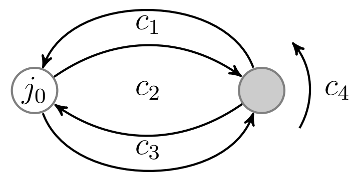
It appears that all three properties of a two-person game form g, Nash-, zero-sum-, and ± 1-solvability, are equivalent to the following property.

To each outcome $a \in A$ let us assign a Boolean variable and denote it, for simplicity, by the same symbol a. Let $g : X \to A$ be a two-person game form, where $I = \{1, 2\}$ and $X = X_1 \times X_2$. We introduce two monotone disjunctive normal forms (DNFs)

$$F_1 = F_1(g) = \bigvee_{x_1 \in X_1} \bigwedge_{x_2 \in X_2} g(x_1, x_2); \quad F_2 = F_2(g) = \bigvee_{x_2 \in X_2} \bigwedge_{x_1 \in X_1} g(x_1, x_2).$$

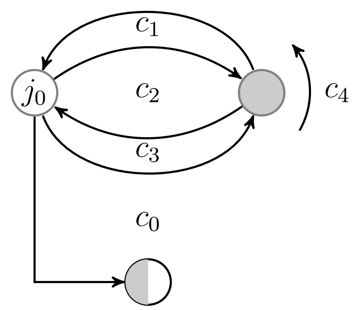
A game form g is called *tight* if these two DNFs define dual monotone Boolean functions, that is, $F_1^d = F_2$. For example, in Figure 1 only the third game form is tight and in Figure 2 the last two game forms are tight, while the first two are not.

(1)



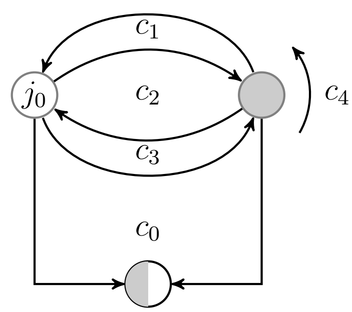
c_1	c_2
c_3	c_4

(2)



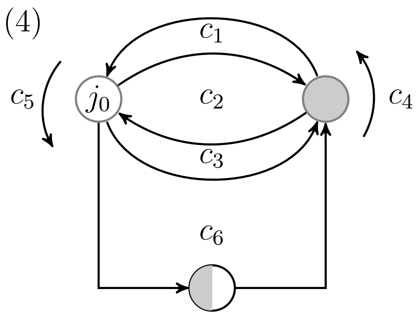
c_0	c_0
c_1	c_2
c_3	c_4

(3)



c_0	c_0	c_0
c_0	c_1	c_2
c_0	c_3	c_4

(4)



c_1	c_2
c_3	c_4
c_5	c_6

Figure 1: Nash-Solvability is monotone and ergodic
(1) $F_1 = c_1c_2 \vee c_3c_4, \quad F_2 = c_1c_3 \vee c_2c_4, \quad F_1^d \neq F_2.$
(2) $F_1 = c_0 \vee c_1c_2 \vee c_3c_4, \quad F_2 = c_0c_1c_3 \vee c_0c_2c_4, \quad F_1^d \neq F_2.$
(3) $F_1 = c_0 \vee c_0c_1c_2 \vee c_0c_3c_4 \approx c_0, \quad F_2 = c_0 \vee c_0c_1c_3 \vee c_0c_2c_4 \approx c_0, \quad F_1^d = F_2.$
(4) $F_1 = c_1c_2 \vee c_3c_4 \vee c_5c_6, \quad F_2 = c_1c_3c_5 \vee c_2c_4c_6, \quad F_1^d \neq F_2.$

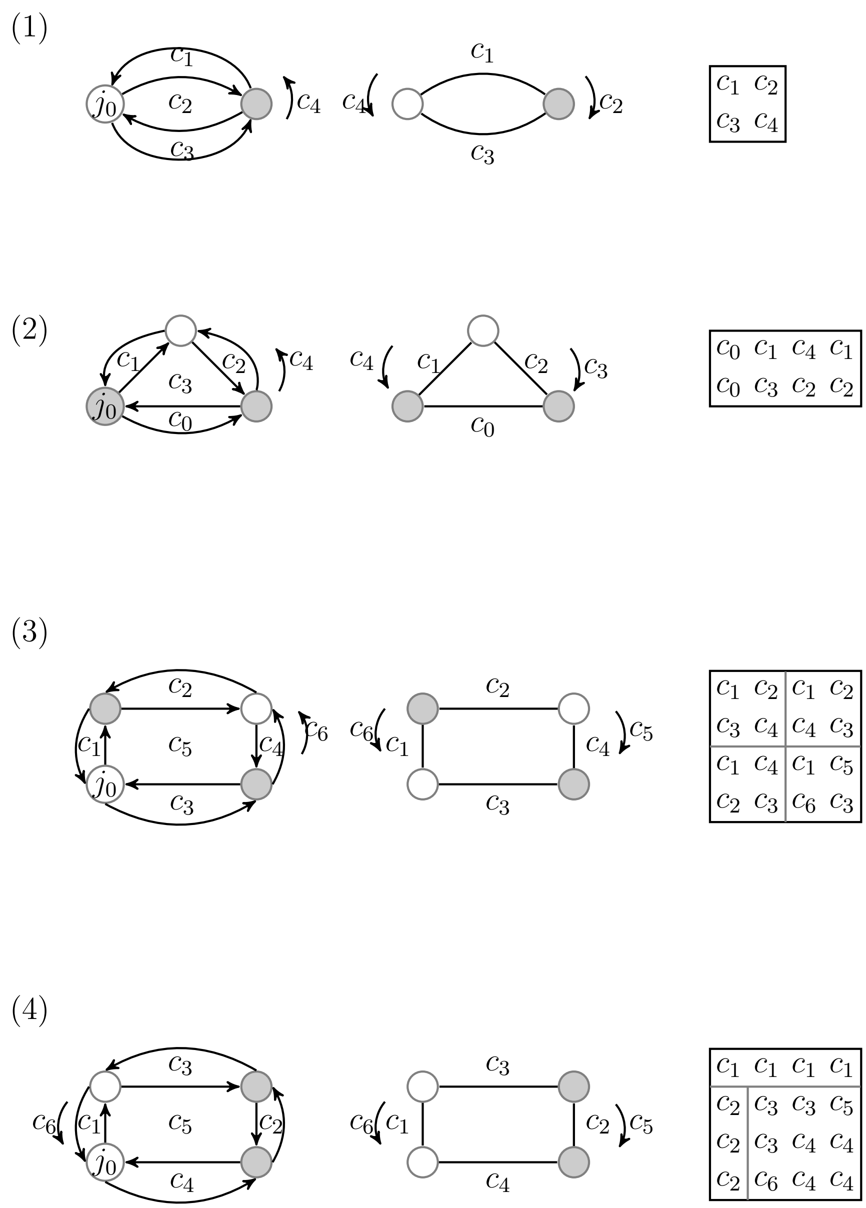


Figure 2: Solvability of cycles
(1) $F_1 = c_1c_2 \vee c_3c_4, \quad F_2 = c_1c_3 \vee c_2c_4, \quad F_1^d \neq F_2.$
(2) $F_1 = c_0c_1c_4 \vee c_0c_2c_3, \quad F_2 = c_0 \vee c_1c_2 \vee c_1c_3 \vee c_2c_4, \quad F_1^d \neq F_2.$
(3) $F_1 = c_1c_2 \vee c_3c_4 \vee c_1c_4c_5 \vee c_2c_3c_6, \quad F_2 = c_1c_3 \vee c_2c_4 \vee c_1c_4c_6 \vee c_2c_3c_5, \quad F_1^d = F_2.$
(4) $F_1 = c_1 \vee c_2c_3c_4 \vee c_2c_3c_5 \vee c_2c_4c_6 = c_1 \vee c_2(c_3c_4 \vee c_3c_5 \vee c_4c_6),$
. $\quad F_2 = c_1c_2 \vee c_1c_3c_4 \vee c_1c_3c_6 \vee c_1c_4c_5 = c_1(c_2 \vee c_3c_4 \vee c_3c_6 \vee c_4c_5), \quad F_1^d = F_2.$

In other words, g is tight if and only if rows and columns of the corresponding matrix form two dual (transversal) hypergraphs on the ground set A. The definition of tightness can be reformulated in several equivalent ways. For example, g is tight if and only if $B_1 \cap B_2 \neq \emptyset$ for any two sets of outcomes $B_1, B_2 \subseteq A$ such that B_1 (respectively, B_2) has an outcome in each column (respectively, row) of the matrix of g. In Appendix 1 we provide several other combinatorial characterizations of tightness and show that, for two-person game forms, tightness is equivalent to every type of solvability.

Theorem 1 *The following properties of two-person game forms are equivalent:*
(i) tightness; (ii) Nash-solvability; (iii) zero-sum solvability; (iv) ± 1*-solvability.*

Game forms satisfying these properties will be called *solvable*.

Equivalence of (i), (iii), and (iv) was first proved in [10], see also [17]. This list was extended by statement (ii) in [18]; see also [19], where it is shown that for three-person game forms tightness is no longer necessary (this observation is due to Danilov, 1988) nor sufficient for Nash-solvability. To make the paper self-contained, we will give these proofs and examples in Appendix 1. Let us note that the proof of Theorem 1 given there (based on some new ideas from [1] and [9]; see also [22]) differs from the original proof [18, 19].

1.2 Positional and normal cyclic game forms

The above general criterion of solvability can be applied to special classes of game forms. In this paper we consider solvability of cyclic game forms.

Given a finite directed graph $\overrightarrow{G}$ in which loops and multiple arcs are allowed, a vertex $j \in V = V(\overrightarrow{G})$ is a *position* and an arc $\overrightarrow{e} = (j, j') \in E(\overrightarrow{G})$ is a *move* from j to j'. A position of out-degree 0, in which there are no moves, is called *terminal*. Let V_T denote the set of all terminal positions. Let us also fix an *initial* position $j_0 \in V$. Furthermore, let us introduce two players $I = \{1, 2\}$ and a partition $P : V = V_1 \cup V_2 \cup V_T$, assuming that player $i \in I$ is in control of all positions of V_i, for $i = 1, 2$.

Let $C = C(\overrightarrow{G})$ denote the set of all simple directed cycles (dicycles) of digraph $\overrightarrow{G}$. In particular, a loop $c_j = (j, j)$ is a dicycle of length 1 and a pair of oppositely directed arcs $\overrightarrow{e} = (j, j')$ and $\overrightarrow{e}' = (j', j)$ is a dicycle of length 2.

The dicycles and terminal positions form the set of outcomes, $A = C \cup V_T$.

Remark 1 *Let us notice that we can easily get rid of all terminal positions* V_T*. To do so, for every such position* $j \in V_T$ *we introduce a new loop* $c_j = (j, j)$*. This loop is an outcome corresponding to* j*. Obviously, after this transformation the sets of outcomes* A *and dicycles* $C = C(\overrightarrow{G})$ *coincide. In the sequel, we assume that* $V_T = \emptyset$*, unless the opposite is mentioned explicitly.*

The triplet $(\overrightarrow{G}, P, j_0)$ will be called a *positional cyclic game form*.

To introduce the corresponding normal game form we need the concept of strategies. A *strategy* of a player $i \in I = \{1, 2\}$ is a mapping x_i that assigns a move $\overrightarrow{e} = (j, j') \in \overrightarrow{E}$ to each position $j \in V_i$.

Remark 2 *In other words, a strategy is a plan choosing a move in every possible position. We assume that this choice is deterministic (not random, that is, we do not consider mixed strategies), the chosen move is unique, and it depends only on the present position (not on the previous positions and moves). In other words, in this paper we restrict ourselves by* pure positional *strategies.*

Let X_i denote the set of strategies of the player $i \in I = \{1, 2\}$ and let $X = X_1 \times X_2$. Obviously, every two strategies (that is, a situation $x = (x_1, x_2) \in X$) determine a unique move in each position $j \in V \setminus V_T$. Furthermore, these moves define a unique path $p = p(x)$ (a *play*) that begins in the initial position j_0 and either results in a dicycle $c \in C$ or ends in a terminal position $j \in V_T$. By Remark 1, in the latter case p also results in a dicycle, namely, in the loop $c_j = (j, j)$. Thus, we obtain a mapping $g(\overrightarrow{G}, P, j_0) : X \to A$ that is called a *normal cyclic game form.*

Given $g = g(\overrightarrow{G}, P, j_0)$, we extend it to a game (g, u) (in normal form) by introducing a utility function $u : I \times C \to \mathbb{R}$, whose value $u(i, c)$ we interpret as a profit of the player $i \in I = \{1, 2\}$ when the play $p = p(x)$ results in the dicycle $c \in C$. In this paper we assume that u is an arbitrary function. (Let us remark, however, that the *additive* utility functions are more frequent in the literature, see Section 1.5.) Let us also recall that cyclic game form g is solvable if for every utility function u the obtained game (g, u) has a Nash equilibrium.

1.3 Properties of bidirected cyclic game forms

Given a digraph $\overrightarrow{G} = (V, \overrightarrow{E})$, for each two vertices $j, j' \in V$ let us denote the number of arcs from j to j' in E by $k(j, j')$. We shall call G *bidirected* if $k(j, j') = k(j', j)$ for each pair $j, j' \in V$.

Remark 3 *We will see in Section 3 that both game forms,* $(\overrightarrow{G}, P, j)$ *and* $(\overrightarrow{G}, P, j')$*, are not solvable if* $k(j, j') \geq 2$ *and* $k(j', j) \geq 2$*. Hence, in a bidirected solvable game form* $k(j, j')$ *can take only values* 0 *and* 1*; in other words, any two parallel edges are oppositely directed. In particular, digraph* $\overrightarrow{G}$ *is not bidirected whenever* $(j, j') \in \overrightarrow{E}$*, while* $(j', j) \notin \overrightarrow{E}$ *for some* $j, j' \in V$*.*

Moreover, in a bidirected digraph set V_T *may contain only isolated vertices. In particular,* $V_T = \emptyset$ *whenever digraph* $\overrightarrow{G}$ *is bidirected and connected.*

To each bidirected digraph $\overrightarrow{G} = (V, \overrightarrow{E})$ we assign a (non-directed) graph $G = (V, E)$ as follows: E contains k (non-directed) edges between j and j' whenever $k(j, j') = k(j', j) = k$; furthermore, to every directed loop in $\overrightarrow{G}$ we assign a non-directed loop in G.

Obviously, the following three properties of a bidirected graph are equivalent:

(i) $\overrightarrow{G}$ is strongly connected, (ii) $\overrightarrow{G}$ is connected, (iii) G is connected.

We shall show that solvability of a connected bidirected game form is an ergodic property, that is, it does not depend on the initial position. Moreover, the following stronger statement holds.

Proposition 1 *Given a strongly connected digraph* $\overrightarrow{G} = (V, \overrightarrow{E})$ *and a partition* $P : V = V_1 \cup V_2$ *(recall that* $V_T = \emptyset$*), solvability of cyclic game forms* $(\overrightarrow{G}, P, j)$ *is an ergodic property, that is, either* $(\overrightarrow{G}, P, j)$ *is solvable for every* $j \in V$ *or for no* $j \in V$.

In the first case pair $(\overrightarrow{G}, P)$ will be called *solvable*.

However, let us remark that, given also a utility function u, the values of games $(\overrightarrow{G}, P, j, u)$ and $(\overrightarrow{G}, P, j', u)$ might differ even in case of ± 1 zero-sum u; see Section 1.6 and Appendix 2 for more details.

In particular, Proposition 1 holds when digraph $\overrightarrow{G}$ is bidirected and the corresponding non-directed graph G is connected. In this case pair (G, P) will be called *solvable* whenever $(\overrightarrow{G}, P)$ is solvable.

Standardly, we say that $G' = (V, E')$ is a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. In this case we use the notation $G' \leq G$ and $G' < G$ if at least one of the above to containments is strict. Furthermore, we say that the pair (G', P') is majorized by (G, P) if $G' = (V, E')$ is a subgraph of $G = (V, E)$ and the partition $P' = V' : V_1' \cup V_2'$ is a subpartition of $P : V = V_1 \cup V_2$ induced by the subset $V' \subseteq V$, that is, $P' : V' = (V_1 \cap V') \cup (V_2 \cap V')$. Respectively, we use notation $(G', P') \leq (G, P)$ when $G' \leq G$ and $(G', P') < (G, P)$ when $G' < G$.

In particular, we can talk about connected and 2-connected components of a pair (G, P). A graph G (and every corresponding pairs (G, P)) is called 2-connected if G is connected and it remains connected after deleting a vertex.

In Section 2 we will prove that solvability is a monotone decreasing property.

Proposition 2 *If pair* (G, P) *is solvable and* $(G', P') \leq (G, P)$ *then pair* (G', P') *is solvable too.*

It is clear that pair (G, P) is solvable if and only if every its connected component is solvable. Hence, without loss of generality we can assume that graph G is connected. Moreover, we can also assume that G is 2-connected, due to the following statement.

Proposition 3 *A pair* (G, P) *is solvable if and only if each its* 2*-connected component is solvable.*

Thus, it is sufficient to characterize all (maximal) 2-connected solvable pairs (G, P). In particular, Proposition 3 implies that pair (G, P) is solvable if it has no 2-connected components

Corollary 1 *A pair* (G, P) *is solvable whenever* G *is a forest.* □

1.4 Criterion of solvability for bidirected cyclic game forms

In this section we introduce a list $\mathcal{L}$ of solvable 2-connected pairs. Our main theorem will claim that an arbitrary pair (G, P) is solvable if and only if each its 2-connected component is in $\mathcal{L}$.

1.4.1 Solvability of simple cycles. Reducing (G, P) to $(\mathcal{G}, \mathcal{P})$

Proposition 4 *Let G be a simple cycle, then a pair (G, P) is solvable if and only if* $(|V_1| > 1$ *and* $|V_2| > 1)$*, or* $(V_1 = V,\ V_2 = \emptyset)$*, or* $(V_2 = V,\ V_1 = \emptyset)$.

In other words we can say that (G, P) is not solvable if and only if

$(|V_1| = 1$ and $V_2 \neq \emptyset)$ or $(|V_2| = 1$ and $V_1 \neq \emptyset)$.

The proof will be given in Section 3. Let us remark, however, that none of these 2-connected solvable pairs is maximal; see Section 1.4.2.

To characterize solvable 2-connected pairs (G, P) we will need one more transformation. To each pair (G, P) with $G = (V, E)$ and $P : V = V_1 \cup V_2$ let us assign another pair $(\mathcal{G}, \mathcal{P})$ with $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and $\mathcal{P} : \mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 = (V_1 \cap \mathcal{V}) \cup (V_2 \cap \mathcal{V})$ defined as follows. Let $\mathcal{V} \subseteq V$ be the set of all vertices of degree at least 3 in G. Such vertices will be called *nodes*. Given two nodes $j, j' \in \mathcal{V}$ and a simple path $p = p(j, j')$ between them such that p contains no other node, we assign an edge $e(p)$ between j and j' and define $\mathcal{E}$ as the set of all such edges.

We will call p a 0-path (respectively, 1-path) if all vertices of p (respectively, all but exactly one), including j and j', are controlled by the same player. The corresponding edge $e(p) \in \mathcal{E}$ will be called a 0-edge (respectively, 1-edge).

Similarly, when $j = j'$, we obtain concepts of 0- and 1-cycle in G by means of which we can reformulate Proposition 4 as follows.

Proposition 5 *If G is a simple cycle then a pair (G, P) is solvable unless G is a* 1*-cycle.*

Obviously, Propositions 4 and 5 are equivalent. Let us mention that pair (G, P) is solvable if G is a 0-cycle, in particular, a loop.

Let us also notice that if G is a simple cycle then $\mathcal{G}$ is empty, i.e., $\mathcal{V} = \emptyset$; if G is any other 2-connected graph then $\mathcal{G}$ is not empty and 2-connected too.

Given a pair $(\mathcal{G}, \mathcal{P})$ with the list of its 0- and 1-edges, we will call it *solvable* if each corresponding pair (G, P) is solvable. We will see that it is sufficient for solvability if at least one corresponding pair (G, P) is solvable. However, let us remark that the list 0- and 1-edges may be essential; see Propositions 6-9 below.

A pair $(\mathcal{G}, \mathcal{P})$ is called *bipartite* if $\mathcal{G}$ is a bipartite graph $\mathcal{G} = (\mathcal{V}_1, \mathcal{V}_2, \mathcal{E})$ and $\mathcal{P} : \mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ is the corresponding partition. In a bipartite game form players 1 and 2 take turns.

In figures, we color positions of players 1 and 2 by white and black, respectively. By the double, black-and-white, coloring we denote ‘uncertain” positions: $j \in \mathcal{V}_1$ or $j \in \mathcal{V}_2$, both options are possible. In other words, a partition $\mathcal{P}$ with ℓ black-and-white positions represents 2^ℓ distinct partitions rather than one. For example, each terminal position $j \in V_T$ can be uncertain, since nothing depends on whether it is white or black.

1.4.2 Solvable θ-pairs

Let us consider the family of bipartite pairs $(\mathcal{G}_K, \mathcal{P}_K) = (\theta_K, \mathcal{P}_K)$ given in Figure 3, where $K = 1, 2, \ldots$. For $K \geq 2$ graphs θ_K contain two types of edges: simple (type 1) and parallel (type 2); see Figure 3. Graph θ_1 consists of two vertices and three parallel edges.

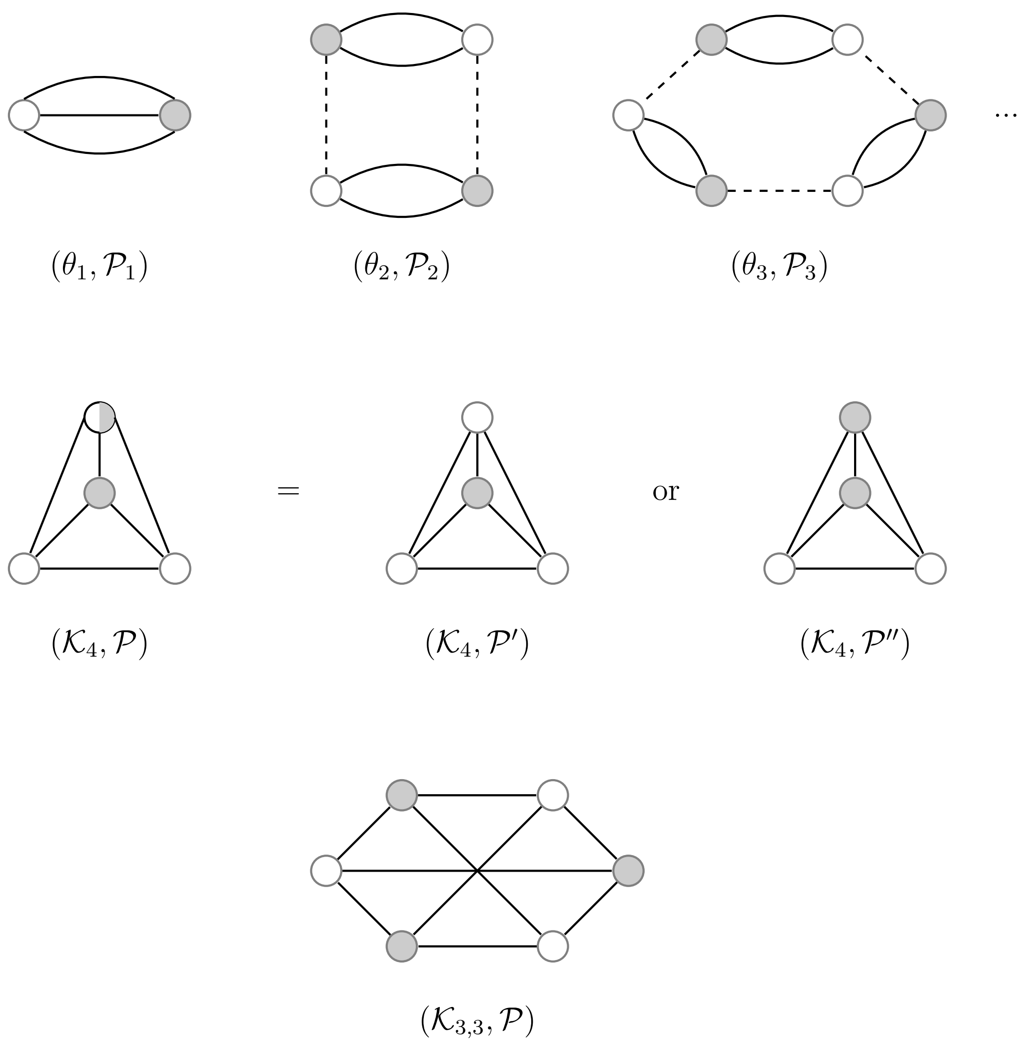


Figure 3: Solvable 2-connected graphs.
Dashed lines can correspond to 1-edges, solid lines cannot.

Proposition 6 *Pair $(\theta_K, \mathcal{P}_K)$ is solvable unless it contains a 1-edge of type 2.*

Let us remark that 1-edges of type 1 do not contradict solvability.

Let us also notice that the non-directed graph G_K corresponding to $\mathcal{G}_K = \theta_K$ contains a simple cycle as a proper (not induced) subgraph. It is easy to see that $(\theta_{K+1}, \mathcal{P}_{K+1}) >$

$(\theta_K, \mathcal{P}_K)$ for all K and, hence, $(\theta_K, \mathcal{P}_K)$ is an infinite chain of solvable 2-connected pairs. This chain has no maximal element.

1.4.3 Solvable pairs $(\mathcal{K}_4, \mathcal{P})$ and $(\mathcal{K}_{3,3}, \mathcal{P})$

Next, let $\mathcal{G} = \mathcal{K}_4$ and $\mathcal{P}$ consist of two white, one black, and one black-and-white positions. In fact, this case represents two subcases: $(\mathcal{K}_4, \mathcal{P}')$ and $(\mathcal{K}_4, \mathcal{P}'')$; see Figure 3.

Proposition 7 *Pair $(\mathcal{K}_4, \mathcal{P})$ is solvable unless it contains a 0- or 1-edge.*

Let us remark that both pairs $(\mathcal{K}_4, \mathcal{P}')$ and $(\mathcal{K}_4, \mathcal{P}'')$ majorize $(\theta_1, \mathcal{P}_1)$.

Now, let us consider the bipartite pair $(\mathcal{K}_{3,3}, \mathcal{P})$ in Figure 3.

Proposition 8 *Pair $(\mathcal{K}_{3,3}, \mathcal{P})$ is solvable unless it contains a 1-edge.*

Let us notice that bipartite pairs cannot contain 0-edges.

Let us also remark that $(\mathcal{K}_{3,3}, \mathcal{P}) > (\mathcal{K}_4, \mathcal{P}'')$. Indeed, $(\mathcal{K}_{3,3} - e, \mathcal{P})$ contains two simple paths of length 2; substitute two edges for them and get $(\mathcal{K}_4, \mathcal{P}'')$. On the other hand, $(\mathcal{K}_{3,3}, \mathcal{P}) \not\geq (\mathcal{K}_4, \mathcal{P}')$.

1.4.4 Solvable monochromatic pairs

A pair $(\mathcal{G}, \mathcal{P})$ is called *monochromatic* if $\mathcal{V} = \mathcal{V}_1,\ \mathcal{V}_2 = \emptyset$ or $\mathcal{V} = \mathcal{V}_2,\ \mathcal{V}_1 = \emptyset$. To characterize solvable monochromatic pairs we introduce one more simple transformation. Given a pair (G, P), let us consider the corresponding pair $(\mathcal{G}, \mathcal{P})$, duplicate every 0-edge in it, and denote the obtained pair by $(\mathcal{G}', \mathcal{P})$.

Proposition 9 *A monochromatic pair $(\mathcal{G}, \mathcal{P})$ is solvable unless $\mathcal{G}'$ contains a 1-edge and two more edge-disjoint simple paths between its ends.*

Several examples of solvable (Y) and not solvable (N) monochromatic pairs are given in Figure 4.

1.4.5 Main result

In Sections 7, we will derive the following criterion of solvability summarizing the above seven propositions. Let us denote by $\mathcal{L}$ the list of 2-connected solvable pairs (G, P) corresponding to pairs $(\mathcal{G}, \mathcal{P})$ from Propositions 4 - 9; see Figures 3 and 4.

Theorem 2 *A pair (G, P) is solvable if and only if every its 2-connected component is in $\mathcal{L}$.*

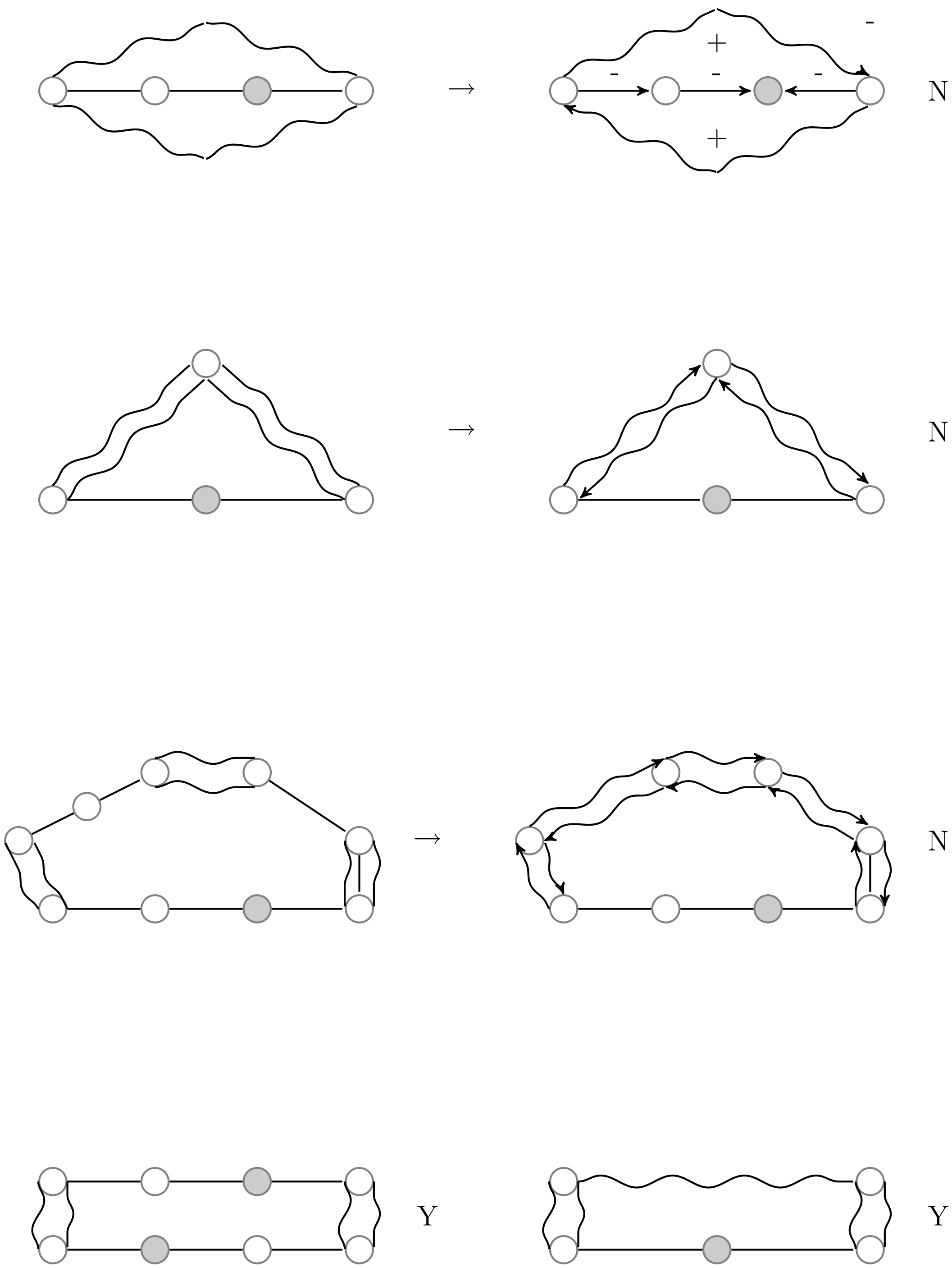


Figure 4: Monochromatic case

This result for the *bipartite* pairs was announced in [14, 15, 16]. In this paper, we extend it to arbitrary pairs and give the complete proof. Thus, we obtain necessary and sufficient conditions of solvability for the bidirected (not necessarily bipartite) cyclic game forms. In general, characterization of solvability still remains an open problem.

By means of Theorem 2 we can verify solvability of a bidirected cyclic game form $(\overrightarrow{G}, P, j_0))$ in polynomial time as follows. First, we check whether digraph $\overrightarrow{G}$ is bidirected. If it is not, Theorem 2 is not applicable; if it is, we construct the corresponding pair (G, P) and then $(\mathcal{G}, \mathcal{P})$. Then we construct all its 2-connected components and verify whether they belong to the list $\mathcal{L}$. If they all do then the considered game form is solvable, otherwise it is not.

1.5 Mean-payoff cyclic games and mean-solvable cyclic game forms

In this paper we study solvability of cyclic game forms $(\overrightarrow{G}, P, j_0)$ with respect to arbitrary utility functions $u : I \times C \to \mathbb{R}$, where $I = \{1, 2\}$ and C is the set of all simple directed cycles (dicycles) of $\overrightarrow{G}$. However, the so-called *additive* utility functions are more frequent in the literature, [2, 3, 4, 11, 12, 24, 26, 29, 31, 32, 36, 38].

A *local* utility function is a mapping $u_\ell : I \times \overrightarrow{E} \to \mathbb{R}$. The value $u_\ell(i, e)$ is interpreted as a profit of player $i \in I = \{1, 2\}$ in case the move $e\overrightarrow{E}$ appears in the play p. Given u_ℓ, the additive utility function $u_a : I \times C \to \mathbb{R}$ is defined by formula: $u_a(i, c) = (|c|^{-1} \sum_{e \in c} u_\ell(i, e)$, where $c \in C$ is a dicycle and $|c|$ is the number of edges (or vertices) in c. It is easy to see that if play p results in a dicycle $c \in C$ then $u_a(i, c)$ is the limit mean profit of the player $i \in I$ per one move. Cyclic games with additive utility functions are known as mean-payoff games. Naturally, we can introduce the concepts of *mean-solvability* and *zero-sum mean-solvability* for cyclic game forms. It appears that the latter concept is trivial.

Theorem 3 *([11, 12, 31, 32, 24]) Each zero-sum mean-payoff game $(\overrightarrow{G}, P, j_0, u)$ has a value (in pure positional strategies). In other words, each cyclic game form $(\overrightarrow{G}, P, j_0)$ is zero-sum mean-solvable.*

This statement was proved in case of a complete bipartite graph $\overrightarrow{G}$ in [31], for bipartite $\overrightarrow{G}$ in [12], and for arbitrary $\overrightarrow{G}$ in [24]. There are several further extensions of this result, [29, 36]. It was recently shown in [4] that mean payoff games can be solved in *expected sub-exponential time*. Then a deterministic sub-exponential algorithm for solving an important special case, the so-called *parity games*, was proposed in [26]. However, the question as to whether this class of games can be solved in polynomial time remains open, even though the corresponding decision problem is in NP $\cap$ co-NP, [29].

In October 2007, Vorobyov obtained a strongly polynomial algorithm [37].

Furthermore, let us notice that, obviously, every solvable cyclic game form is mean-solvable. However, solvability and zero-sum solvability are equivalent, while mean-solvability and zero-sum mean-solvability are not. In other words, not all cyclic game forms are mean-solvable. For example, $(\overrightarrow{G}, P, j_0)$ is not mean-solvable when $\overrightarrow{G}$ is the bidirected complete

3×3 bipartite digraph, [20, 24]. In fact, this example is minimal, since $(\overrightarrow{G}, P, j_0)$ is mean-solvable whenever $\overrightarrow{G}$ is a bidirected $2 \times n$ bipartite digraph, [21]. However, no general characterization of the mean-solvable cyclic game forms is known.

Recently, a similar special type of payoff was considered in [5, 7].

Let $(\overrightarrow{G}, P, j_0)$ be a game form of $n \geq 2$ players $I = \{1, \ldots, n\}$ and X_i be the set of (pure positional) strategies of a player $i \in I$. Each situation $x = (x_1, \ldots x_n) \in X_1 \times \ldots \times X_n = X$ defines a play $p = p(x)$ that can terminate in a final position $j \in V_T$ or result in a dicycle $c \in C$. In the former case we introduce a standard additive payoff $u(i, p) = \sum_{e \in p} u_\ell(i, e)$, where $u_\ell : I \times \overrightarrow{E} \to \mathbb{R}$ is a given local payoff, while in the latter case we define $u(i, c) = -\infty$ for each player $i \in I$. In other words, we assume that cycles are possible but extremely unattractive for all players. Thus, we obtain a utility function $u : I \times A \to \mathbb{R}$, where $A = C \cup V_T$ is the set of outcomes.

In [7], it was conjectured that the obtained game $(\overrightarrow{G}, P, j_0, u)$ has a Nash equilibrium whenever $u \geq 0$, or in other words, each game form $(\overrightarrow{G}, P, j_0)$ is Nash-solvable with respect to non-negative utility functions. Partial results in this direction and examples showing that non-negativity of u is essential are given in [7].

1.6 Criteria of ergodicity for cyclic game forms

As we mentioned in Section 1.3, solvability of a bidirected pair (G, P) is an ergodic property, that is, if G is bidirected then solvability of a cyclic game form (G, P, j) does not depend on j.

However, the result of the game (G, P, j, u) can depend on j.

As we also mentioned, any cyclic game form $(\overrightarrow{G}, P, j)$ is zero-sum mean-solvable, that is, for every zero-sum mean payoff u the corresponding game $(\overrightarrow{G}, P, j, u)$ has a value $v(\overrightarrow{G}, P, j, u)$ (in pure positional strategies). Yet, this value might depend on j. If it does not, then pair $(\overrightarrow{G}, P)$ is called *ergodic*.

Furthermore, given two positions $j', j'' \in V$, we will use notation $j' \leq j''$ if $v' = v(\overrightarrow{G}, P, j', u) \leq v(\overrightarrow{G}, P, j'', u) = v''$ for each zero-sum mean payoff u.

Proposition 10 *Pair $(\overrightarrow{G}, P)$ is ergodic if and only if $j' \leq j''$ for all $j', j'' \in V$.*

Proof By definition, $(\overrightarrow{G}, P)$ is ergodic if and only if $v' = v''$ for all $j', j'' \in V$. Hence, it is sufficient to notice that if $j' \leq j''$ and $j'' \leq j'$ then $v' = v''$. □

Simple necessary and sufficient conditions for ergodicity, as well as for $j' \leq j''$ were obtained in [25].

Given a pair $(\overrightarrow{G}, P)$, where $\overrightarrow{G} = (V, \overrightarrow{E})$ is a digraph and $P : V = V_1 \cup V_2$ is a partition of its positions, we assume that each position $j \in V$ has a strictly positive out-degree, or in other words, there is no dead-end in $\overrightarrow{G}$.

Let $Q : V = V^1 \cup V^2$ be another partition of V with the following properties:

(i) Both sets V^1 and V^2 are not empty.

(ii) There is no arc $e = (j_1, j_2) \in \overrightarrow{E}$ such that $j_1 \in V^2 \cap V_1, j_2 \in V^1$ or $j_1 \in V^1 \cap V_2, j_2 \in V^2$. In other words, player 1 (respectively, player 2) cannot leave V^2 for V^1 (respectively, V^1 for V^2).

(iii) For each $j_1 \in V^1 \cap V_1$ (respectively, $j_2 \in V^2 \cap V_2$) there is an arc $e = (j_1, j_1') \in \overrightarrow{E}$ such that $j_1' \in V^1$ (respectively, $e = (j_2, j_2') \in \overrightarrow{E}$ such that $j_2' \in V^2$). In other words, player 1 (respectively, player 2) cannot be forced to leave V^1 for V^2 (respectively, V^2 for V^1).

In particular, (iii) implies that both induced digraphs $\overrightarrow{G}[V^1]$ and $\overrightarrow{G}[V^2]$ have no dead-ends.

Given a partition $Q : V = V^1 \cup V^2$ satisfying (i, ii, iii), we will say that player 1 (respectively, 2) is in control of V^1 (respectively, of V^2) and call Q a *contra-ergodic partition* of the pair $(\overrightarrow{G}, P)$.

Proposition 11 *([25]). (a) A pair $(\overrightarrow{G}, P)$ is ergodic if and only if it admits no contra-ergodic partition. (b) Furthermore, $j' \leq j''$ if and only if there is no contra-ergodic partition $Q : V = V^1 \cup V^2$ such that $j' \in V^1$ and $j'' \in V^2$.*

Proof "If parts of (a) and (b)". Given a pair $(\overrightarrow{G}, P)$, let us suppose that a contra-ergodic partition $Q : V = V^1 \cup V^2$ exists and define a zero-sum local payoff $u_\ell : I \times \overrightarrow{E} \to \mathbb{R}$ as follows. For an arc $e = \overrightarrow{E}$ from j to j' we set $u_\ell(e) = +1$ if $j, j' \in V^1$, $u_\ell(e) = -1$ if $j, j' \in V^2$, and otherwise $u_\ell(e)$ can take any value. (Standardly, for a zero-sum payoff we denote $u_\ell(1, e) = -u_\ell(2, e)$ by $u_\ell(e)$.) Obviously, $v(\overrightarrow{G}, P, j', u) = +1$ for $j' \in V^1$ and $v(\overrightarrow{G}, P, j'', u) = -1$ for $j'' \in V^2$. Indeed, by (i,ii,iii), each player $i \in I = \{1, 2\}$ can lock the opponent $3 - i$ within V^i and, by this, guarantee profit $3 - 2i$ per each move. Thus, value depends on the initial position and, hence, pair $(\overrightarrow{G}, P)$ is not ergodic. Moreover, relation $j' \leq j''$ does not hold.

"Only if part of (a) and (b)". Let us suppose that $(\overrightarrow{G}, P)$ is not ergodic, or in other words, $j' \leq j''$ does not hold for some $j', j'' \in V$ and a zero-sum payoff $u : I \times C \to \mathbb{R}$, that is, $v' = v(\overrightarrow{G}, P, j', u) > v(\overrightarrow{G}, P, j'', u) = v''$. Let us choose a number v such that $v' > v > v''$ and set $V^1 = \{j \in V \mid v(\overrightarrow{G}, P, j, u) > v\} \subseteq V$ and $V^2 = \{j \in V \mid v(\overrightarrow{G}, P, j, u) \leq v\} \subseteq V$. By this definition, $j' \in V^1 \neq \emptyset$, $j'' \in V^2 \neq \emptyset$, $V^1 \cap V^2 = \emptyset$, and $V^1 \cup V^2 = V$; in other words, $Q : V = V^1 \cup V^2$ is a partition of V satisfying (i). Moreover, (ii) and (iii) hold for Q too. Indeed, (iii) follows, since each player $i \in I = \{1, 2\}$ in each position $j \in V_i$ has a move (j, ℓ) that keeps the value, that is, $v(\overrightarrow{G}, P, j, u) = v(\overrightarrow{G}, P, \ell, u)$. Also (ii) follows, since in each position player 1 (respectively, 2) has no move that increase (respectively, decrease) the value; see Section 2.1 for more details. Thus, Q is a contra-ergodic partition and, moreover, $j' \leq j''$ does not hold. □

Let us remark that both "only if parts" are shown for arbitrary (not necessarily additive) utility functions. Hence, the following claim holds.

Corollary 2 *Given a digraph* $(\overrightarrow{G} = (V, \overrightarrow{E})$, *partition* $P : V = V_1 \cup V_2$, *positions* $j', j'' \in V$, *and a zero-sum (not necessarily additive) payoff* $u : C \to \mathbb{R}$, *if* v' *and* v'' *exist and* $v' > v''$ *then there is a contra-ergodic partition* $Q : V = V^1 \cap V^2$ *such that* $j' \in V^1$ *and* $j'' \in V^2$. □

Example 1.1 *Let us consider four bipartite bidirected pairs such that the corresponding non-directed graphs are* c_2, c_4, c_6, *and* p_4. *It is easy to verify that the first two pairs are ergodic, while the last two are not; they admit three and one contra-ergodic partitions, respectively.*

Although conditions (i, ii, iii) look simple, yet, somewhat surprisingly, given a *bipartite* pair $(\overrightarrow{G}, P)$ (and positions $j', j'' \in V$), it is NP-hard to verify whether a contra-ergodic partition $Q : V = V^1 \cup V^2$ (such that $j' \in V^1$ and $j'' \in V^2$) exist.

Proposition 12 *([25]). The following two decision problems are co-NP-complete already for bipartite pairs:*

(a) whether $(\overrightarrow{G}, P)$ *is ergodic ? (b) whether* $j' \leq j''$ *?*

Moreover, each of the four subproblems of (b) defined by the extra assumptions:

(b11) $j' \in V_1,\ j'' \in V_1$, *(b12)* $j' \in V_1,\ j'' \in V_2$,

(b21) $j' \in V_2,\ j'' \in V_1$, *(b22)* $j' \in V_2,\ j'' \in V_2$,

is co-NP-complete too.

These claims were also shown in [25]. Yet, due to space limits (two pages for the whole paper), the proofs were only sketched. We will give complete proofs in Appendix 2.

Now we will derive several simple corollaries of Proposition 11 for the bipartite case.

Corollary 3 *Let* $(\overrightarrow{G}, P)$ *a bipartite pair in which* $\overrightarrow{G} = (V, \overrightarrow{E})$ *and* $P : V = V_1 \cup V_2$; *furthermore, let* $Q : V = V^1 \cup V^2$ *be a contra-ergodic partition in it. Then* $V_i \cap V^\ell \neq \emptyset$ *for all four cases* $i, \ell = 1, 2$; *moreover, in* $\overrightarrow{E}$ *there is no arc between* $V^1 \cap V_2$ *and* $V^2 \cap V_1$.

Proof It follows, since $(\overrightarrow{G}, P)$ is bipartite and (i,ii) hold for Q. □

A bipartite pair $(\overrightarrow{G}, P)$ will be called *semi-complete* if between every two vertices $j_1 \in V_1$ and $j_2 \in V_2$ there is an arc, from j_1 to j_2 or from j_2 to j_1.

Corollary 4 *Each semi-complete bipartite pair is ergodic.*

Proof It follows immediately from Corollary 3. □

Corollary 5 *A bidirected bipartite pair* $(\overrightarrow{G}, P)$ *is ergodic if and only if the corresponding contra-directed bipartite graph* G *is complete.*

Proof "If part". Obviously, if G is complete then no partition Q can satisfy (i) and (ii). "Only if part". Assume indirectly that G is not complete, that is, there are two vertices $j_1 \in V_1$ and $j_2 \in V_2$ such that there is no arcs between j_1 and j_2 in $\overrightarrow{E}$. Then let us set $V^1 = (V_2 \setminus \{j_2\}) \cup \{j_1\}$ and $V^2 = (V_1 \setminus \{j_1\}) \cup \{j_2\})$ to obtain a contra-ergodic partition $Q : V = V^1 \cup V^2$. □

Now let us consider ergodicity for digraphs with loops.

Corollary 6 *Given a pair* $(\overrightarrow{G}, P)$ *such that* $\overrightarrow{G}$ *has a loop* $e_j = (j, j)$ *at a vertex* $j \in V_i$, *where* $i \in I = \{1, 2\}$, *if* $(\overrightarrow{G}, P)$ *is ergodic then player* i *has a strategy that results in* e_j *for every initial position* $j_0 \in V$.

Proof Without loss of generality, assume that $i = 1$. Let $V^j \subseteq V$ denote the set of all positions from which player 1 can enforce e_j. More precisely, let $V_0^j = \{j\}$; furthermore, let $V_1^j \subseteq V_1$ denote the set of all positions from which there is a move to V_0^j; let $V_2^j \subseteq V_2$ be the set of all positions from which each move results in $V_0^j \cup V_1^j$; let $V_3^j \subseteq V_1$ be the set of all positions from which there is a move to $V_0^j \cup V_1^j \cup V_2^j$; let $V_4^j \subseteq V_2$ be the set of all positions from which each move results in $V_0^j \cup V_1^j \cup V_2^j \cup V_3^j$; etc. Finally, let us define V_j as the union of all obtained sets $V_j = V_0^j \cup V_1^j \cup V_2^j \cup V_3^j \ldots$.

It is easy to see that partition $Q : V = V^j \cup V \setminus V^j$ satisfies (ii) and (iii). Moreover, $V^j \neq \emptyset$, since $j \in V^j$. Hence, Q is contra-ergodic unless $V^j = V$. It is also clear that player i has a strategy that results in e_j whenever the game begins in a position from V^j. □

Corollary 7 *A pair* $(\overrightarrow{G}, P)$ *is not ergodic whenever* $\overrightarrow{G}$ *has two loops* e_{j_1} *and* e_{j_2} *such that* $j_1 \in V_1$ *and* $j_2 \in V_2$.

2 Proofs of Propositions 1, 2, and 3

2.1 Maxmin and minmax depend on the initial position

Given a cyclic game $(\overrightarrow{G}, P, j, u)$ with a zero-sum utility function u, although we cannot guarantee that the value $v = v(\overrightarrow{G}, P, j, u)$ exists, yet, in every game there exist maxmin $v_1 = v_1(\overrightarrow{G}, P, j, u)$ and minmax $v_2 = v_2(\overrightarrow{G}, P, j, u)$. They represent the values that player 1 and, respectively, player 2 can guarantee. Inequality $v_1 \leq v_2$ holds for every fixed $\overrightarrow{G}, P, u$, and j. Furthermore, given $\overrightarrow{G}, P$, and u, maxmin and minmax can depend on the initial position $j \in V$, that is, $v_1 = v_1(j)$ and $v_2 = v_2(j)$.

Lemma 1 *For each move from* j *to* j' *we have:*

(i) $v_1(j') \leq v_1(j)$ *and* $v_2(j') \leq v_2(j)$ *whenever* $j \in V_1$ *and*

(ii) $v_1(j') \geq v_1(j)$ *and* $v_2(j') \geq v_2(j)$ *whenever* $j \in V_2$.

Moreover, for every position $j \in V = V_1 \cup V_2$ *there are moves* (j, j') *and* (j, j'') *such that* *(i')* $v_1(j') = v_1(j)$ *and* *(ii')* $v_2(j'') = v_2(j)$.

Proof Indeed, by definition of maxmin and minmax, we have:

$v_1(j) = \max\{v_1(j') \mid (j, j') \in \overrightarrow{E}\}$,
$v_2(j) = \max\{v_2(j'') \mid (j, j'') \in \overrightarrow{E}\}$ for $j \in V_1$;
$v_1(j) = \min\{v_1(j') \mid (j, j') \in \overrightarrow{E}\}$,
$v_2(j) = \min\{v_2(j'') \mid (j, j'') \in \overrightarrow{E}\}$ for $j \in V_2$.

These equalities imply (i,i') and (ii,ii'), respectively. □

In other words, both maxmin and minmax do not increase (respectively, decrease) with any move of player 1 (respectively, 2); moreover, in each position there are two moves that keep unchanged maxmin in minmax, respectively. However, these two moves can be distinct, since we cannot guarantee that (i') and (ii') hold simultaneously. For example, let $j \in V_1$ and there are exactly two moves (j, j') and (j, j'') from j which result in positions j' and j'' such that $v_1(j') = v_2(j') = 0$, $v_1(j'') = -1$, $v_2(j'') = 1$. Then $v_1(j) = 0$ and $v_2(j) = 1$.

Let us recall Theorem 1 and from now on (except Appendix 1 and Section 3.2) restrict ourselves by the zero-sum ± 1 payoffs. Then maxmin $v_1(j)$ and minmax $v_2(j)$ take only values ± 1, too. Since $v_1(j) \leq v_2(j)$ for all $j \in V$, pairs $(v_1(j), v_2(j))$ can take only three pairs of values: $(-1, -1)$, $(1, 1)$, and $(-1, 1)$ that define the partition $V = V_- \cup V_+ \cup V_\pm$. Let us consider three induced subgraphs $\overrightarrow{G}_- = \overrightarrow{G}[V_-]$, $\overrightarrow{G}_+ = \overrightarrow{G}[V_+]$, $\overrightarrow{G}_\pm = \overrightarrow{G}[V_\pm]$ and three partitions $P_-, P_+, P_\pm$ induced on sets $V_-, V_+, V_\pm$ by the original partition $P : V = V_1 \cup V_2$.

For zero-sum ± 1 games we can reformulate Lemma 1 as follows.

Lemma 2 *All three digraphs $\overrightarrow{G}_- = \overrightarrow{G}[V_-]$, $\overrightarrow{G}_+ = \overrightarrow{G}[V_+]$, and $\overrightarrow{G}_\pm = \overrightarrow{G}[V_\pm]$ have no dead-ends. Furthermore, player 1 has no moves from V_- to $V_\pm$, from $V_\pm$ to V_+, and from V_- to V_+, while player 2 has no moves from V_+ to $V_\pm$, from $V_\pm$ to V_-, and from V_+ to V_-.* □

In other words, in the sequence $V_-, V_\pm, V_+$ player 1 can move only from right to left, while player 2 only from left to right, and each player can always stay in the same set. These observations easily imply the following two claims.

Corollary 8 *Triplets $(\overrightarrow{G}_-, P_-, j)$, $(\overrightarrow{G}_+, P_+, j)$, and $(\overrightarrow{G}_\pm, P_\pm, j)$ form cyclic game forms if the initial position j belongs to V_-, V_+, and $V_\pm$, respectively.*

Corollary 9 *The original pair $(\overrightarrow{G}, P)$ is not ergodic whenever at least two of the three sets V_-, V_+, and $V_\pm$ are not empty.*

Proof First claim follows from Lemma 2 immediately. The second one follows from Lemma 2 and criterion of ergodicity given by Proposition 11. Indeed, $Q : V = V^1 \cup V^2$ is a contra-ergodic partition whenever both sets are not empty and $V^1 = V_+, V^2 = V_- \cup V_\pm$ or $V^1 = V_+ \cup V_\pm, V^2 = V_-$. Obviously, V^1 and V^2 are not empty if and only if at least two of the three sets V_-, V_+, and $V_\pm$ are not empty. □

2.2 Uniformly non-solvable pairs

By definition and Theorem 2, cyclic game form $(\overrightarrow{G}, P, j)$ is not solvable if and only if there is a utility functions u such that $j \in V_{\pm}$ in the obtained game $(\overrightarrow{G}, P, j, u)$. We can strengthen this claim as follows. A pair $(\overrightarrow{G}, P)$ will be called *uniformly non-solvable* if there is a utility function u such that for every initial position $j \in V$ the obtained game $(\overrightarrow{G}, P, j, u)$ is not solvable, that is, $-1 = v_1(j) < v_2(j) = 1$ for each $j \in V$, or in other words, $V_{\pm} = V$.

Lemma 3 *Pair $(\overrightarrow{G}_{\pm}, P_{\pm})$ is uniformly non-solvable.*

Proof Suppose that $V_{\pm} \neq \emptyset$ for some payoff u and consider the subgame $(\overrightarrow{G}_{\pm}, P_{\pm}, j, u_{\pm})$, where $u_{\pm} : C(\overrightarrow{G}_{\pm}) \to \{-1, +1\}$ is the restriction of u to the dicycles of digraph $\overrightarrow{G}_{\pm}$. By definition of $V_{\pm}$, we have $-1 = maxmin = v_1(j) < v_2(j) = minmax = 1$ for every $j \in V_{\pm}$. In other words, game form $(\overrightarrow{G}_{\pm}, P_{\pm}, j)$ is not solvable for each j, that is, pair $(\overrightarrow{G}_{\pm}, P_{\pm})$ is uniformly non-solvable. □

Lemma 4 *Furthermore, let us choose two arbitrary vertices $j', j'' \in V_{\pm}$ and add the extra arc $e = (j', j'')$ to digraph $\overrightarrow{G}_{\pm}$. The obtained pair $(\overrightarrow{G}'_{\pm}, P_{\pm})$ is uniformly non-solvable.*

Proof Without loss of generality we can assume that $j' \in V_1$. In this case let us extend $u_{\pm}$ to $u'_{\pm} : C(\overrightarrow{G}'_{\pm}) \to \{-1, +1\}$ by setting $u'_{\pm}(c) = -1$ for every dicycle c that contains the new arc e. Let us consider the obtained game $(\overrightarrow{G}'_{\pm}, P_{\pm}, j, u'_{\pm})$ and show that $-1 = v'_1(j) < v'_2(j) = 1$ for every initial position j. The second equality is obvious, since the only new move $e = (j', j'')$ belongs to player 1, while the set of strategies of player 2 remains the same.

Yet, we have to show that $v'_1(j) = -1$, that is, player 1 cannot use the new move e in a strategy that will guarantee the result $+1$. This is not fully obvious. Although, by definition, $u'_{\pm}(c) = -1$ for every dicycle c that contains e, yet, perhaps, e could create a vital "communication" for player 1. However, this cannot happen either. Indeed, obviously, $v'_1(j'') = -1$, since if $j = j''$ is the initial position then player 1 will immediately lose after move $e = (j', j'')$. Furthermore, since $v'_1(j'') = -1$ and $e = (j', j'')$ results in j'', this move is useless as a communication. Thus, $v'_1(j) = -1$, while $v'_2(j) = 1$ for all $j \in V_{\pm}$. □

Similarly to the non-directed case, we will say that pair $(\overrightarrow{G}, P)$ is majorized by $\overrightarrow{G}', P')$ if the digraph $\overrightarrow{G} = (V, \overrightarrow{E})$ is a subgraph of $(\overrightarrow{G}' = (V', \overrightarrow{E}')$ (that is, $V \subseteq V'$, $\overrightarrow{E} \subseteq \overrightarrow{E}'$) and partition $P : V = V_1 \cup V_2$ is a subpartition of $P' = V' : V'_1 \cup V'_2$ induced by the subset $V \subseteq V'$ (that is, $P : V = (V'_1 \cap V) \cup (V'_2 \cap V)$). Standardly, we use the notation $\overrightarrow{G} \leq \overrightarrow{G}'$ and $(\overrightarrow{G}, P) \leq (\overrightarrow{G}', P')$.

Theorem 4 *If (i) cyclic game form $(\overrightarrow{G}, P, j_0)$ is not solvable, (ii) $(\overrightarrow{G}, P) \leq (\overrightarrow{G}', P')$, and (iii) for every position $j' \in V'$ in $\overrightarrow{G}'$ there is a directed path from j' to j_0, then (iv) $(\overrightarrow{G}', P')$ is uniformly non-solvable.*

Proof Since $(\overrightarrow{G}, P, j_0)$ is not solvable, there is a utility function $u : C(\overrightarrow{G}) \to \{-1, +1\}$ such that the corresponding game $(\overrightarrow{G}, P, j_0, u)$ is not solvable either, that is, $-1 = maxmin = v_1(j_0) < v_2(j_0) = minmax = 1$. In other words, $V_\pm \neq \emptyset$; in particular, $j_0 \in V_\pm$. Then, by Lemma 3, the induced pair $(\overrightarrow{G}_\pm, P_\pm)$ is uniformly non-solvable, that is, game form $(\overrightarrow{G}_\pm, P_\pm, j)$ is not solvable for every j. In other words, $-1 = maxminv_1(j) < v_2(j) = minmax = 1$ for every initial position $j \in V_\pm$ of the game $(\overrightarrow{G}_\pm, P_\pm, j, u_\pm)$, where $u_\pm : C(\overrightarrow{G}_\pm) \to \{-1, +1\}$ is the restriction of u to $C(\overrightarrow{G}_\pm)$.

To prove that pair $(\overrightarrow{G}', P')$ is uniformly non-solvable we will define a similar utility function $u' : C(\overrightarrow{G}') \to \{-1, +1\}$ such that which $-1 = maxmin = v'_1(j) < v'_2(j) = minmax = 1$ for every $j \in V'$. To get u' we will extend $u_\pm$ from $C(\overrightarrow{G}_\pm)$ to $C(\overrightarrow{G}')$. To do so, we set $u'(c) = u_\pm(c)$ for each dicycle c in $C(\overrightarrow{G}_\pm)$. Now we will extend, step by step, digraph $\overrightarrow{G}_\pm$ to $\overrightarrow{G}'$, so that in each step we obtain a uniformly non-solvable pair. Thus, to get u' from u, first, we reduce u from $C(\overrightarrow{G}_)$ to $C(\overrightarrow{G}_\pm)$ getting $u_\pm$ and then extend it to $C(\overrightarrow{G}')$.

Step A1. Let us add to digraph $\overrightarrow{G}_\pm$ all arcs $(j', j'') \in \overrightarrow{E}'$ such that $j', j'' \in V_\pm$. The obtained pair $(\overrightarrow{G}_1, P_1)$ is uniformly non-solvable, by Lemma 4.

Step B1. By condition (iii) of the Theorem, since $j_0 \in V_\pm$, there is an arc (j', j'') in $\overrightarrow{G}'$ such that $j'' \in V_\pm$, while $j' \notin V_\pm$. Let us add this arc (together with vertex j') to digraph $(\overrightarrow{G}_1)$. Obviously, the obtained pair $(\overrightarrow{G}_2, P_2)$ is uniformly non-solvable, since move in position j' is forced.

Step A2. Now let us add to digraph $\overrightarrow{G}_2$ all arcs $(j, j') \in \overrightarrow{E}'$ for $j \in V_\pm$. By Lemma 4, the obtained pair $(\overrightarrow{G}'_2, P_2)$ is uniformly non-solvable.

By this, we extend u' from $C(\overrightarrow{G}_1) = C(\overrightarrow{G}_2)$ to $C(\overrightarrow{G}'_2)$ as follows. If C contains (j, j') then $u'(C) = -1$ for $j \in V_1$ and $u'(C) = +1$ for $j \in V_2$.

Now we can proceed with Step B2, etc., until we obtain the final digraph $C(\overrightarrow{G}')$ and show that pair $(\overrightarrow{G}', P')$ is uniformly non-solvable. □

Example 2.1 *As an illustration, let us consider four pairs* $(\overrightarrow{G}_k, P_k)$, $k = 1, 2, 3, 4$ *in Figure 1. The first and the last pairs are uniformly non-solvable;* $(\overrightarrow{G}_3, P_3)$ *is uniformly solvable, moreover, it is ergodic; finally,* $(\overrightarrow{G}_2, P_2)$ *is not solvable, yet, not uniformly. Let us notice that* $(\overrightarrow{G}_1, P_1) < (\overrightarrow{G}_2, P_2) < (\overrightarrow{G}_3, P_3)$.

This example shows that condition (iii) of Theorem 4 is essential. Indeed, both extensions $(\overrightarrow{G}_1, P_1)$ *to* $(\overrightarrow{G}_2, P_2)$ *and* $(\overrightarrow{G}_2, P_2)$ *to* $(\overrightarrow{G}_3, P_3)$ *satisfy (i) and (ii) but (iii) and (iv) fail.*

2.3 Propositions 1, 2, and 3 follow from Theorem 4

Let us notice, however, that condition (iii) automatically holds if digraph $\overrightarrow{G}'$ is strongly connected, or in particular, if $\overrightarrow{G}'$ is bidirected and the corresponding graph G' is connected. Hence, Theorem 4 implies Proposition 2.

Furthermore, Proposition 1 follows from Theorem 4, too. Indeed, conditions (ii) and (iii) of the theorem hold whenever $\overrightarrow{G} = \overrightarrow{G}'$ is a strongly connected digraph. Hence, in this case solvability does not depend on the initial position. Moreover, Proposition 1 can be strengthened as follows.

Proposition 13 *If digraph $\overrightarrow{G}$ is strongly connected then a pair $(\overrightarrow{G}, P)$ is either solvable or uniformly non-solvable.* □

Finally, let us derive Proposition 3 from Theorem 4. Again, we will prove a stronger claim : Given K digraphs $\overrightarrow{G_k} = (V_k, \overrightarrow{E}_k)$, where $k \in [K] = \{1, \ldots, K\}$, with a unique common vertex j_0, that is, $V_{k'} \cap V_{k''} = \{j_0\}$ for each two distinct $k', k'' \in [K]$, then obviously, K arc-sets are pairwise disjoint. Let $\overrightarrow{G} = (V, \overrightarrow{E})$ be the union of these K digraphs, that is, $V = \cup_{k=1}^{K} V_k$ and $\overrightarrow{E} = \cup_{k=1}^{K} \overrightarrow{E}_k$.

Lemma 5 *Digraph $\overrightarrow{G}$ is strongly connected if and only if all digraphs $\overrightarrow{G}_k$ are strongly connected for $k = 1, \ldots, K$.*

Proof Each of the above digraphs is strongly connected if and only if each its vertex can be reached by a (simple) directed path from j_0 and, vice versa, j_0 can be reached by a (simple) directed path from each vertex. Clearly, this property holds for $\overrightarrow{G}$ if and only if it holds for $\overrightarrow{G}_k$ for all $k \in [K] = \{1, \ldots, K\}$. □

In the rest of this section, we will assume that digraphs $\overrightarrow{G}_k$ are strongly connected for $k = 1, \ldots, K$. Hence, digraph $\overrightarrow{G}$ is strongly connected, too.

Furthermore, let $P_k : V_k = V_1^k \cup V_2^k$ be K partitions such that $j_0 \cap_{k=1}^{K} V_i^k$ either for $i = 1$ or for $i = 2$, in other words, position j_0 belongs to the same player, 1 or 2, in all partitions. Let $P : V = V_1 \cup V_2$ be the union of these partitions, that is, $V_1 = \cup_{k=1}^{K} V_1^k$ and $V_2 = \cup_{k=1}^{K} V_2^k$.

Lemma 6 *Cyclic game form $(\overrightarrow{G}, P, j_0)$ is solvable if and only if $(\overrightarrow{G}_k, P_k, j_0)$ are solvable for all $k = 1, \ldots, K$.*

Proof Given an arbitrary zero-sum payoff u, the following formulas, obviously, hold for maxmin and minmax:

$v_i = \max(v_i^k \mid k = 1, \ldots, K)$ if $j_0 \in V_1$ and

$v_i = \min(v_i^k \mid k = 1, \ldots, K)$ if $j_0 \in V_2$ for $i \in I = \{1, 2\}$,

where $v_i = v_i(\overrightarrow{G}, P, j_0)$ and $v_i^k = v_i^k(\overrightarrow{G}_k, P_k, j_0)$ for $k = 1, \ldots, K$ are maxmin if $i = 1$ and minmax if $i = 2$. Now, it is easy to see that maxmin and minmax are equal for all u in game $(\overrightarrow{G}, P, j_0, u)$ if and only if they are equal for all u in all games $(\overrightarrow{G}_k, P_k, j_0, u)$ for $k = 1, \ldots, K$. □

Let us recall that, by Proposition 13, solvability is ergodic, that is, it does not depend on the initial position. Hence, we can strengthen Lemma 6 as follows.

Lemma 7 *Pair $(\overrightarrow{G}, P)$ is solvable if the pairs $(\overrightarrow{G}_k, P_k)$ are solvable for all $k = 1, \ldots, K$, otherwise pair $(\overrightarrow{G}, P)$ is uniformly non-solvable.* □

Let us summarize. We have defined an operation of the union of pairs with a unique common position and proved that the resulting pair is solvable (and strongly connected) if and only if all involved pairs are solvable (and strongly connected). Obviously, this operation can be applied several times successively and the same conclusion can be proved by induction.

Let us apply this construction to pairs whose digraphs are bidirected (and the corresponding non-directed graphs are connected). It is easy to see that each connected graph can be obtained in this way from its 2-connected components and some edges. Obviously, a pair corresponding to a single edge is solvable. This and Lemma 7 imply Proposition 3 and, in particular, Corollary 1. □

3 Solvability of cycles. Proof of Proposition 5

As an introductory example let us study solvability of a bidirected cyclic game form $(\overrightarrow{G}, P, j_0)$ such that the corresponding graph G is a simple cycle, $G = C_K$. By Proposition 1, solvability in this case is ergodic, does not depend on j_0. Hence, in fact, we study solvability of the pair (G, P). Four cases are given in Figure 1, where $G_1 = C_2, G_2 = C_3$, and $G_3 = G_4 = C_4$, respectively. It is not difficult to verify that the last two game forms are solvable, while the first two are not. To see this, let us construct the corresponding normal game forms and check that the last two are tight, while the first two are not; see Figure 1.

Obviously, pair (G, P) is solvable when G is a loop. Moreover, (G, P) is also solvable when G is a 0-cycle, that is, $V_1 = \emptyset, V_2 = V$ or $V_2 = \emptyset, V_1 = V$. Indeed, in this case one player, 1 or 2, respectively, is a dummy.

3.1 1-cycles are not solvable

The simplest non-solvable pair is $(G, P) = (C_2, P)$; see Figure 1. Graph $G = (V, E)$ consists of two vertices and two edges; players, 1 and 2, control one position each. The corresponding pair $(\overrightarrow{G}, P)$ is given in Figure 1 (1). Digraph $\overrightarrow{G}$ contains four dicycles. Each player has two strategies. Thus, all four outcomes of the corresponding 2×2 normal game form are distinct. Hence, it is not solvable.

More generally, let $(G, P) = (C_K, P)$ be a 1-cycle, say, $V_1 = \{j_0\}$, $V_2 = V \setminus \{j_0\}$, and $|V| = K \geq 2$. We will show that (G, P) is not solvable. The corresponding digraph $\overrightarrow{G}$ contains $2K + 2$ dicycles: K "short", of length 2 each, and two "long", of length K each, the "clockwise" and "counter-clockwise" dicycles. Let us introduce a payoff u as follows: $u(c) = -1$ if cycle c is long and $u(c) = +1$ if c is short. Player 1 controls only the initial position j_0 and has two strategies: to begin "clockwise or counter-clockwise". Player 2 has 2^{K-1} strategies, yet, all of them, but two, are definitely losing, since they always result in a short cycle. (Recall that player 2 is the minimizer.) Only two strategies, the clockwise and counter-clockwise, can be winning. Yet, there is no guarantee. The clockwise or counter-clockwise strategy of player 2 wins only if player 1 begins correspondingly. In these two cases two long cycles appear; otherwise (if layer 1 begins clockwise and 2 proceeds counter-clockwise, or vice versa) a short cycle appears and player 1 wins. Thus, removing all dominated strategies of player 2, we reduce normal form of the original game to the 2×2 matrix with $a_{1,1} = a_{2,2} = +1$ and $a_{1,2} = a_{2,1} = -1$. This matrix has no saddle point in pure strategies.

Remark 4 *It is important to notice that both players are restricted to their positional strategies, that is, the move in a position can depend only on this position but not on the preceding positions or moves. By this assumption, player* 2 *is not aware of the move of player* 1 *in* j_0*. In other words, both players choose their (positional) strategies simultaneously.*

3.2 Passing through a simple cycle

Given a pair (G, P) such that $G = C_K = (V, E)$ is a simple cycle of length K (that is, $V = \{j_0, j_1, \ldots, j_{K-1}\}$ (by convention, $j_K = j_0$) and $E = \{(j_{k-1}, j_k), k \in [K] = \{1, \ldots, K\}\}$, the corresponding digraph $\overrightarrow{G} = (V, \overrightarrow{E})$ contains K short dicycles $\{c_k = ((j_{k-1}, j_k), (j_k, j_{k-1})), k \in [K]\}$ that are in one-to-one correspondence with E and two long dicycles directed clockwise c_L and counter-clockwise c_R. As before, we assume that $j_0 \in V_1$ is the initial position.

Now let us make an extra assumption that player 1 begins clockwise, by move (j_0, j_1), and that player 2 knows it. The obtained game can be easily solved in positional strategies, since it is reduced to a finite positional game with perfect information whose tree is a caterpillar. Indeed, in each position $j_k \in V_i$, where $k = 1, \ldots, K-1$, the corresponding player $i \in I = \{1, 2\}$ has two olptions: either to return to j_{k-1} (and by this finish the game in c_k) or to proceed with j_{k+1}. If $k + 1 = K$ then the game is over; it results in the long cycle c_L and, hence, dicycles c_R and c_K cannot appear at all.

Now let us consider a zero-sum utility function u_L such that $u_L(c_L) = 0$ (that is, c_L is a draw) and $u_L(c_k)$ take values -1 and $+1$ for $k = 1, \ldots, K-1$. It is easy to see that all these $K - 1$ values are uniquely determined if we assume that the game is a draw, that is, the optimal strategies result in c_L. Indeed, in this case we must have

$$u_L(c_k) = (-1)^i \text{ whenever } j_k \in V_i, \text{ where } i = 1, 2; \text{ and } k = 1, \ldots, K-1. \tag{1}$$

It is easy to see that otherwise one of the players, 1 or 2, wins.

As we already mentioned, the value $u_L(c_K)$ is irrelevant.

Now let us assume alternatively that player 1 begins counter-clockwise, by move (j_K, j_{K-1}), and again player 2 knows it. Now in each position $j_k \in V_i$, where $k = 1, \ldots, K-1$, the corresponding player $i \in I = \{1, 2\}$ has two options: either to return to j_{k+1} (and by this finish the game in c_{k+1}) or to proceed with j_{k-1}. If $k - 1 = 0$ then the game is over; it results in the long dicycle c_R and, hence, dicycles c_L and c_1 cannot appear at all.

Now let us consider a zero-sum utility function u_R such that $u_R(c_R) = 0$ (that is, c_R is a draw) and $u_R(c_k)$ take values -1 and $+1$ for $k = 2, \ldots, K$. All these $K-1$ values are uniquely determined if the game is a draw, that is, the optimal strategies result in c_R. Indeed, in this case instead of (1) we obtain

$$u_R(c_k) = (-1)^i \text{ whenever } j_{k-1} \in V_i, \text{ where } i = 1, 2; \text{ and } k = K, \ldots, 2. \tag{2}$$

Now, the value $u_R(c_1)$ is irrelevant.

Lemma 8 *Equations (1) and (2) hold simultaneously, that is,* $u_L(c_k) = u_R(c_k)$ *for* $k = 2, \ldots, K-1$*, if and only if*

(i) pair (C_K, P) *is a* 0*-cycle (with* $V_1 = V$, $V_2 = \emptyset$*) and*
$u_L(c_1) = u_L(c_k) = u_R(c_k) = u_R(c_K) = -1$ *for* $k = 2, \ldots, K-1$, *or*

(ii) pair (C_K, P) *is a* 1*-cycle (with* $V_1 = \{j_0\}$, $V_2 = V \setminus \{j_0\}$*) and*
$u_L(c_1) = u_L(c_k) = u_R(c_k) = u_R(c_K) = +1$ *for* $k = 2, \ldots, K-1$.

Proof "If parts". It is easy to verify that in both cases equations (1), (2) hold and, moreover, $u_L(c_k) = u_R(c_k)$ for $k = 2, \ldots, K-1$.

"Only if parts". Obviously, equations (1), (2) and $u_L(c_k) = u_R(c_k)$ for $k = 2, \ldots, K-1$ imply that $u_L(c_1) = u_R(c_2) = u_L(c_2) = u_R(c_3) = \ldots = u_L(c_{k-1}) = u_R(c_k) = \ldots = u_L(c_{K-1}) = u_R(c_K)$. Hence, by equations (1) and (2), all positions $j_1, \ldots, j_K - 1$ must belong to the same player $i \in I = \{1, 2\}$. Obviously, cases (i) and (ii) appear for $i = 1$ and $i = 2$, respectively. □

Remark 5 *In Section 3.1 we have already seen that game* (C_K, P, j_0, u) *is not solvable if pair* (C_K, P) *is a* 1*-cycle,* $V_1 = \{j_0\}$, $V_2 = V \setminus \{j_0\}$*, and* $u(c_k) = +1$ *for all* $k = 1, \ldots, K$*, while* $u(c_L) = u(c_R) = -1$ *(or* 0*). This is Case (ii) of Lemma 8. In this case, cycle* $G = C_K$ *can be passed both ways, clockwise and counter-clockwise. The Lemma is instrumental not only in the proof of Proposition 4 but of Theorem 2 as well; see Sections 5-7.*

3.3 All cycles are solvable, except 1-cycles

Let (C_K, P, j_0, u) be a cyclic game, where $j_0 \in V_1$, and u be a utility function. (In particular, now $u(c_L)$ and $u(c_R)$ can take only values ± 1, not 0.) The following case analysis will complete the proof of Proposition 4.

Clearly, player 1 wins if he can begin clockwise, with (j_0, j_1), (respectively, counter-clockwise, with (j_K, j_{K-1})) and force a winning short cycle or c_L, provided $u(c_L) = 1$ (respectively, or c_R, provided $u(c_R) = 1$).

We have to show that if player 1 cannot do this then player 2 wins, unless (G, P) is a 1-cycle. This is not obvious, since player 2 does not know whether player 1 begins clockwise or counter-clockwise.

Let us partition the set of strategies of player 1 in two subsets, $X_1 = X_1^L \cup X_1^R$, where the strategies of X_1^L (respectively, of X_1^R) choose move (j_0, j_1) (respectively, (j_K, j_{K-1})) in the initial position $j_0 = j_K$, and consider the corresponding partition of the normal form:

$$X = X_1 \times X_2 = (X_1^L \times X_2) \cup (X_1^R \times X_2) = X^L \cup X^R.$$

As we already mentioned in Section 3.1, each of two subgames, defined by X^L and X^R, is the normal form of a finite positional game with perfect information. Hence, player 2 has a winning strategy in each subgame, since player 1 has not. However, the corresponding two winning strategies x_2^ℓ and x_2^r may differ, $x_2^\ell \neq x_2^r$, and we have yet to prove that player 2 has a strategy $x_2 \in X_2$ winning in both subgames X^L and X^R simultaneously, or in other words, winning in the total game X, unless (G, P) is a 1-cycle.

Let us consider two special strategies x_2^L and x_2^R of player 2 that in all positions of V_2 choose to move clockwise and counter-clockwise, respectively. We prove Proposition 5 by the following case analysis.

Case 1: x_2^L and x_2^R are unique winning strategies in X^L and X^R, respectively.

In this case we have to consider two subcases: $V_1 = \{j_0\}$ and $|V_1| > 1$.

Case 1.1 : $V_1 = \{j_0\}, V_2 = V \setminus \{j_0\}$. In this case we obtain a 1-cycle, which is not solvable, according to Section 3.1. More precisely, there is a unique payoff, $u(c_k) = +1$ for $k = 1, \ldots, K$ and $u(c_L) = u(c_R) = -1$, such that in the obtained game no player has a winning strategy.

Case 1.2 : $|V_1| > 1$; in other words, there is an $m \in \{1, \ldots, K-1\}$ such that $j_m \in V_1$. In this case we can define a uniformly winning strategy x_2 by setting $x_2(j_k) = x_2^L(j_k)$ for $k < m$ and $x_2(j_k) = x_2^R(j_k)$ for $k > m$. Of course, $j_k \in V_2$ in both cases.

Case 2 : player 2 has winning strategies x_2^ℓ in X^L and x_2^r in X^R distinct from x_2^L and x_2^R, respectively. In other words, x_2^ℓ (respectively, x_2^r) chooses the counter-clockwise (respectively, clockwise) move in some position j_{k_L} (respectively, j_{k_R}). Let us assume that we have fixed the minimum such k_L and the maximum such k_R and consider the following three subcases.

Case 2.1 : $k_L < k_R$. Let us define strategy x_2 as follows: $x_2(j_k) = x_2^L(j_k)$ for $k \leq k_L$, $x_2(j_k) = x_2^R(j_k)$ for $k \geq k_R$, and $x_2(j_k)$ is arbitrary when $k_L < k < k_R$.

Case 2.2 : $k_L > k_R$. Let us define x_2 as follows: $x_2(j_k) = x_2^L(j_k)$ for $k \leq k_L$ and $x_2(j_k) = x_2^R(j_k)$ for $k \geq k_L$.

Case 2.3 : $k_L = k_R = m$. Then we have to consider two subsubcases: $V_2 = \{j_m\}$ and $|V_2| > 1$.

Case 2.3.1 : $V_2 = \{j_m\}$. Again we obtain a 1-cycle, which is not solvable. There is a unique payoff ($u(c_k) = -1$ for $k = 1, \ldots, K$ and $u(c_L) = u(c_R) = +1$) such that in the obtained game no player has a winning strategy.

Case 2.3.2 : $|V_2| > 1$. Then, except j_m, there is another position $j_n \in V_2$. Without loss of generality, we can assume that $n < m$. By definition of m, strategy x_2^L chooses the clockwise move (j_k, j_{k+1}) in every position j_k such that $k < m$, in particular, in j_n. Hence, we can define x_2 as follows: $x_2(j_k) = x_2^L(j_k)$ for $k \leq m$ and $x_2(j_k) = x_2^R(j_k)$ for $k > m$.

Case 3 : player 2 has a winning strategy x_2^r in X^R distinct from x_2^R, while x_2^L is the unique winning strategy in X^L.

Since $x_2^r \neq x_2^R$, there is an $m \in \{1, \ldots, K-1\}$ such that $j_m \in V_2$ and x_2^r (as well as x_2^L chooses the clockwise move (j_m, j_{m+1}) in j_m. Let us assume that we have fixed the maximum such m. In this case we can define a uniformly winning strategy x_2 by setting $x_2(j_k) = x_2^r(j_k)$ for $k \geq m$ and $x_2(j_k) = x_2^L(j_k)$ for $k \leq m$; of course, $j_k \in V_2$ in both cases.

Case $3'$: player 2 has a winning strategy x_2^ℓ in X^L distinct from x_2^L, while x_2^R is the unique winning strategy in X^R.

Due to obvious symmetry, cases 3 and $3'$ are equivalent.

Since, in each case we got either a uniformly optimal strategy x_2 or a 1-cycle, Proposition 4 follows. □

4 Passing through a simple path

4.1 Main lemma for simple paths

Here we derive for simple paths a result similar to Lemma 8 for simple cycles.

Let $G = P_K = (V, E)$ be a simple path of length K in which $V = \{j_0, j_1, \ldots, j_K\}$ (now $j_0 \neq j_K$) and $E = \{(j_{k-1}, j_k); k = 1, \ldots, K\}$.

The corresponding digraph $\overrightarrow{G} = (V, \overrightarrow{E})$ contains K short dicycles, $\mathcal{C}_K = \{c_k = ((j_{k-1}, j_k), (j_k, j_{k-1})); k \in [K]\}$, that are in one-to-one correspondence with E, and no other dicycles. Let us add to G one loop c_L at j_K (respectively, c_R at j_0) and denote the obtained graph G_L (respectively, G_R). Given a partition $P : V = V_1 \cup V_2$, let us consider two bidirected zero-sum cyclic games (G_L, P, j_0, u_L) and (G_R, P, j_K, u_R) whose utility functions are defined as follows: $u_L(c_L) = u_R(c_R) = 0$, while $u_L : \mathcal{C}_K \to \{-1, +1\}$ and $u_R : \mathcal{C}_K \to \{-1, +1\}$ are arbitrary functions defined on $\mathcal{C}_K = \{c_1, \ldots, c_K\}$.

Lemma 9 *(i) Equations*

$$u_L(c_k) = (-1)^i \text{ whenever } j_k \in V_i, \text{ where } i = 1, 2; \text{ and } k = 1, \ldots, K, \tag{3}$$

define a unique payoff u_L such that the obtained game (G_L, P, j_0, u_L) is a draw, that is, it results in c_L.

(ii) Respectively, equations

$$u_R(c_k) = (-1)^i \text{ whenever } j_{k-1} \in V_i, \text{ where } i = 1, 2; \text{ and } k = 0, \ldots, K-1, \tag{4}$$

define a unique payoff u_R such that the obtained game (G_R, P, j_K, u_R) is a draw, that is, it results in c_R.

(iii) Equations (3) and (4) hold simultaneously, or more precisely, there is a utility function $u = u_L = u_R$ satisfying (3) and (4) if and only if

(a) $V_1 = V, V_2 = \emptyset$ *and* $u(c_k) = -1$ *for all* $k = 1, \ldots, K$ *or*
(b) $V_2 = V, V_1 = \emptyset$ *and* $u(c_k) = +1$ *for all* $k = 1, \ldots, K$.
In both cases graph (G, P) *is a* 0*-path.*

Proof Let us consider first (i) and (ii). It is easy to see that (G_L, P, j_0, u_L) (respectively, (G_R, P, j_K, u_R)) is a positional game with perfect information whose tree is a caterpillar. Indeed, in each position j_k the corresponding player i (such that $j_k \in V_i$) has two options: either to return to j_{k-1} (respectively, to j_{k+1}) and by this finish the game in c_k (respectively, in c_{k+1}) or to proceed with j_{k+1} (respectively, with j_{k-1}). It is also easy to see that the first options is winning whenever (3) (respectively, (4)) does not hold. Thus the play results in c_L (respectively, in c_R), and the game is a draw, if and only if (3) (respectively, (4)) holds for all k.

(iii) "If part". It is easy to verify that in both cases (a) and (b) equations (3) and (4) hold and, moreover, $u_L(c_k) = u_R(c_k)$ for $k = 1, \ldots, K$.

"Only if part". Obviously, equations (1), (2) and $u_L(c_k) = u_R(c_k)$ for $k = 1, \ldots, K$ imply that $u_L(c_1) = u_R(c_2) = u_L(c_2) = u_R(c_3) = \ldots = u_L(c_{k-1}) = u_R(c_k) = \ldots = u_L(c_{K-1}) = u_R(c_K)$.

Hence, (3) and (4) imply that all positions $j_0, \ldots, j_K$ must belong to the same player $i \in I = \{1, 2\}$ and, moreover, that $u_L(c_k) = u_R(c_k) = u(c_k) = (-1)^i$ for all $k = 1, \ldots, K$. It is easy to see that cases (a) and (b) appear for $i = 1$ and $i = 2$, respectively. □

In particular, Lemma 9 (iii) shows that path P_K can be passed through both ways if and only if it is a 0-path. However, in the next section we show that 1-paths have a similar, just slightly weaker, property.

4.2 Special properties of 1-paths

Let us consider a 1-path (G, P) in which $V_{3-i} = \{k_0\}$, $V_i = V \setminus \{k_0\}$, where $k_0 \in \{1, \ldots, K-1$ and $i \in I = \{1, 2\}$. Furthermore, let $u(c_k) = (-1)^i$ for all $k \in [K] = \{1, \ldots, K\}$. Obviously, in both games (G, P, j_0, u) and (G, P, j_K, u) player i has a winning strategy: (j_{k_0}, j_{k_0-1}) and (j_{k_0}, j_{k_0+1}), respectively. However, if $(G, P) < (G', P')$ then in a larger game (G', P', j', u') player i does not know whether the play enters path (G, P) in j_0 or in j_K and, hence, he does not know how to play in j_{k_0}. One of the two available moves is winning, while the other one might be losing.

Similar situation can appear for $k_0 = 0$ or $k_0 = K$. For example, let $k_0 = 0$ and again $u(c_k) = (-1)^i$ for all $k \in [K]$. Obviously, move (j_0, j_1) is winning whenever the play enters path (G, P) in j_K and then comes to j_0. However, (j_0, j_1) can be a losing move if the play enters j_0 not from j_1;

Example 4.1 *Let us consider a game* (G, P, j, u) *in Figure 5*

whose graph G *consists of the middle path* p_K *and two more paths* p' *and* p''*. All three paths are between* j_0 *and* j_K *and have no other common vertices. Furthermore, let* p_K *be a*

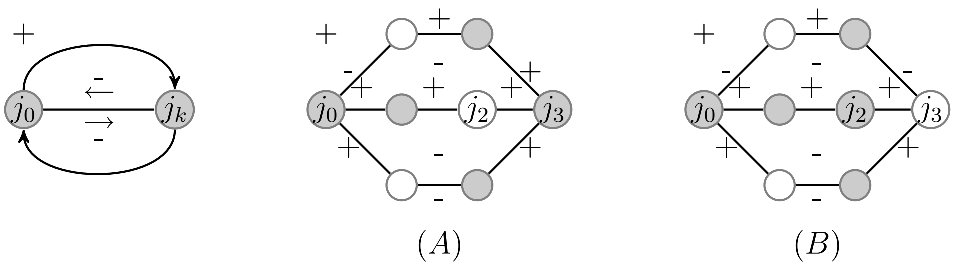


Figure 5:

1-*path, where player* 1 *controls only one position* j_{k_0}, $k_0 \in \{0, 1, \ldots, K\}$ *and player* 2 *controls all other positions. We will say that a* 1-*path* p_K *is of type* A *if* $0 < k_0 < K$ *and* p_K *is of type* B *if* $k_0 = 0$ *or* $k_0 = K$.

The payoff u *is chosen as follows. For the short dicycles of* p' *and* p'' *we define* u *in such a (unique) way that* p' *and* p'' *will be passed clockwise and player* 1 *wins in the obtained long dicycle* c, *that is,* $u(c) = +1$. *Furthermore, let* $u(c_k) = +1$ *for all short dicycles* c_k, $k \in [K]$, *of path* p_K. *Then, as we already mentioned,* p_K *could be passed both ways. Respectively, two more clockwise long dicycles* c' *and* c'' *(formed by* (p', p_K) *and* (p'', p_K)*) can appear. Let us set* $u(c') = u(c'') = -1$, *that is, player* 2 *wins in both cases.*

Let us set $K = 3$ *and consider the following two examples in Figure 5.*

(A): $k_0 = 2$. *The play can come to position* $j_{k_0} = j_2$ *in two ways: from* j_1 *or* j_3. *In both cases player* 1 *can win; it is enough to return to the same position, where the play came from, since* $u(c_2) = u(c_3) = +1$. *Yet, to guarantee the victory player* 1 *must know the strategy of the opponent, and, by our assumptions, he does not. Thus, player* 2 *should not surrender. Instead, she can try to enter* p_K *and approach* j_2 *either from* j_0 *or from* j_3. *Although, in both cases player* 1 *can win moving in* j_2 *left or right, respectively. Yet, if his guess is wrong then a long dicycle* c' *or* c'' *appears and player* 2 *wins, since* $u(c') = u(c'') = -1$.

(B): $k_0 = 3$. *Again, the play can come to position* $j_{k_0} = j_3$ *in two ways: from* j_2 *or by* p'. *Respectively, player* 1 *should return to* j_2 *or proceed with path* p''. *In both cases he wins, since* $u(c) = u(c_3) = +1$. *However, again he cannot guarantee the victory, because he is not aware of opponent's strategy. Hence, player* 2 *should not surrender. Instead, in position* j_0 *she can try either to enter (and then pass through) path* p_K *or proceed with* p'. *In each case player* 2 *can win. Respectively, in* j_3 *he should return to* j_2 *or proceed with* p''. *Yet, if his guess is wrong then a long cycle* c' *or* c'' *appears and player* 2 *wins, since* $u(c') = u(c'') = -1$.

In both cases (A) and (B) p_K *is a* 1-*path, of type* A *and* B, *respectively.*

It is not difficult to verify that both above games are not solvable. Moreover, they are uniformly non-solvable with respect to the considered payoffs.

Remark 6 *Every* 1-*path* p_K *(as well as any other path) is standardly one-way oriented by a payoff satisfying (3) or (4). A "problem" with the orientation appears when a* 1-*path is obtained from a* 0-*path as follows. Given a* 0-*path* p^0_K *controlled by a player* $i \in I = \{1, 2\}$

and constant payoff, $u(c_k) = (-1)^i$ *for all* $k \in \{1, \dots, K\}$*, let us fix a* $k_0 \in \{0, 1, \dots, K\}$ *and switch position* j_{k_0} *to player* $3 - i$*. Then the obtained* 1*-path* p_K *can be passed through both ways.*

However, nothing of that sort can happen when each player controls at least two positions of a simple path.

Let (G, P) be a pair in which $G = P_K = (V, E)$ is a simple path with vertices $V = \{j_0, \dots, j_K\}$ and $P : V = V_1 \cup V_2$ be a partition such that $|V_1| \geq 2$ and $|V_2| \geq 2$. Furthermore, let $u : C_k \to \{-1, +1\}$ be a utility function.

Lemma 10 *If a player, say, player* 1*, has winning strategies* x_1^L *and* x_1^R *in games* (G, P, j_0, u) *and* (G, P, j_K, u)*, respectively, then he has a strategy* x_1 *that is winning in both games simultaneously.*

Proof We proceed with a case analysis similar to one of Section 3.3.

Case 1 : Strategy x_1^L (respectively, x_1^R) in each position $j_k \in V_1$ chooses the right move (j_k, j_{k+1}) (respectively, left one (j_k, j_{k-1})).

In this case we simply set $x_1 = x_1^L$ (respectively, $x_1 = x_1^R$).

Case 2 : There are positions j_{k_L} and j_{k_R} in which strategies x_1^L and x_1^R prescribe the left and right move, respectively. Let us fix the minimum such k_L and the maximum such k_R and consider the following subcases.

Case 2.1 : $k_L < k_R$. Let us define x_1 as follows: $x_1(j_k) = x_1^L(j_k)$ for $k \leq k_L$, $x_1(j_k) = x_1^R(j_k)$ for $k \geq k_R$, and $x_1(j_k)$ is arbitrary when $(k_L < k < k_R)$.

Case 2.2 : $k_L > k_R$. Then we define x_1 as follows: $x_1(j_k) = x_1^L(j_k)$ for $k \leq k_L$ and $x_1(j_k) = x_1^R(j_k)$ for $k \geq k_R$.

Case 2.3 : $k_L = k_R = m$. We have to consider two subsubcases: $V_1 = \{j_m\}$ and $|V_1| > 1$.

Case 2.3.1 : $V_1 = \{j_m\}$. Again we obtain a 1-cycle, which is not solvable. There is a unique payoff, $u(c_k) = -1$ for $k = 1, \dots, K$ and $u(c_L) = u(c_R) = +1$, such that in the obtained game no player has a winning strategy.

Case 2.3.2 : $|V_1| > 1$. Then, except j_m, there is another position $j_n \in V_1$. Without loss of generality, we can assume that $n < m$. By definition of m, strategy x_1^L chooses the right move (j_k, j_{k+1}) in every position j_k such that $k < m$, in particular, in j_n. Hence, we can define x_1 as follows: $x_1(j_k) = x_1^L(j_k)$ for $k \leq m$ and $x_1(j_k) = x_1^R(j_k)$ for $k > m$.

Since, in each case we got either a uniformly optimal strategy x_1 or a 1-cycle, Lemma 10 follows. □

4.3 List of options for simple paths

Given a cyclic game (G, P, j, u) and a simple path p_K between two nodes j_0 and j_K of G (recall that $deg(j_0) \geq 3$ and $deg(j_K) \geq 3$), the following options can take place.

(a). Player i wins whenever i' enters p_K at j_0 (respectively, at j_K). All eight combinations j_0, j_K and $i, i' \in I = \{1, 2\}$ are possible. By Lemma 10, if the same player wins in each case, when p_K is entered from j_0 or from j_K, then this player has a uniformly winning strategy, which wins in both cases simultaneously, unless p_k is a 1-cycle and u is the corresponding constant.

(b). None of two players wins whenever a player $i \in I = \{1, 2\}$ enters p_K at j_0 (respectively, at j_K). In this case the play can pass through p_k from j_0 to j_K (respectively, from j_K to j_0) and exit p_K. By Lemma 9, this option takes place for both j_0 and j_K if and only if p_K is a 0-path and u is the corresponding constant.

5 Criteria of solvability of pair (G, P) based on orientations of its paths

5.1 Pairs $(G, P), (\mathcal{G}, \mathcal{P})$, and $(\overrightarrow{\mathcal{G}}, \mathcal{P})$

Given a pair (G, P), the corresponding pair $(\mathcal{G}, \mathcal{P})$ was introduced in Section 1.4.1. Now we introduce one more transformation. Let us substitute each 0-edge (j, j') in $\mathcal{G}$ by two oppositely oriented arcs $(j, j'), (j', j)$, then orient arbitrarily all other edges of $\mathcal{G}$, and denote the obtained digraph by $\overrightarrow{\mathcal{G}}$ and pair by $(\overrightarrow{\mathcal{G}}, \mathcal{P})$.

Similarly to (G, P), we will call pair $(\overrightarrow{\mathcal{G}}, \mathcal{P})$ *solvable* (respectively, *uniformly non-solvable*) if the corresponding zero-sum game $(\overrightarrow{\mathcal{G}}, \mathcal{P}, j, \sqcap)$ is solvable (respectively, not solvable) for each initial position j of digraph $\overrightarrow{\mathcal{G}}$ and for every (respectively, for some) payoff $\mathcal{U} : C(\overrightarrow{\mathcal{G}}) \to \{-1, +1\}$ satisfying the following two extra conditions:

First, we assume that $\mathcal{U}(c) = (-1)^i$ for each dicycle $c = ((j, j'), (j', j))$ corresponding to a 0-edge (j, j') in $\mathcal{G}$ controlled by player $i \in I = \{1, 2\}$. This restriction is natural, since otherwise player i would immediately win on c.

Then, let us notice that digraph $\overrightarrow{\mathcal{G}}$ may have dead-ends. We assume that a player wins whenever the opponent cannot move.

Thus, we have defined three successive transformations. Given a bidirected pair $(\overrightarrow{G}, P)$, first, we introduce (G, P), then $(\mathcal{G}, \mathcal{P})$, and finally, consider all orientations $(\overrightarrow{\mathcal{G}}, \mathcal{P})$ of the latter.

Proposition 14 *(i) If pair (G, P) (respectively, cyclic game form (G, P, j)) is solvable then for every orientation $\overrightarrow{\mathcal{G}}$ of $\mathcal{G}$ the obtained pair $(\overrightarrow{\mathcal{G}}, \mathcal{P})$ (respectively, cyclic game form $(\overrightarrow{\mathcal{G}}, \mathcal{P}, j)$) is also solvable.*

(ii) A pair (G, P) (respectively, cyclic game form (G, P, j)) is solvable whenever it contains no 1-paths and the corresponding pair $(\overrightarrow{\mathcal{G}}, \mathcal{P})$ (respectively, cyclic game form $(\overrightarrow{\mathcal{G}}, \mathcal{P}, j)$) is solvable for every orientation $\overrightarrow{\mathcal{G}}$ of $\mathcal{G}$.

(i') If pair (G, P) is uniformly non-solvable and it contains no 1-paths then there is an orientation $\overrightarrow{\mathcal{G}}$ of $\mathcal{G}$ such that the obtained pair $(\overrightarrow{\mathcal{G}}, \mathcal{P})$ is uniformly non-solvable too.

(ii') A pair (G, P) is uniformly non-solvable whenever there is an orientation $\overrightarrow{\mathcal{G}}$ of $\mathcal{G}$ such that the obtained pair $(\overrightarrow{\mathcal{G}}, \mathcal{P})$ is uniformly non-solvable.

Proof First, let us recall that:

(a) By Proposition 3, pair (G, P) is solvable if and only if each its 2-connected component is solvable. Hence, without loss of generality, we can assume that graph G is 2-connected; in particular, it contains no vertex of degree 1 and all vertices of graph $\mathcal{G}$ are of degree at least 3.

(b) By Proposition 13, pair (G, P) is either solvable or uniformly non-solvable, whenever graph G is connected.

(c) By definition, pairs (G, P) and $(\overrightarrow{\mathcal{G}}, \mathcal{P})$ are solvable (respectively, uniformly non-solvable) if and only if there are utility functions u and $\mathcal{U}$ such that for each initial positions j the obtained games (G, P, j, u) and $(\overrightarrow{\mathcal{G}}, \mathcal{P}, j, \mathcal{U})$ are solvable (respectively, uniformly non-solvable).

It follows from (b) that parts (i) and (ii'), as well as (ii) and (i') are equivalent.

Moreover, it is sufficient to prove parts (i) and (ii) only for cyclic game forms; then, the corresponding statements for pairs will follow, by (c).

Part (i). Given a pair (G, P) and initial position j in graph G of degree at least 3, assume indirectly that there is an orientation $\overrightarrow{\mathcal{G}}$ of $\mathcal{G}$ and a payoff $\mathcal{U} : C(\overrightarrow{\mathcal{G}}) \to \{-1, +1\}$ such that the obtained game $(\overrightarrow{\mathcal{G}}, \mathcal{P}, j, \mathcal{U})$ is not solvable. We have to construct a payoff $u : C(\overrightarrow{G}) \to \{-1, +1\}$ such that game (G, P, j, u) is not solvable, either.

Since each dicycle $c \in C(\overrightarrow{\mathcal{G}})$ is naturally assigned to a long dicycle of the original digraph $\overrightarrow{G}$, let us set $u(c) = \mathcal{U}(c)$ for every such dicycle c.

As for the short dicycles of $\overrightarrow{G}$, the corresponding payoffs are determined by the orientations of the simple paths of graph G induced by digraph $\overrightarrow{\mathcal{G}}$.

Let $P_K = (V_K, E_K)$ be a simple path in G between nodes $j_0, j_K \in V(G)$. Standardly, we assume that $deg_G(j_0) \geq 3$, $deg_G(j_K) \geq 3$ (that is, $j_0, j_K \in V(\mathcal{G})$), and $V_K = \{j_k;\ k = 0, 1, \ldots, K\}$, $E_K = \{e_k = (j_k - 1, j_k);\ k = 1, \ldots, K\}$. As before, to each edge $e_k \in E_K$ we assign a short dicycle $c_k = ((j_{k-1}, j_k), (j_k, j_{k-1})$ in digraph $\overrightarrow{G}$ and denote by $\mathcal{C}_K$ the set of these k cycles, $\mathcal{C}_K = \{c_k;\ k = 1, \ldots, K\}$.

If p_K is a 0-path of (G, P), that is, $V_i = V_K$, $V_{3-i} = \emptyset$ for some $i \in I = \{1, 2\}$, then we set $u(c_k) = (-1)^i$ for all $k = 1, \ldots, K\}$. In this case player i can pass through p_K in both ways, yet, no short cycle $c_k \in \mathcal{C}_K$ is winning for i.

Furthermore, given a path p_K between nodes j_0 to j_K in G. Without loss of generality, we can assume that the corresponding edge (j_0, j_K) of graph $\mathcal{G}$ is oriented in digraph $\overrightarrow{\mathcal{G}}$ from j_0 to j_K. By Lemma 9, for a path P_K oriented from j_0 to j_K, there is a (unique) utility function $u_L : \mathcal{C}_K \to \{-1, +1\}$ such that no player wins. (In other words, the players will pass through p_K from j_0 to j_K.) Let us set $u(c_k) = u_L(c_k)$ for all $k = 1, \ldots, K\}$.

For any remaining dicycle $c \in C(\overrightarrow{G})$ let us assign $u(c) = -1$ or $u(c) = +1$, arbitrarily. By the above definition of u, games $(\overrightarrow{\mathcal{G}}, \mathcal{P}, j, \mathcal{U})$ and (G, P, j, u) are equivalent for every initial position j in $\mathcal{G}$. □

Part (ii). Assume indirectly that cyclic game form (G, P, j) is not solvable. Then, by Proposition 13, pair (G, P) is uniformly non-solvable (whenever G is connected), i.e., there is a payoff $u : C(\overrightarrow{G}) \to \{-1, +1\}$ such that for every initial position j in graph G the obtained game (G, P, j, u) is not solvable. We have to prove that then pair $(\mathcal{G}, \mathcal{P})$ contains a 1-edge or there is an orientation $\overrightarrow{\mathcal{G}}$ of graph $\mathcal{G}$ such that pair $(\overrightarrow{\mathcal{G}}, \mathcal{P})$ is not solvable. In fact, we will prove that it is uniformly non-solvable, i.e., there is a payoff $\mathcal{U} : C(\overrightarrow{\mathcal{G}}) \to \{-1, +1\}$ such that for every initial position $j \in V(\mathcal{G})$ the obtained game $(\overrightarrow{\mathcal{G}}, \mathcal{P}, j, \mathcal{U})$ is not solvable. Given (G, P) and u, we define $\overrightarrow{\mathcal{G}}$ and $\sqcap$ as follows.

Let us recall all options listed in Section 4.3. Given a path p_K in graph G, we will keep our standard notation and also say that a player $i \in I = \{1, 2\}$ wins (respectively, loses) *within* p_K if player i (respectively, the opponent $3-i$) can guarantee that the play will result in a dicycle $c_k \in \mathcal{C}_k$ such that $u(c_k) = (-1)^{i+1}$. (Recall, that player 1 is the maximizer, while 2 is the minimizer.)

Case A. If at j_0 or j_K a player i can enter p_K and win within it, then the corresponding game, (G, P, j_0, u) or (G, P, j_K, u), is solvable and we get a contradiction.

This case takes place whenever p_K is a 0-path controlled by a player $i \in I = \{1, 2\}$, unless $u(c_k) = (-1)^{i+1}$ for all $k \in [K] = \{1, \ldots, K\}$. In the latter case we set $\mathcal{U}(c) = (-1)^{i+1}$ for the short dicycle $c = ((j_0, j_K), (j_K, j_0))$ of digraph $\overrightarrow{\mathcal{G}}$. Let us recall that, by definitions of graph $\mathcal{G}$ and digraph $\overrightarrow{\mathcal{G}}$, we substitute a 0-path between j_0 and j_K in G by two parallel edges (j_0, j_K) and (j_K, j_0) in $\mathcal{G}$ and then orient them oppositely in $\overrightarrow{\mathcal{G}}$.

Case B is considered in the following statement.

Lemma 11 *If each player loses within p_K whenever he enters it, from j_0 or j_K, then for each interior position j_k in p_K (i.e., for $k \in \{1, \ldots, K-1\}$) the obtained game (G, P, j_k, u) is solvable, unless P_K is a 1-path.*

Proof At first, let us notice that the assumption of the lemma cannot hold for 0-paths. Indeed, let p_K be a 0-path controlled by a player $i \in I = \{1, 2\}$. Then i will not lose within p_K, since in any case i can pass through p_K (in both ways) and leave it. (Let us recall that $deg_G(j_0) \geq 3$ and $deg_G(j_K) \geq 3$.

At second, let us notice that the conclusion of the lemma may not hold for 1-paths; see Example 4.1. However, by Lemma 10, there is no other exception.

Finally , it is obvious that $K \geq 3$ whenever p_K is not a 1-path. Hence, game (G, P, j_k, u) is solvable when $k = 1$ or $k = 2$. □

Case C. Neither of two players can win within p_K, whenever a player i enters p_K at j_0 (respectively, at j_K). By Lemma 9 (i,ii), this case takes place if and only if equation (3) (respectively, (4) holds for u. Then we orient the corresponding edge (j_0, j_K) of graph $\mathcal{G}$ from j_0 to j_K (respectively, from j_K to j_0).

Furthermore, by Lemma 9 (iii), equations (3) and (4) hold simultaneously if and only if p_K is a 0-path controlled by a player $i \in I = \{1, 2\}$ and $u(c_k) = (-1)^{i+1}$ for all $k \in [K] = \{1, \ldots, K\}$. This option was already considered; see Case A above.

Since, by our assumptions, pair (G, P) is uniformly non-solvable and there is no 1-paths in G, all edges of $\mathcal{G}$ will be oriented: every 0-edge in both directions and every other edge in one direction, according to the above rule.

Thus, an orientation $\overrightarrow{\mathcal{G}}$ of graph $\mathcal{G}$ is defined.

It is not difficult to see that a player i that makes a move opposite to this orientation in a simple path p_K in (G, P) will lose within p_K immediately after this move, since a short cycle c_k will appear such that $u(c_k) = (-1)^i$.

It is also clear that each dicycle $c \in \mathcal{C}(\overrightarrow{\mathcal{G}})$ is naturally assigned either to a 0-path in G or to a long dicycle c' of the original digraph $\overrightarrow{G}$. In the former case the value $\mathcal{U}(c)$ was already defined and in the latter case let us set $\mathcal{U}(c) = u(c')$. For any remaining dicycle c in $\overrightarrow{\mathcal{G}}$ let us assign $u(c) = -1$ or $u(c) = +1$, arbitrarily.

Finally, it is not difficult to verify that the obtained game $(\overrightarrow{\mathcal{G}}, \mathcal{P}, j, \mathcal{U})$ is equivalent to the original game (G, P, j, u) and hence, they are both not solvable for every initial position $j \in V(\mathcal{G})$. □

5.2 Uniform non-solvability is a monotone property

Let us recall that, by Propositions 2, 13, and Theorem 4, if pair (G, P) is solvable (respectively, uniformly non-solvable) and $(G', P') \leq (G, P)$ (respectively, $(G', P') \geq (G, P)$), then (G', P') is solvable (respectively, uniformly non-solvable), too. In other words, solvability (respectively, uniform non-solvability) is a monotone decreasing (respectively, increasing) property. Proposition 14 implies a similar claim.

Proposition 15 *If $(\mathcal{G}, \mathcal{P}) \leq (\mathcal{G}', \mathcal{P}')$ and there is an orientation $\overrightarrow{\mathcal{G}}$ of $\mathcal{G}$ and payoff $\mathcal{U} : C(\overrightarrow{\mathcal{G}}) \to \{-1, +1\}$ such that pair $(\overrightarrow{\mathcal{G}}, \mathcal{P})$ is uniformly non-solvable with respect to $\sqcap$, then there is an extension $\overrightarrow{\mathcal{G}}'$ of $\overrightarrow{\mathcal{G}}$ and extension $\mathcal{U}' : C(\overrightarrow{\mathcal{G}'}) \to \{-1, +1\}$ of $\mathcal{U}$ such that pair $(\overrightarrow{\mathcal{G}}', \mathcal{P}')$ is uniformly non-solvable with respect to $\mathcal{U}'$.*

Proof This claim follows immediately from Proposition 14. Also it is similar to Theorem 4 and can be proved by the same arguments. □

In its turn, Proposition 15 implies the following criterion of uniform non-solvability. Let us consider a (uniformly non-solvable) pair $(\overrightarrow{\mathcal{G}}, \mathcal{P})$ represented in Figure 6.

It consists of four vertices j_1, j_1', j_2, j_2' and six simple directed paths: p_1 from j_1' to j_2, p_2 from j_2' to j_1, p_1' and p_1'' from j_1 to j_1', p_2' and p_2'' from j_2 to j_2'. We assume that, except

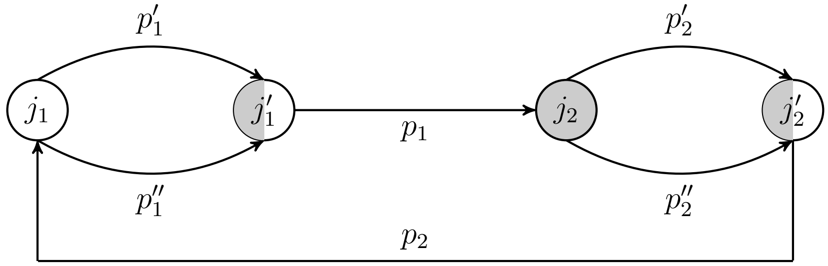


Figure 6: Uniformly non-solvable pair

j_1, j_1', j_2, j_2', these paths have no vertices in common. We also assume that positions j_1 and j_2 are controlled by players 1 and 2, respectively (as in the Figure), or vice versa. As for j_1' and j_2', they can be controlled by any players. Let us remark that, in fact, $(\overrightarrow{\mathcal{G}}, \mathcal{P})$ is not a pair but a family of pairs. Let us also remark that path p_1 (respectively, p_2) can be reduced to a single vertex; then $j_1' = j_2$ (respectively, $j_2' = j_1$). In particular, the first digraph in Figure 2 belongs to the considered family.

It is easy to see that pair $(\overrightarrow{\mathcal{G}}, \mathcal{P})$ is uniformly non-solvable. Indeed, digraph $\overrightarrow{\mathcal{G}}$ contains four directed cycles and each player has two strategies. Hence, the corresponding 2×2 normal game form is not tight, since it contains four distinct outcomes; see Figure 2.1. This simple claim can be strengthened as follows.

Proposition 16 *Let $(\mathcal{G}, \mathcal{P}) \leq (\mathcal{G}', \mathcal{P}')$ and there is an orientation $\overrightarrow{\mathcal{G}}$ of digraph $\mathcal{G}$ such that the obtained pair $(\overrightarrow{\mathcal{G}}, \mathcal{P})$ belongs to the family presented in Figure 6. Then both pairs $(\mathcal{G}, \mathcal{P})$ and $(\mathcal{G}', \mathcal{P}')$ are uniformly non-solvable. Moreover, the corresponding orientation $\overrightarrow{\mathcal{G}}'$ is an extension of $\overrightarrow{\mathcal{G}}$ and payoff $\mathcal{U}' : C(\overrightarrow{\mathcal{G}}') \to \{-1, +1\}$ is an extension of $\mathcal{U} : C(\overrightarrow{\mathcal{G}}) \to \{-1, +1\}$.*

Proof We already demonstrated that pair $(\mathcal{G}, \mathcal{P})$ is uniformly non-solvable. For pair $(\mathcal{G}', \mathcal{P}')$ it follows immediately from Proposition 15. $\square$

This necessary conditions of solvability will be frequently used in Section 6.

5.3 Treating 1-paths

Let us notice that the absence of 1-edges in $\mathcal{G}$ is an essential condition of parts (ii) and (i'); see Example 4.1. Another example is given by Proposition 7, where 1-edges of type 1 (respectively, of type 2) are (respectively, are not) in conflict with solvability of pairs $(\theta_k, \mathcal{P})$. Thus, 1-paths must be treated separately. The following condition is sufficient for solvability.

Given a pair $(\mathcal{G}, \mathcal{P})$ and a 1-edge $e = (j_0, j_K)$ in it, we shall say that e is of type A if $j_0, j_K \in V_i$ and e is of type B if $j_0 \in V_i, j_K \in V_{3-i}$, where $i \in I = \{1, 2\}$. Let us consider the simple 1-path $p = p(j_0, j_K)$ in graph G corresponding to edge e. In other words we can say that e and p are of type A (respectively, of type B) if one of two players controls all position of path p, except only one interior position; see Example 4.1 (A) (respectively, except only one terminal position, j_0 or j_K; see Example 4.1 (B)).

Given a pair $(\mathcal{G}, \mathcal{P})$ that has no 0-edges and 1-edges of type A, let us substitute by two oppositely oriented arcs not only every 0-edge of $\mathcal{G}$ but also every 1-edge of type B. Then let us orient arbitrarily all remaining edges of $\mathcal{G}$ and denote the obtained digraph by $\overrightarrow{\mathcal{G}}_1^B$ and pair by $(\overrightarrow{\mathcal{G}}_1^B, \mathcal{P})$.

Proposition 17 *A pair (G, P) is solvable whenever pair $(\overrightarrow{\mathcal{G}}_1^B, \mathcal{P})$ is solvable for each defined above orientation $\overrightarrow{\mathcal{G}}_1^B$ of graph $\mathcal{G}$.*

Proof We use the same arguments as in the proof of Proposition 15 (ii). Assume indirectly that pair (G, P) is not solvable. Then it is uniformly non-solvable, i.e., there is a payoff $u : C(\overrightarrow{G}) \to \{-1, +1\}$ such that for each $j \in V(G)$ the obtained game (G, P, j, u) is not solvable. Moreover, we can assume that (G, P) is a minimal not solvable pair, that is, (G', P') is solvable whenever $(G', P') < (G, P)$. We have to prove that then there is an orientation $\overrightarrow{\mathcal{G}}_1^B$ of graph $\mathcal{G}$ and payoff $\mathcal{U} : C(\overrightarrow{\mathcal{G}}_1^B) \to \{-1, +1\}$ such that for each $j \in V(\mathcal{G})$ the obtained game $(\overrightarrow{\mathcal{G}}_1^B, \mathcal{P}, j, \mathcal{U})$ is not solvable. We define this orientation and payoff as in Proposition 15 (ii); only for 1-paths we need a modification.

Let $p = p(j_0, j_K)$ be a 1-path of type B in (G, P); say, $j_0 \in V_1$ and $j_1, \ldots, j_K \in V_2$. Then we can assume that (a) $u(c_k) = +1$ for all $k = 1, \ldots, K$ or (b) $u(c_1) = +1$ and $u(c_k) = -1$ for $k = 2, \ldots, K$, since otherwise game (G, P, j_k, u) would be solvable for some k. In case (b) equation 3 holds and hence, players can pass through p from j_0 to j_K only. In case (a) players can pass through p both ways, since none of them knows opponent's strategy.

Respectively, in $\overrightarrow{\mathcal{G}}_1^B$ we substitute two oppositely oriented arcs (j_0, j_K) and (j_K, j_0) for 1-path $p(j_0, j_K)$. These two arcs form a short dicycle $c(j_0, j_K)$. Obviously, player 1 must win on this cycle, that is, we set $\mathcal{U}'(c(j_0, j_K)) = +1$.

For all other dicycles of $\overrightarrow{\mathcal{G}}_1^B$ define $\sqcap'$ as in the proof of proposition 15 (ii). It is not difficult to see that for each $j \in V(\mathcal{G})$ the obtained game $(\overrightarrow{\mathcal{G}}_1^B, \mathcal{P}, j, \mathcal{U})$ is equivalent to the original game $(\overrightarrow{G}, P, j, u)$. Since the second one is not solvable, the first one is not solvable either. $\square$

Remark 7 *By Theorem 2, the inverse claim holds, too. Moreover, both claims hold even if pair (G, P) can contain 0-paths. However, we need Proposition 17 to prove Theorem 2 and for this the present version is sufficient.*

Yet, let us notice that for 1-edges of type A the similar claim fails. Indeed, Example 4.1 (A) shows that pair (G, P) can be not solvable when the corresponding pair $(\mathcal{G}, \mathcal{P})$ is monochromatic but contains a 1-edge of type A. Since $(\mathcal{G}, \mathcal{P})$ is monochromatic, pair $(\overrightarrow{\mathcal{G}}_1^A, \mathcal{P})$ is monochromatic too. Hence, it is solvable, because one of two players is a dummy.

6 List $\mathcal{L}$ of solvable pairs; proof of Propositions 6 - 9

The analysis goes the same line for every pair $(\mathcal{G}, \mathcal{P})$ considered below. First, we assume that there are no 0- and 1-edges and verify that every orientation $(\overrightarrow{\mathcal{G}}, \mathcal{P})$ of pair $(\mathcal{G}, \mathcal{P})$ is solvable. By Propositions 14 and 15, it is sufficient to check that it cannot be uniformly non-solvable. In particular, we can ignore an orientation if it has a dead-end or one of two players is a dummy.

Then we assume that there is a 0- or 1-edge and produce a uniformly non-solvable orientation $(\overrightarrow{\mathcal{G}}, \mathcal{P})$ of pair $(\mathcal{G}, \mathcal{P})$. To prove non-solvability of $(\overrightarrow{\mathcal{G}}, \mathcal{P})$ we find out a uniformly non-solvable subpair $(\overrightarrow{\mathcal{G}}', \mathcal{P}') < (\overrightarrow{\mathcal{G}}, \mathcal{P})$. In many cases we use three such standard subpairs: one from Proposition 17, see Figure 6, and two from Example 4.1 (A) and (B), see Figures 6 (a) and (b).

6.1 Pairs $(\theta_K, \mathcal{P}_K)$; proof of Proposition 6

Let us recall bipartite pairs $(\theta_K, \mathcal{P}_K)$ for $K = 1, 2, \ldots,$ in Figure 3 and 7.

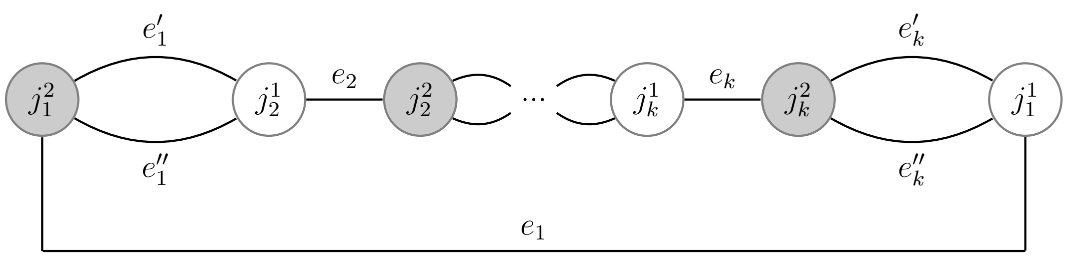


Figure 7: Pair$(\theta_K, \mathcal{P}_K)$

Graph $\theta_K = (V_K, E_K)$ contains $2K$ vertices (positions) and $3K$ edges (moves).

Positions $V_K = \{j_k^1, j_k^2 \mid k \in [K] = \{1, \ldots K\}\}$ are partitioned in two subsets, $\mathcal{P}_K : V_K = V_1^K \cup V_2^K$, where $V_i^K = \{j_k^i \mid k \in [K]\}$ are K positions of player $i \in I = \{1, 2\}$. In Figure 3 these positions are colored in white and black, respectively.

Furthermore, set $E_K = \{e_k, e'_k, e''_k;\ k \in [K]\}$ consists of $3K$ edges: K of type 1, $e_k = (j_k^1, j_k^2)$, and $2K$ of type 2, $e'_k = (j_k^2, j_{k+1}^1)'$, $e''_k = (j_k^2, j_{k+1}^2)''$, where $k \in [K]$ and $K + 1 = 1$, by convention. Let us notice that case $K = 1$ is "slightly degenerated", since θ_1 consists of three parallel edges.

Let us also recall that θ_K is a proper subgraph of θ_{K+1}; moreover, it is easy to see that $(\theta_{K+1}, \mathcal{P}_{K+1}) > (\theta_K, \mathcal{P}_K)$ for all K and, hence, pairs $(\theta_K, \mathcal{P}_K)$ form an infinite chain of solvable 2-connected pairs that has no maximal element.

Finally, since pairs $(\theta_K, \mathcal{P}_K)$ are bipartite, they cannot contain 0-edges, yet, can contain 1-edges.

By Proposition 6, pair $(\theta_K, \mathcal{P}_K)$ is solvable, unless it contains a 1-edge of type 2. We will prove it by induction on K. Let us begin with $K = 1$. Clearly, if pair $(\theta_1, \mathcal{P}_1)$ contains no 1-edges then it is solvable, since for each orientation of graph θ_1 one of two players is a dummy.

Yet, pair $(\theta_1, \mathcal{P}_1)$ is not solvable if it contains a 1-edge. Moreover, for each $K \geq 1$ pair $(\theta_K, \mathcal{P}_K)$ is uniformly non-solvable whenever it contains a 1-edge. To see this we can just refer to Example 4.1 (B) and Proposition 15.

Now let us fix an arbitrary integral $K \geq 1$ and consider pair $(\theta_K, \mathcal{P}_K)$. First, we will suppose that it contains no 1-edges. Let us assume indirectly that pair $(\theta_K, \mathcal{P}_K)$ is uniformly non-solvable, i.e., there is an orientation $\overrightarrow{\theta}_K$ of θ_K and payoff $\mathcal{U} : C(\overrightarrow{\theta}_K) \to \{-1, +1\}$ such that for each initial position $j \in V_K$ the obtained game $(\theta_K, \mathcal{P}_K, j, \mathcal{U})$ is not solvable. The following case analysis results in contradiction.

Case A: there is a $k \in [K]$ such that edges e_k and e_{k+1} are oriented oppositely, one towards the other, that is, $\overrightarrow{e_k} = (j_k^1, j_k^2)$, while $\overrightarrow{e}_{k+1} = (j_{k+1}^2, j_{k+1}^1)$. Clearly, in this case game $(\theta_K, \mathcal{P}_K, j, \mathcal{U})$ is solvable when $j = j_k^2$ or $j = j_{k+1}^1$ and we get a contradiction.

Case B: there is a $k \in [K]\}$ such that edges e_k and e_{k+1} are oriented oppositely, one from the other, that is, $\overrightarrow{e}_k = (j_k^2, j_k^1)$, while $\overrightarrow{e}_{k+1} = (j_{k+1}^1, j_{k+1}^2)$. It is easy to see that this case can be reduced to Case A. Indeed, if conditions of Case B hold for some $k \in [K]$ then conditions of Case A hold for some other $k' \in [K]$.

Thus, we can conclude that all edges of type 1 are oriented in the same way. Let us assume, without loss of generality, that they all are oriented clockwise, that is, $\overrightarrow{e}_k = (j_k^1, j_k^2)$ for all $k \in [K]$. Now let us consider the edges of type 2.

Case C: there is a $k \in K$ such that both edges e'_k and e''_k are oriented counter-clockwise, that is, from j_{k+1}^1 to j_k^2. In this case, game $(\theta_K, \mathcal{P}_K, j_k^2, \mathcal{U})$ is obviously solvable, since position j_k^2 is the dead-end.

Case D: there is a $k \in K$ such that edges e'_k and e''_k are oppositely oriented, say, $\overrightarrow{e}'_k = (j_k^2, j_{k+1}^1)$, while $\overrightarrow{e}''_k = (j_{k+1}^1, j_k^2)$. Then these two arcs form a short dicycle c. Let us also note that $\overrightarrow{e}'_k$ is the forced move in position j_k^2 and consider two subcases.

Subcase D1: $\mathcal{U}'(c) = +1$. Then, obviously, player 1 wins in game $(\theta_K, \mathcal{P}_K, j, \mathcal{U})$ when $j = j_k^2$ or $j = j_{k+1}^1$.

Subcase D2: $\mathcal{U}'(c) = -1$. In this case $\overrightarrow{e}''_k = (j_{k+1}^1, j_k^2)$ is definitely a losing move for player 1. Hence, we can delete this arc from digraph $\overrightarrow{\theta_K}$. Then, obviously, the obtained pair is equivalent to $(\overrightarrow{\theta}_{K-1}, \mathcal{P}_{K-1})$ and this pair is solvable by the induction hypothesis.

Thus, we can conclude that all edges are oriented clockwise. Yet, in this case game $(\theta_K, \mathcal{P}_K, j, \mathcal{U})$ is obviously solvable for all j, since player 1 is a dummy. □

Now let us consider pairs $(\theta_K, \mathcal{P}_K)$ that have 1-edges of type 1 but none of type 2. Let us notice that there are no such pairs when $K = 1$, since in this (degenerate) case each of three edges of θ_1 is of type 1 and 2 simultaneously. From Proposition 17 we will derive that all these pairs are solvable when $K \geq 2$.

Let us substitute each 1-edge of type 1 by two oppositely directed arcs and then orient arbitrarily all remaining edges of type 1 and all edges of type 2. Each pair $e'_k, e''_k,\ k \in [K]$, of parallel edges of type 2 can be oriented either oppositely or one way. Thus, the whole long cycle is partitioned in oppositely oriented parallel arcs (1-edges of type 1 and oppositely oriented pairs of type 2) and one way oriented arcs, single (regular edges of type 1) and parallel (one way oriented pairs of edges of type 2).

Case A : there are no one way oriented arcs at all. Then we obtain a cycle of oppositely oriented arcs. This case was considered in Section 3.

Hence, we can assume that there are one way oriented arcs. Let us show that all these arcs must be oriented in the same way, say, clockwise.

Case B : there are two of these arcs oriented oppositely, say, $e^1 = (j^1, j^2)$ and $e^2 = (j^3, j^4)$. Then it is easy to see that the considered game is solvable when the initial position is between j^2 and j^4.

Now, let us consider oppositely oriented parallel pairs of arcs. Several such successive pairs form an interval. We can repeat the arguments of case D and substitute every such maximal interval by one directed arc. Moreover, all these arcs must be oriented clockwise too, since otherwise the obtained game would be solvable for some initial positions. Thus, we come to the following assumptions.

Case C : all arcs are oriented clockwise; there are parallel arcs, which correspond to 1-edges of type 1 and intervals of clockwise oriented single arcs between them. In this case the game is solvable too, since player 1 is a dummy.

This completes the proof of Proposition 6. □

6.2 Pairs $(\mathcal{K}_4, \mathcal{P}')$ and $(\mathcal{K}_4, \mathcal{P}'')$; proof of Proposition 7

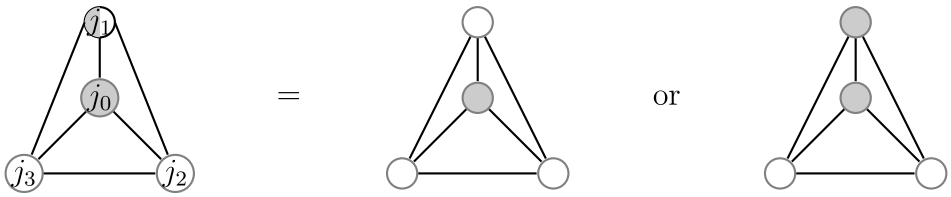
We will prove that each of these two pairs is solvable, unless it contains a 0- or 1-edge. For brevity we can represent both pairs by one $(\mathcal{K}_4, \mathcal{P})$ in which j_1 is an uncertain position; see Figure 8.

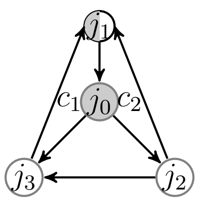
First, let us suppose that there are no 0- and 1-edges and show that all orientations $(\overrightarrow{\mathcal{K}}_4, \mathcal{P})$ of $(\mathcal{K}_4, \mathcal{P})$ are solvable.

Let us assume indirectly that there is a not solvable orientation $(\overrightarrow{\mathcal{K}}_4, \mathcal{P})$. Then it is uniformly non-solvable. In particular, there are no-dead-ends and no player is a dummy. The following simple case analysis shows that there are only three such orientations; see Figure 8.

First let us show that out-degree of each vertex j_k is 1 or 2. Indeed, if it is 0 then j_k is a dead-end and if it is 3 then there are two options: (a) one of the remaining three vertices is a dead-end, or (b) the orientation forms a simple cycle on these three vertices. Yet, in case (b) the player that controls j_k has three strategies and his opponent is a dummy.

Since in $\mathcal{K}_4$ each vertex is of degree 3, simple counting arguments show that there are two vertices, say, j_0 and j_2, of out-degree 2 and in-degree 1 and the remaining two, j_1 and j_3, of out-degree 1 and in-degree 2. It is clear that j_0 and j_2 cannot belong to the same player, since then the opponent is a dummy.





	↙	↘	
↖	c_1	c_2	(j_2, j_1)
←	c_1	c_3	(j_2, j_3)
	(j_0, j_3)	(j_0, j_2)	

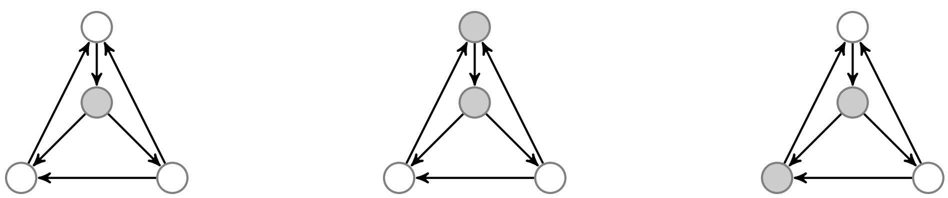


Figure 8: $F_1 = c_1c_2 \vee c_1c_3, \quad F_2 = c_1 \vee c_2c_3, \quad F_1 = F_2^d$.
$c_1 = ((j_0, j_3), (j_3, j_1), (j_1, j_0))$,
$c_2 = ((j_0, j_2), (j_2, j_1), (j_1, j_0))$,
$c_3 = ((j_0, j_2), (j_2, j_3), (j_3, j_1), (j_1, j_0))$.

Thus we obtain the pair $(\overrightarrow{K}_4, \mathcal{P})$ in which each player has two strategies and the corresponding normal game form is tight; see Figure 8.

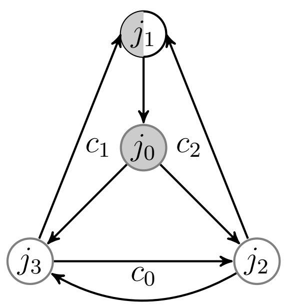
Let us remark that we can "recolor" vertices j_1 and j_3 and get $j_1 \in V_1$, while j_3 becomes uncertain. However, this transformation does not change the normal game form, since both vertices j_1 and j_3 are of out-degree 1, i.e., there is only one (forced) move in each of these two positions.

Assigning a player to each uncertain position we obtain three slightly different pairs; yet, all three have the same normal game form; see Figure 8.

Now, let us assume that pair $(\mathcal{K}_4, \mathcal{P})$ contains a 1-edge e. It may be of type A or B.

Without loss of generality, assume that (a) (j_0, j_2) is a 1-edge of type A or (b) (j_2, j_3) is a 1-edge of type B. In each case it is easy to show that the considered pair is uniformly non-solvable. Indeed, in case (a) (respectively (b)) it is enough to delete edge (j_0, j_1) (respectively, (j_1, j_3)) and recall that the obtained subpair was already considered in Example 4.1 (A) (respectively, (B)).

Finally, let us assume that pair $(\mathcal{K}_4, \mathcal{P})$ contains a 0-edge e. Without loss of generality, we can assume that $e = (j_2, j_3)$. Let us substitute edge e by two oppositely oriented arcs and orient all other edges as in Figure 9.



↙	c_0	c_3	c_1	c_1	(j_0, j_3)
↘	c_0	c_2	c_4	c_2	(j_0, j_2)
	(j_2, j_3) (j_3, j_2)	(j_2, j_1) (j_3, j_2)	(j_2, j_3) (j_3, j_1)	(j_2, j_1) (j_3, j_1)	

Figure 9: $F_1 = c_0c_1c_3 \vee c_0c_2c_4$, $\quad F_2 = c_0 \vee c_1c_2 \vee c_1c_4 \vee c_2c_3$, $\quad F_1 \neq F_2^d$.
$c_0 = ((j_2, j_3), (j_3, j_2))$,
$c_1 = ((j_0, j_3), (j_3, j_1), (j_1, j_0))$,
$c_2 = ((j_0, j_2), (j_2, j_1), (j_1, j_0))$,
$c_3 = ((j_0, j_3), (j_3, j_2), (j_2, j_1), (j_1, j_0))$,
$c_4 = ((j_0, j_2), (j_2, j_3), (j_3, j_1), (j_1, j_0))$.

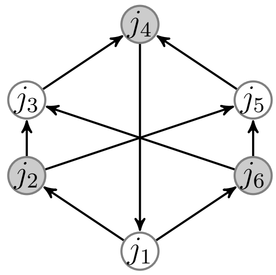
It is easy to see that the corresponding normal game form does not depend on the initial position and it is not tight. This completes the proof of Proposition 7. □

6.3 Pair $(\mathcal{K}_{3,3}, \mathcal{P})$; proof of Proposition 8

We will prove that this pair is solvable unless it contains a 1-edge. (Clearly, it cannot contain 0-edges, since it is bipartite.) First, let us suppose that there is no 1-edge and show that all orientations $(\overrightarrow{K}_{3,3}, \mathcal{P})$ of $(\mathcal{K}_{3,3}, \mathcal{P})$ are solvable.

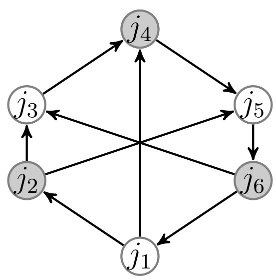
Let us assume indirectly that there is a not solvable orientation $(\overrightarrow{\mathcal{K}}_{3,3}, \mathcal{P})$. Then it is uniformly non-solvable. In particular, there is no-dead-end and no player is a dummy. The following simple case analysis shows that there are only two such orientations; see Figure 10.

First let us show that out-degree of each vertex j_k is 1 or 2. Indeed, if it is 0 then j_k is a dead-end and we get a contradiction. If it is 3 then j_k is transient position. In this case we can reduce $\mathcal{K}_{3,3}$ to $\mathcal{K}_{2,3}$ by deleting j_k. Furthermore, it is easy to see that the bipartite pair



	(j_2,j_3) (j_6,j_3)	(j_2,j_3) (j_6,j_5)	(j_2,j_5) (j_6,j_3)	(j_2,j_5) (j_6,j_5)
(j_1,j_2)	c_1	c_1	c_2	c_2
(j_1,j_6)	c_3	c_4	c_3	c_4

(a)



	(j_2,j_3) (j_6,j_3)	(j_2,j_5) (j_6,j_3)	(j_2,j_3) (j_6,j_1)	(j_2,j_5) (j_6,j_1)
(j_1,j_2)	c_1	c_1	c_2	c_3
(j_1,j_4)	c_1	c_1	c_4	c_4

(b)

Figure 10:
$(a) F_1 = c_1c_2 \vee c_3c_4, \quad F_2 = c_1c_3 \vee c_1c_4 \vee c_2c_3 \vee c_2c_4, \quad F_1 = F_2^d;$
$c_1 = ((j_1,j_2),(j_2,j_3),(j_3,j_4),(j_4,j_1)),$
$c_2 = ((j_1,j_2),(j_2,j_5),(j_5,j_4),(j_4,j_1)),$
$c_3 = ((j_1,j_6),(j_6,j_3),(j_3,j_4),(j_4,j_1)),$
$c_4 = ((j_1,j_6),(j_6,j_5),(j_5,j_4),(j_4,j_1)).$
$(b) F_1 = c_1(c_2c_3 \vee c_4), \quad F_2 = c_1 \vee c_2c_4 \vee c_3c_4, \quad F_1 = F_2^d;$
$c_1 = ((j_3,j_4),(j_4,j_5),(j_5,j_6),(j_6,j_1)),$
$c_2 = ((j_1,j_2),(j_2,j_3),(j_3,j_4),(j_4,j_5),(j_5,j_6),(j_6,j_1)),$
$c_3 = ((j_1,j_2),(j_2,j_5),(j_5,j_6),(j_6,j_1)),$
$c_4 = ((j_1,j_4),(j_4,j_5),(j_5,j_6),(j_6,j_1)).$

$(\mathcal{K}_{2,3},\mathcal{P})$ is, in fact, equivalent to the monochromatic pair $\theta_1,\mathcal{P}'$. By Proposition 9, this pair is solvable, unless it contains a 1-edge.

By simple counting arguments, we prove that there are three positions of out-degree 2 and three of out-degree 1. If the first three belong to one player and the last three to the other one then the latter player is a dummy. Hence, without loss of generality we can assume that $j_1, j_3, j_5 \in V_1$ and $j_2, j_4, j_6 \in V_2$, j_1, j_2, j_6 has out-degree 2, while j_3, j_4, j_5 has out-degree 1. It is easy to verify that, up to an isomorphism, there are only two such orientations of $\mathcal{K}_{3,3}$. They are given in Figure 10. The corresponding two normal game forms are tight.

Now let us show that pair $(\mathcal{K}_{3,3}, \mathcal{P})$ is not solvable whenever it contains a 1-edge. Due to symmetry, we can assume without loss of generality that (j_1, j_2) is such an edge. In this case we can delete edges (j_4, j_5) and (j_3, j_6) and obtain a pair from Example 4.1 (B), which, as we already know, is not solvable.

This completes the proof of Proposition 8. □

6.4 Monochromatic pairs; proof of Proposition 9

Let $(\mathcal{G}, \mathcal{P})$ be a monochromatic pair, that is, all nodes of graph $\mathcal{G}$ are controlled by the same player, say, player 2. Then, obviously, $(\mathcal{G}, \mathcal{P})$ cannot contain 1-edges of type B. For every its 0-edge we substitute two parallel edges and denote the obtained pair by $(\mathcal{G}', \mathcal{P})$. Proposition 9 states that $(\mathcal{G}, \mathcal{P})$ is not solvable if and only if $(\mathcal{G}', \mathcal{P})$ contains a 1-edge (of type A) and two more edge-disjoint simple paths between its ends.

Without loss of generality we can assume that graph $\mathcal{G}$ is 2-connected.

“If part”. We will keep notation of Example 4.1 part A and repeat similar arguments. Let us consider a pair (G, P) corresponding to $(\mathcal{G}', \mathcal{P})$. By our assumption, (G, P) contains a 1-path P_K between vertices j_0 and j_K and two more paths p' and p'' between the same vertices. Furthermore, let P_K and p', as well as P_K and p'', have no vertices in common, except j_0 and j_K, while p' and p'' are edge-disjoint but, in addition to j_0 and j_K, they might have more common vertices. (Let us remark that in Example 4.1 it was assumed that three paths p', p'' and p are pairwise vertex-disjoint; more precisely, they have no common vertices, except j_0 and j_K.) Obviously, the 1-path $\mathcal{P}_K$ is of type A (since pair $(\mathcal{G}, \mathcal{P})$ is monochromatic), i.e., there is a unique $k_0 \in \{1, \ldots, K-1\}$ such that $j_{k_0} \in V_1$, while $j_k \in V_2$ for each $k \in \{0, \ldots, K\} \setminus \{k_0\}$.

Similarly to example 4.1 A, we will define a payoff $u : C(\overrightarrow{G}) \to \{-1, +1\}$ such that for every initial position $j \in V(G)$ the obtained game $G, P, j, u)$ is not solvable. Let us set $u(c_k) = +1$ for every short dicycle c_k formed by path P_K for $k = 1, \ldots, k$. Then, as we know, path P_K can be passed through in both ways.

Now, let us consider paths p' and p'' in graph $\mathcal{G}$ and the corresponding paths P' and P'' in graph G. The last two paths may have common vertices and edges. Yet, obviously, their intersection can be partitioned in 0-paths and isolated vertices. It is also clear that all their common vertices are controlled by player 2.

For every short cycle C of a 0-path let us set $u(C) = +1$. Then, as we know, each 0-path can be passed through in both ways. Furthermore, for all remaining short cycles of P' and P'', let us define payoff u such that P' becomes oriented from j_0 to j_K, while P'' from j_K to j_0. By Lemma 9, such payoffs are uniquely defined.

Now, let us define function u for long dicycles of graph $\overrightarrow{G}$. Some of these cycles are formed by paths P' and P'' with orientations defined above. Let us set $u(c) = +1$ for each such dicycle C. Let us notice that in Example 4.1 there is only one such cycle.

Finally, there are only two more long dicycles, c' and c'', in digraph $\overrightarrow{G}$. The first one we get passing through P' from j_0 to j_K and then through P_K from j_K to j_0. Respectively, to get the second one we pass through P'' from j_K to j_0 and then through P_K from j_0 to j_K. Let us set $u(c') = u(c'') = -1$.

Now we can repeat the arguments of Example 4.1 A and show that game $G, P, j, u)$ is not solvable for any j, in other words, pair (G, P) is uniformly non-solvable.

"Only if part". Given a pair $(\mathcal{G}, \mathcal{P})$, let $\mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2$ be a partition of its edge-set in 0-edges, 1-edges, and all other edges. Obviously, a monochromatic pair $(\mathcal{G}, \mathcal{P})$ without 1-edges is solvable. In other words, if $(\mathcal{G}, \mathcal{P})$ is a not solvable monochromatic pair then $\mathcal{E}_1 \neq \emptyset$.

Furthermore, let $C = C(\mathcal{G})$ denote the set of all simple cycles of graph $\mathcal{G}$. Each of these cycles can be oriented in two ways. Let us denote by $\mathcal{C} \times 2$ the obtained set of dicycles. Now we can introduce a utility function as a mapping $u : \mathcal{C} \times 2 \to \{-1, +1\}$ and denote by $\mathcal{C} \times 2 = \mathcal{C}^+ \cup \mathcal{C}^-$ the corresponding partition.

We assume that pair $(\mathcal{G}, \mathcal{P})$ is not solvable. Then there is a utility function u and an orientation $(\overrightarrow{\mathcal{G}})$ with the following two properties.

(i) Every dicycle $c \in \mathcal{C}^-$ contains a 1-edge.

Let us change orientations of some 1-edges $\mathcal{E}_1' \subseteq \mathcal{E}_1$ in $\overrightarrow{\mathcal{G}}$ and denote the obtained digraph by $\overrightarrow{\mathcal{G}}^1$.

(ii) For every $\mathcal{E}_1' \subseteq \mathcal{E}_1$ there is a dicycle from $\mathcal{C}^-$ in digraph $\overrightarrow{\mathcal{G}}'$.

Statement (i) (respectively, (ii)) means that player 2 (respectively, 1) cannot win.

Lemma 12 *Statements (i) and (ii) imply that there exist two dicycles $c', c'' \in \mathcal{C}^-$ and 1-edge $e \in \mathcal{E}_1$ such that for both orientations of e the corresponding arcs e' and e'' belong to c' and c'', respectively.*

Proof Let us introduce a partition $\mathcal{E}_1 = \mathcal{E}_1^0 \cup \mathcal{E}_1^1 \cup \mathcal{E}_1^2$ such that $e \in \mathcal{E}_1^k$ when exactly k orientations of e belong to a simple directed cycle from $\mathcal{C}^-$; where $k = 0, 1$, or 2. As we already mentioned, $\mathcal{E}_1 \neq \emptyset$. If $e \in \mathcal{E}_1^0$, we can orient e arbitrarily. If $e \in \mathcal{E}_1^1$, let us choose the orientation of e such that the obtained arc belongs to no $C \in C^-$. Let us assume that $\mathcal{E}_1^2 = \emptyset$ and denote the obtained orientation of $\mathcal{G}$ by $\overrightarrow{\mathcal{G}}^2$. By (i), $\overrightarrow{\mathcal{G}}^2$ contains no cycle from $\mathcal{C}^-$; in contrast, by (ii), it must contain such a cycle. Thus, (i) and (ii) imply that $\mathcal{E}_1^2 \neq \emptyset$. $\square$

Let us consider the obtained cycles C', C'' and edge e in graph $\mathcal{G}$. (Since this graph is not directed, we ignore the orientations.) Furthermore, let us delete from C' and C'' all their common edges, except e. It is easy to demonstrate that we obtain two simple edge disjoint paths between two ends of e. (Yet, to show this, we have to recall that cycles C' and C'' were directed so that edge e had opposite orientations in them.) This completes the proof of Proposition 9. □

Remark 8 *Let us also recall that in digraph* $(\overrightarrow{\mathcal{G}})$ *we substitute two oppositely directed arcs for each* 0*-edge of graph* $\mathcal{G}$ *and define* $u(C) = +1$ *for the obtained directed cycle of length* 2*. Furthermore, we add a loop* ℓ *to every dead-end of digraph* $\overrightarrow{\mathcal{G}}'$ *and define* $u(\ell) = +1$*. Without these two conventions player* 2*, who controls all positions of* $\mathcal{G}$*, would win trivially, while we assume that pair* $(\mathcal{G}, \mathcal{P})$ *is not solvable.*

7 Proof of Theorem 2

We have to prove that a 2-connected pair (G, P) is solvable only if it belongs to the list $\mathcal{L}$ of solvable pairs given by Propositions 4-9.

7.1 Generating all 2-connected pairs by ear extensions

By Lovasz' "Ear-Decomposition" [30], each 2-connected graph $(\mathcal{G}, \mathcal{P})$ can be obtained from a loop by successive addition of new vertices and edges. By each step $k = 1, 2, \ldots$, we add to a current graph G_{k-1} at most two new vertices and one new edge $e_k = (j'_k, j''_k)$. There are the following three options:

(a) j'_k and j''_k are two "old" vertices of graph $\mathcal{G}_k$.

(b) j'_k is an old, while j''_k is a new vertex subdividing an edge of $\mathcal{G}_k$.

(c) both j'_k and j''_k are new vertices that subdivide an edge or two distinct edges of $\mathcal{G}_k$.

In all cases e_k is not a loop, that is, $j' \neq j''$. This inequality automatically holds for (b) and we assume that it holds for (a) and (c). Then, after each step, we obtain a 2-connected graph. To generate all 2-connected pairs $(\mathcal{G}, \mathcal{P})$ we should assign a player, 1 or 2, to every new vertex, in cases (b) and (c).

We start with a loop $\mathcal{G}_0$. Strictly speaking, a loop has one edge and one vertex of degree 2, while, by definition of graph $\mathcal{G}$, all its vertices are of degree at least 3. For example, let G_0 be a simple cycle, which is the simplest 2-connected graph. In this case, by Proposition 5, a pair (G_0, P) is solvable, unless it is a 1-cycle. Yet, the corresponding pair $(\mathcal{G}_0, \mathcal{P})$ has no vertices. Let us say that $\mathcal{G}_0$ is a *vertex-less* loop. It admits a unique ear extension. Indeed, options (a) and (b) are not applicable but (c) works and we obtain graph θ_1 that consists of two vertices and three parallel edges between them.

There are two pairs corresponding to this graph: $(\theta_1, \mathcal{P}_1)$, whose two vertices belong to two distinct players, and the monochromatic pair $(\theta_1, \mathcal{P}_0)$, whose two vertices belong to the same player, 1 or 2; see Figure 11.

By Propositions 7 and 9, each of these two pairs is solvable unless it contains a 1-edge. Thus, by the first step, we got two solvable pairs.

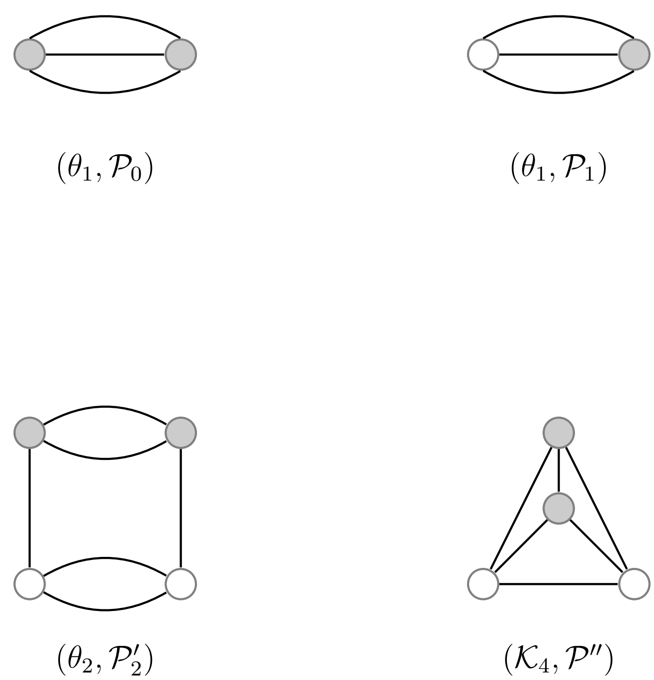


Figure 11: Minimal ear extensions of monochromatic pairs

General step is similar to the first one. Given a list of solvable pairs, we apply all ear extensions to them all and get a list of new pairs. Some of them are not solvable. There is no need to consider their extensions, since, by Proposition 2, solvability is anti-monotone.

Furthermore, by Proposition 9, every monochromatic pair is solvable (unless it contains no 1-edge). We will consider extensions of all monochromatic pairs separately, in Section 7.3; before that we treat them as terminal pairs, as well as we do with non-solvable pairs. We use Proposition 16 as a certificate of non-solvability; see Figure 6.

7.2 Ear extensions of graphs $\mathcal{K}_4$, $\mathcal{K}_{3,3}$, and θ_K

As the second step, let us consider all ear extensions of graph θ_1. There are four of them: $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3 = \mathcal{K}_4$, and $\mathcal{G}_4 = \theta_2$; see Figure 12. Clearly, by Proposition 16, pairs $(\mathcal{G}_1, \mathcal{P}_1)$ and $(\mathcal{G}_2, \mathcal{P}_2)$ are not solvable, unless they are monochromatic. However, this is not the case with pairs $(\mathcal{G}_3, \mathcal{P}_3)$ and $(\mathcal{G}_4, \mathcal{P}_4)$.

We already know that for every partition $\mathcal{P}$ pair $(\mathcal{K}_4, \mathcal{P})$ is solvable unless it contains a 0- or 1-edge. Clearly, we can restrict ourselves by two partitions $\mathcal{P}'$ and $\mathcal{P}''$; the corresponding pairs $(\mathcal{K}_4, \mathcal{P}')$ and $(\mathcal{K}_4, \mathcal{P}'')$ are given in Figure 8; see also Figure 3.

Now let us consider graph $G_4 = \theta_2$ and orient its edges as shown in Figure 12.4. By Proposition 16, a pair $(\theta_2, \mathcal{P})$ is not solvable whenever vertices x and y belong to two distinct players. Hence, pair $(\theta_2, \mathcal{P})$ can be solvable only if it is monochromatic or bipartite. Indeed,

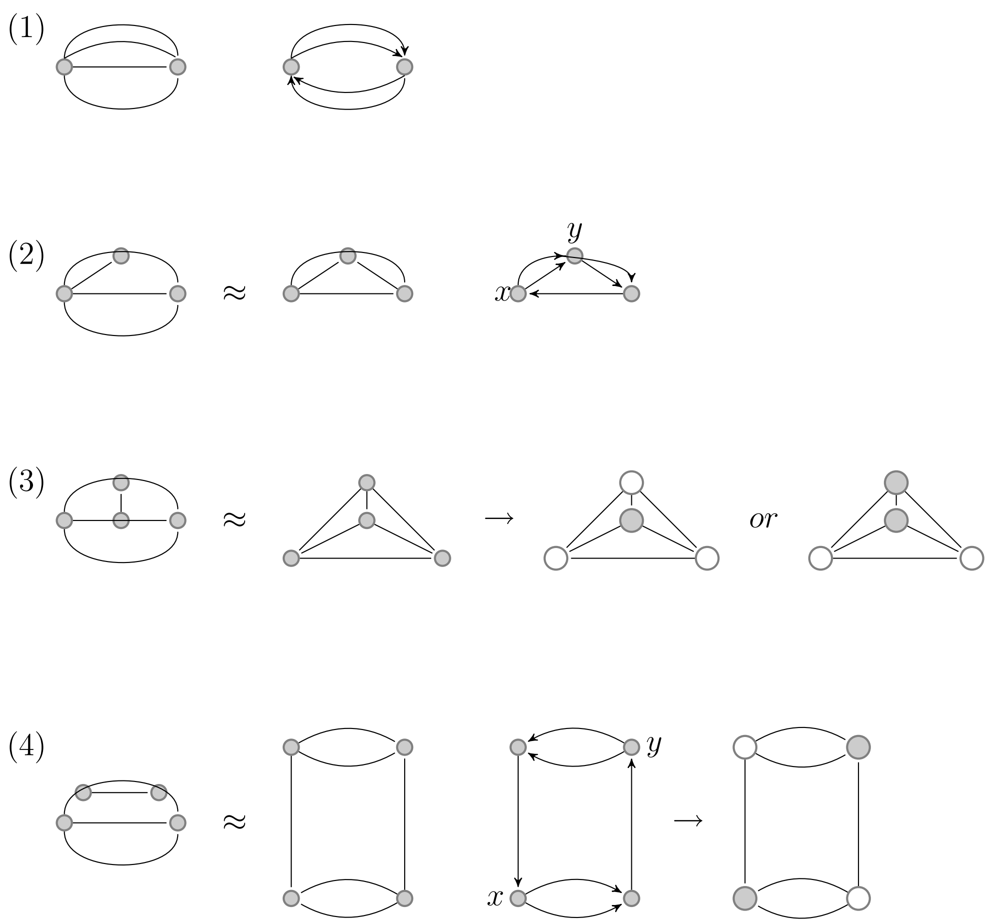


Figure 12: All ear-extensions of θ_1

by Proposition 6, the bipartite pair $(\theta_2, \mathcal{P}_2)$ is solvable unless it has a 1-edge of type 2. In contrast, pair $(\theta_2, \mathcal{P}'_2)$ in Figure 11 is not solvable, by Proposition 16.

All ear extensions of $\mathcal{K}_4$ are given in Figure 13. There are six of them, $\mathcal{G}_k;\ k = 1, \dots, 6$, from which the first five are non-solvable. More precisely, for $k = 1, \dots, 5$, no pair $(\mathcal{G}_k, \mathcal{P})$ is solvable unless it is monochromatic. Indeed, let us consider orientations given in Figures 13.1-5. Again, by Proposition 16, a pair from this set is not solvable unless vertices x and y belong to the same player. In this case, by symmetry and transitivity, we can conclude that all vertices must belong to one player.

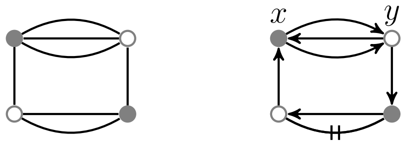
However, this is not the case with $k = 6$. It is easy to see that $\mathcal{G}_6 = \mathcal{K}_{3,3}$ and that Proposition 16 is not applicable to this graph. Instead, let us consider its orientation given in Figure 13. It results in a bidirected digraph $\overrightarrow{G}$ such that the corresponding graph $G = c_3$ is a simple cycle with three vertices x, y, z. Hence, each pair (G, P) is either 0- or 1-cycle. The latter is not solvable, by Proposition 5. Hence, pair $(\mathcal{K}_{3,3}, \mathcal{P})$ can be solvable only if it is either monochromatic or bipartite, since positions x, y, z must belong to the same player.

Figure 13: All ear-extensions of $\mathcal{K}_4$

By Proposition 8, the bipartite pair $(\mathcal{K}_{3,3}, \mathcal{P})$ is solvable unless it has a 0- or 1-edge.

All ear extensions of the bipartite pair $(\theta_2, \mathcal{P}_2)$ are given in Figure 14.
Cases 10, 11, 12, and 13 are equivalent to cases 4, 5, 4 of $\mathcal{K}_4$ and to case 9, respectively. There are 13 of them, $\mathcal{G}_k$; $k = 1, \ldots, 13$; see Figure 14. In case $k = 8$ we obtain the solvable bipartite pair $(\theta_3, \mathcal{P}_3)$. Any other pair $(\mathcal{G}_k, \mathcal{P}_k)$; $k = 1, \ldots, 7, 9, \ldots, 13$ can be solvable only

(1)



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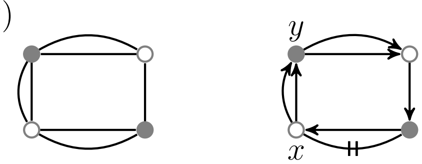

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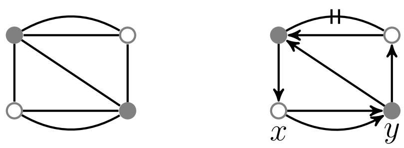

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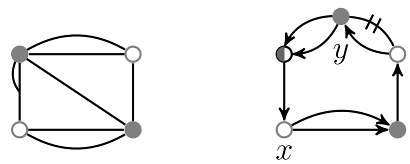

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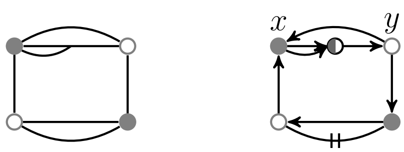
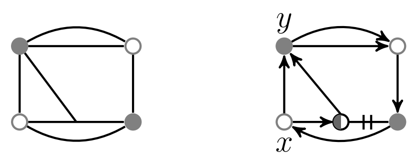


(6)



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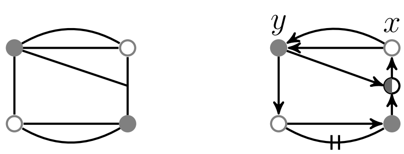
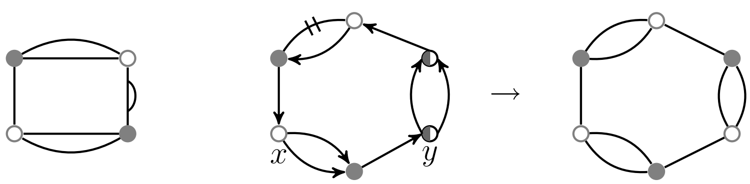


(8)



(9)

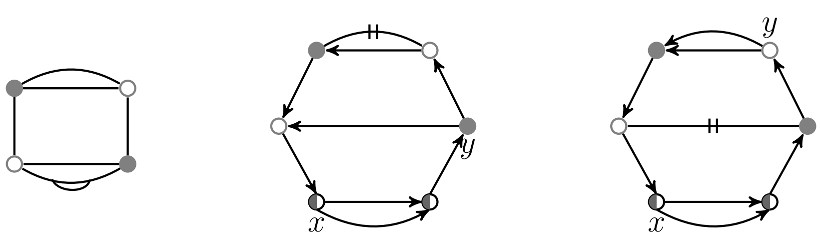


Figure 14: All ear - extensions of θ_2

.

if it is monochromatic. This follows standardly from Proposition 16. Let us notice that we skip the analysis of four cases, $10 \leq k \leq 13$, since they are not new. Indeed, graphs $\mathcal{G}_{13}$ and $\mathcal{G}_9$ are isomorphic; furthermore, $\mathcal{G}_{10}$ and $\mathcal{G}_{12}$ are isomorphic to $\mathcal{G}_4$ in Figure 13 and $\mathcal{G}_{11}$ to $\mathcal{G}_5$ in Figure 13.

Similarly, we can consider all ear extensions of pair $(\theta_K, \mathcal{P}_K)$ for arbitrary $K \geq 3$. One of them results in the next bipartite pair $(\theta_{K+1}, \mathcal{P}_{K+1})$, while any other can be solvable only if it is monochromatic. The proof immediately follows, since $(\theta_2, \mathcal{P}_2) \leq (\theta_K, \mathcal{P}_K)$ for all $K \geq 2$ and case $K = 2$ was already considered.

Now let us consider all ear extensions of graph $\mathcal{K}_{3,3}$. There are seven of them, $\mathcal{G}_k$; $k = 1, \ldots, 7$; see Figure 15.

For each $k = 1, \ldots, 7$ the corresponding pair $(\mathcal{G}_k, \mathcal{P}_k)$ can be solvable only if it is monochromatic. This follows standardly, from Proposition 16.

7.3 Ear extensions of monochromatic pairs

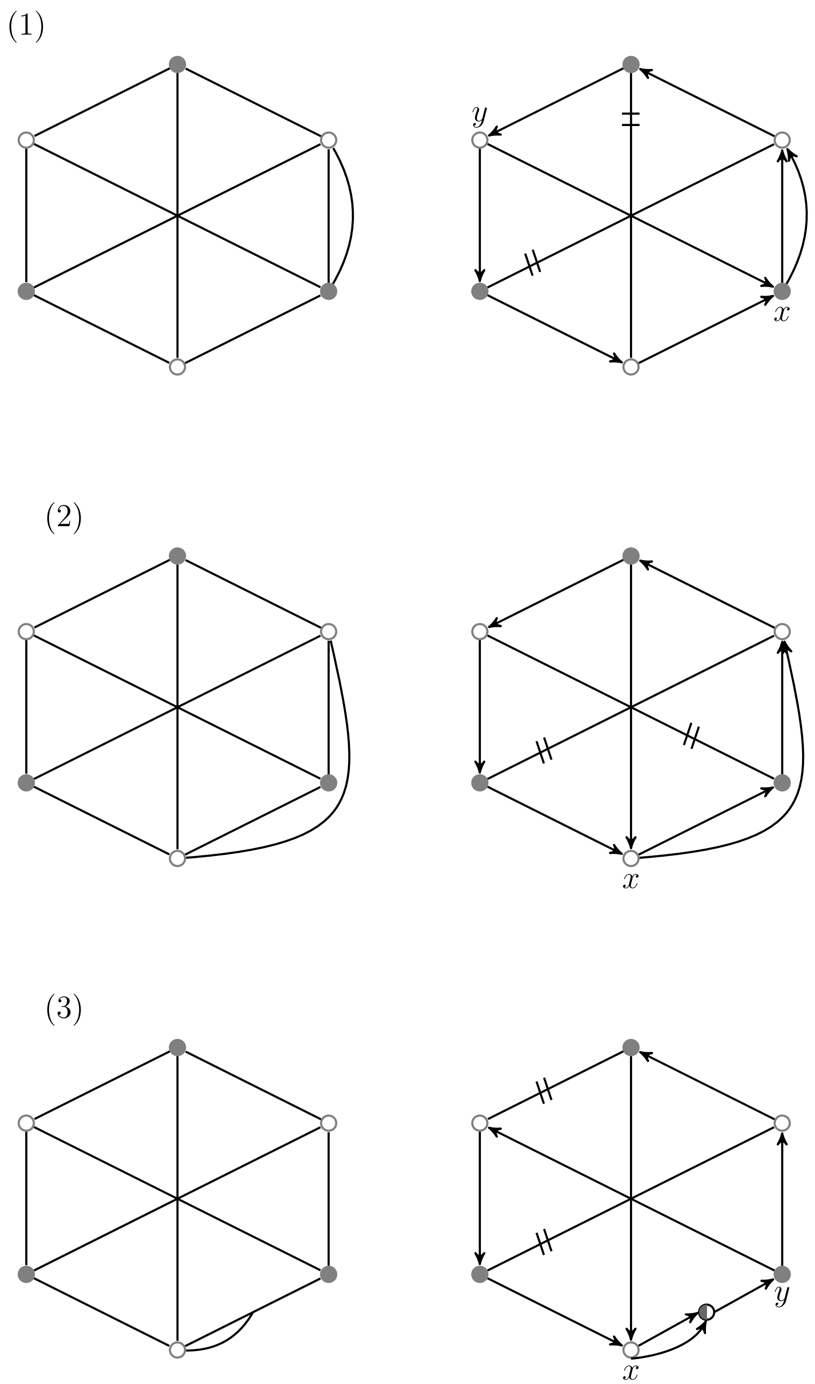
It is easy to see that all solvable pairs obtained by the above ear extensions form the list $\mathcal{L}$ defined by Propositions 5-9. Yet, we did not consider ear extensions of the monochromatic pairs. To finish the proof of Theorem 2 we have to show that these extensions cannot produce any new solvable pair. This is implied by the following claim.

Lemma 13 *If a solvable pair $(\mathcal{G}, \mathcal{P})$ is obtained by an ear extensions of a monochromatic pair $(\mathcal{G}_0, \mathcal{P}_0)$ then either pair $(\mathcal{G}, \mathcal{P})$ is monochromatic itself, or it can be alternatively obtained by some ear extensions of the pair $(\theta_1, \mathcal{P}_1)$ or $(\mathcal{K}_4, \mathcal{P}'')$ given in Figure 3; in other words, $(\mathcal{G}, \mathcal{P}) \geq (\mathcal{K}_4, \mathcal{P}'')$ or $(\mathcal{G}, \mathcal{P}) \geq (\theta_1, \mathcal{P}_1)$.*

Proof Let $(\mathcal{G}_0, \mathcal{P}_0)$ be a monochromatic pair all whose positions belong to the same player, say, to player 1, and let $(\mathcal{G}, \mathcal{P})$ be its ear extension by one new edge $e = (j', j'')$. Let us recall three options (a), (b), and (c) from Section 7.1. For (b) or (c) we will assume that all new positions (one and two, respectively) belong to player 2.

Obviously, in case (a) pair $(\mathcal{G}, \mathcal{P})$ still remains monochromatic. Let us notice that pair $(\mathcal{G}, \mathcal{P})$ can remain monochromatic even in case (c). This happens if and only if graph $\mathcal{G}_0$ is a vertex-less loop; see Section 7.1. Then $(\mathcal{G}, \mathcal{P})$ is a monochromatic pair, since both its vertices belong to player 2, and pair $(\mathcal{G}_0, \mathcal{P}_0)$ can be viewed as monochromatic, too.

Yet, we will prove that for every other 2-connected monochromatic pair $(\mathcal{G}_0, \mathcal{P}_0)$ each its ear extension by an edge $e = (j', j'')$ of type (b) or (c) results in a pair $(\mathcal{G}, \mathcal{P})$ such that $(\mathcal{G}, \mathcal{P}) \geq (\mathcal{K}_4, \mathcal{P}'')$, or $(\mathcal{G}, \mathcal{P}) \geq (\theta_1, \mathcal{P}_1)$, or $(\mathcal{G}, \mathcal{P}) \geq (\theta_2, \mathcal{P}_2')$; see Figure 11. Let us notice



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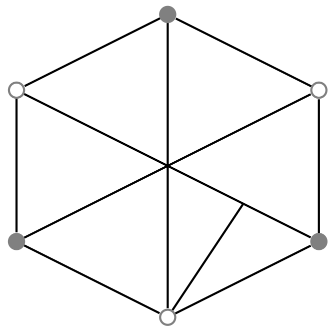

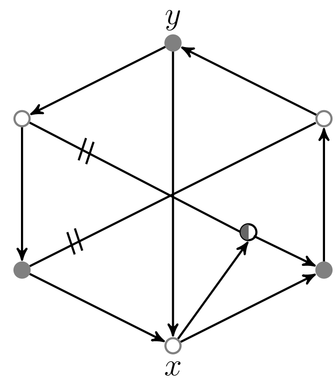

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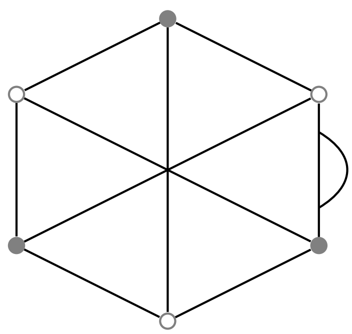

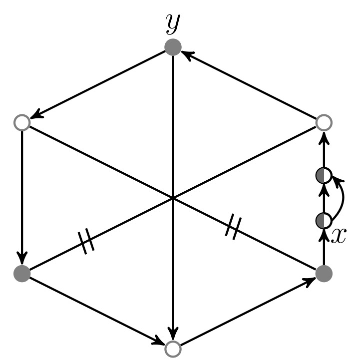

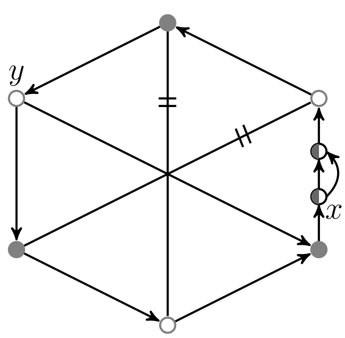

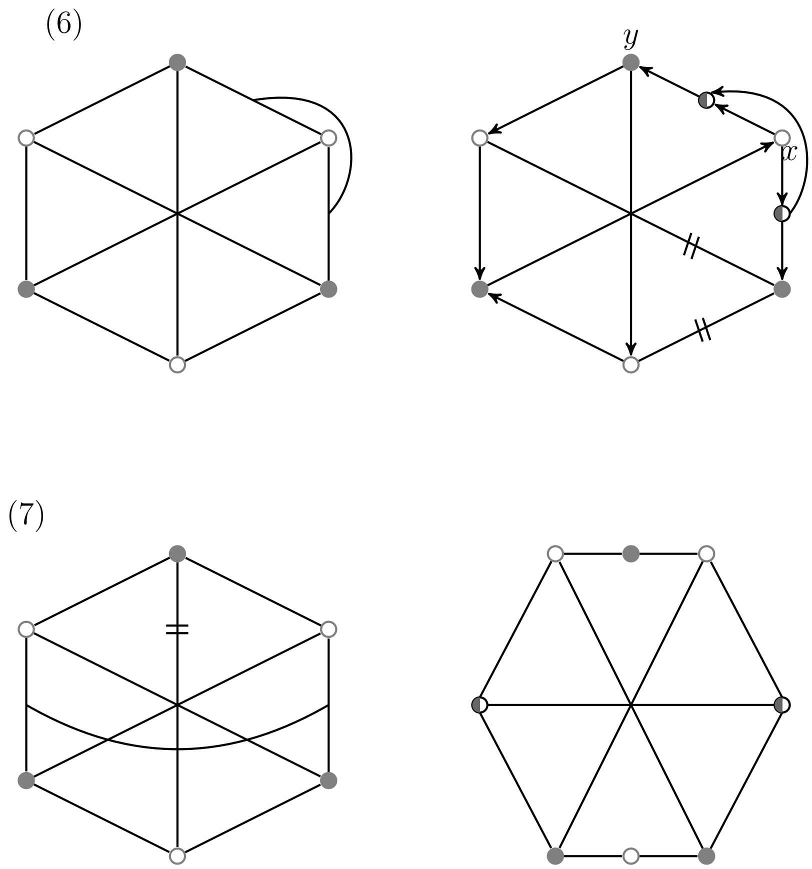

Figure 15: All ear - extensions of $\mathcal{K}_{3,3}$

that in the last case pair $(\mathcal{G}, \mathcal{P})$ is not solvable, since $(\theta_2, \mathcal{P}_2')$ is not solvable, by Proposition 16.

First, let us assume that $e = (j', j'')$ is of type (b), that is, positions j' and j'' belong to players 1 and 2, respectively. Since graph $\mathcal{G}_0$ is 2-connected, there is a simple cycle c in $\mathcal{G}$ that contains both vertices j' and j'' but does not contain edge $e = (j', j'')$. Obviously, in this case $(\mathcal{G}, \mathcal{P}) \geq (\theta_1, \mathcal{P}_1)$.

Now, let us assume that $e = (j', j'')$ is of type (c), that is, both positions j' and j'' belong to player 2. Again, since graph $\mathcal{G}$ is 2-connected, there is a simple cycle c in it that contains j' and j''. Obviously, c contains at least one more position j_1. It is also clear that j_1 belongs to player 1 and $deg_{\mathcal{G}}(j_1) \geq 3$. Then , since graph $\mathcal{G}_0$ is 2-connected, there is a simple path between j_1 and another vertex $j_2 \neq j_1$ in C. If $j_2 = j'$ or $j_2 = j'$ then $(\mathcal{G}, \mathcal{P}) \geq (\theta_1, \mathcal{P}_1)$. If $j_2 \neq j'$ and $j_2 \neq j''$ then j_2 belongs to player 1 and $(\mathcal{G}, \mathcal{P}) \geq (\mathcal{K}_4, \mathcal{P}'')$ or $(\mathcal{G}, \mathcal{P}) \geq (\theta_2, \mathcal{P}_2')$. □

This completes the proof of Theorem 2. □

8 Appendix 1. Proof of Theorem 1 and its limits

8.1 Tight two-person game forms

Let $g : X_1 \times X_2 \to A$ be a two-person game form. By definitions of Section **??**, g is tight if $F_1^d = F_2$, or in other words, if two hypergraphs H_1 and H_2 on the ground set A defined by the rows and columns of g are transversal (dual); see 8 examples in Figures 1 and 2.

More reformulations of tightness are possible. Let us consider an arbitrary *reply mapping* $\phi_1 : X_2 \to X_1$ that assigns a strategy of player 1 (a row) to each strategy of player 2 (a column). In the special case, when this functions takes a unique value $x_1 \in X_1$, we will use the notation $\phi_1^0 : X_2 \to \{x_1\}$. Let $gr(\phi_1) \subseteq X = X_1 \times X_2$ be the graph of ϕ_1 in X and $[\phi_1] = g(gr(\phi_1)) \subseteq A$ be the corresponding set of outcomes. Similarly we define $[\phi_1^0]$, $[\phi_2]$, and $[\phi_2^0]$.

Proposition 18 *The following properties of a game form are equivalent:*

(j) For each ϕ_1 there exists a ϕ_1^0 such that $[\phi_1^0] \subseteq [\phi_1]$;

(jj) For each ϕ_2 there exists a ϕ_2^0 such that $[\phi_2^0] \subseteq [\phi_2]$;

(jjj) For each ϕ_1 and ϕ_2 we have $[\phi_1] \cap [\phi_2] \neq \emptyset$.

Proof (j) $\Rightarrow$ (jjj). Assume indirectly that (j) holds and (jjj) does not. The latter means that there exist ϕ_1 and ϕ_2 such that $[\phi_1] \cap [\phi_2] = \emptyset$, while by (j), there exists a ϕ_1^0 such that $[\phi_1^0] \subseteq [\phi_1]$. Hence, $[\phi_1^0] \cap [\phi_2] = \emptyset$. However, this is impossible, since clearly, $gr(\phi_1^0) \cap gr(\phi_2) \neq \emptyset$ for every ϕ_1^0 and ϕ_2.

(jjj) $\Rightarrow$ (j). Suppose that (j) does not hold, that is, there is a ϕ_1 such that $[\phi_1^0] \subseteq [\phi_1]$ for no ϕ_1^0. Choosing an outcome from $[\phi_1^0] \setminus [\phi_1]$ for each ϕ_1^0 we get a mapping ϕ_2 such that $[\phi_1] \cap [\phi_2] \neq \emptyset$. Hence, (jjj) does not hold either.

Thus, (j) and (jjj) are equivalent. Similarly, (jj) and (jjj) are equivalent. To come to this conclusion it is enough to rename the players 1 and 2. □

It is also clear that these three claims are equivalent to tightness of g. Indeed, (j) and (jj) mean that $H_1^d = H_2$ and $H_1 = H_2^d$, respectively. We will need one more reformulation of tightness in terms of effectivity functions (EFF).

Given a two-person game form $g : X_1 \times X_2 \to A$ and a subset of outcomes $B \subseteq A$, for each player $i \in \{1, 2\}$ define $\mathcal{E}_g(i, B) = 1$ if there is a strategy $x_i \in X_i$ such that $g(x_i, x_{3-i}) \in B$ for each strategy $x_{3-i} \in X_{3-i}$ of the opponent; otherwise $\mathcal{E}_g(i, B) = 0$. Respectively, we say that player i is effective or not effective for $B \subseteq A$. Let us note that $\mathcal{E}_g(1, *)$ and $\mathcal{E}_g(2, *) : 2^A \to \{0, 1\}$ are two Boolean functions whose variables are the outcomes $a \in A$.

Proposition 19 *Implication $\mathcal{E}_g(i, B) = 1 \Rightarrow \mathcal{E}_g(3 - i, A \setminus B) = 0$ holds for every game form g, while the inverse implication $\mathcal{E}_g(i, B) = 1 \Leftarrow \mathcal{E}_g(3 - i, A \setminus B) = 0$ holds if and only if g is tight.*

Proof . Let us assume indirectly that $\mathcal{E}_g(1,B) = \mathcal{E}_g(2, A \setminus B) = 1$; Then there exist two strategies $x_1 \in X_1$ and $x_2 \in X_2$ such that $g(x_1, x_2) \in B \cap (A \setminus B) = \emptyset$ and we get a a contradiction.

Now let us consider the inverse implication $\mathcal{E}_g(2,B) = 1 \Leftarrow \mathcal{E}_g(1, A \setminus B) = 0$. Let player 1 be not effective for $A \setminus B$, that is, for each strategy $x_1 \in X_1$ there is $x_2 \in X_2$ such that $g(x_1, x_2) \in B$; in other words, there exists a function ϕ_2 such that $[\phi_2] \subseteq B$. Furthermore, by (jj), g is tight if and only if there is a function ϕ_2^0 such that $[\phi_2^0] \subseteq \phi_2^0$. Hence, player 2 is effective for B, that is, $\mathcal{E}_g(2,B) = 1$ and the implication holds, if and only if g is tight. □

8.2 Tightness and zero-sum-solvability

Let us recall that by definition, a game form g is zero-sum-solvable if for each utility function $u : A \to \mathbb{R}$ the obtained normal form game (g, u) is solvable, that is, has a saddle points (in pure strategies). It is well-known that the latter property holds if and only if maxmin and minmax are equal, that is, if

$$v_1 = max_{x_1 \in X_1} min_{x_2 \in X_2} u(g(x_1, x_2)) = min_{x_2 \in X_2} max_{x_1 \in X_1} u(g(x_1, x_2)) = v_2.$$

Proposition 20 *([10, 17]). (i) If game form g is tight then it is zero-sum-solvable; (ii) if g is not tight then it is not ±1-solvable.*

Proof . Suppose that g is not tight. Then, by (jjj), there exist ϕ_1 and ϕ_2 such that $[\phi_1] \cap [\phi_2] = \emptyset$. Let us set $u(a) = 1$ for $a \in [\phi_1]$, $u(a) = -1$ for $a \in [\phi_2]$, and $u(a) = 1$ or $u(a) = -1$, arbitrarily, for all remaining $a \in A$. Obviously, for this u we obtain $-1 = v_1 < v_2 = 1$ and hence, there is no saddle point in game (g, u). Thus, game form g is not ±1-solvable.

Suppose that g is not zero-sum-solvable; i.e., there is a payoff $u : A \to \mathbb{R}$ such that the normal form game (g, u) is not solvable, i.e., $v_1 < v_2$. Furthermore, for every $x_1 \in X_1$ there is an $x_2 \in X_2$ such that $u(g(x_1, x_2)) = v_1$ and for every $x_2 \in X_2$ there is an $x_1 \in X_1$ such that $u(g(x_1, x_2)) = v_2$. In particular, this implies that there exist ϕ_1 and ϕ_2 such that $[\phi_1] \cap [\phi_2] = \emptyset$. Hence, g is not tight, by (jjj). □

8.3 Tightness implies Nash-solvability

Still we have to prove that g is Nash-solvable (not only zero-sum-solvable) whenever it is tight. We will partition the set of outcomes A in three pairwise disjoint subsets $A = B \cup B_1 \cup B_2$ such that

$(p1)$ $u(1, b) \geq u(1, b_1)$ for every $b \in B, b_1 \in B_1$ and

$(p2)$ $u(2, b) \geq u(2, b_2)$ for every $b \in B, b_2 \in B_2$.

Condition $p1$ (respectively, $p2$) means that any outcome of B_1 for player 1 (respectively, of B_2 for player 2) is not better than any outcome of B. We also assume that that the following two conditions hold for $A = B \cup B_1 \cup B_2$ too:

$(q1)$ $\mathcal{E}(1, B_2) = 0$ and $(q2)$ $\mathcal{E}(2, B_1) = 0$.

In other words, player 1 (respectively, 2) cannot "punish" the opponent by forcing B_2 (respectively, B_1). Assuming that g is tight we can rewrite these two conditions as follows:

$(q1')$ $\mathcal{E}(1, B \cup B_1) = 1$ and $(q2')$ $\mathcal{E}(2, B \cup B_2) = 1$.

Our proof is "dynamic". We will reduce the set B by sending its outcomes to B_1 and B_2 in such a way that all four above conditions hold. Let us note that we cannot get $B = \emptyset$, since in this case conditions $q1'$ and $q2'$ imply that $\mathcal{E}(1, B_1) = \mathcal{E}(2, B_2) = 1$ in contradiction to $B_1 \cap B_2 = \emptyset$. (This is the only place where we make use of the tightness of g.)

Thus, there is a partition $A = B \cup B_1 \cup B_2$ such that B cannot be reduced any longer. Let us fix such a partition and let a be the worst outcome for player 1 in B_1, that is, $u(1, a) \leq u(1, b)$ for every $b \in B_1$. We know that we cannot send a from B to B_1, although this operation would be OK with $(p1)$. Clearly, it can contradict only $(q2)$ and this happens indeed if $\mathcal{E}(2, (B_1 \cup \{a\})) = 1$.

Furthermore, let B_2^a denote the set of all outcomes of B_2 that are not better than a for player 2, that is, $u(2, b) \leq u(2, a)$ for every $b \in B_2^a$; in particular, $a \in B_2^a$. We know that we cannot send B_2^a from B to B_2, although this operation would be OK with $(p2)$. Clearly, it can contradict only $(q1)$ and this happens indeed if $\mathcal{E}(1, (B_2 \cup B_2^a)) = 1$.

Thus, we obtain $\mathcal{E}(2, (B_1 \cup \{a\})) = \mathcal{E}(1, (B_2 \cup B_2^a)) = 1$. By the definition of $\mathcal{E}_g$, there are strategies $x_1^0 \in X_1$ and $x_2^0 \in X_2$ such that $g(x_1^0, x_2) \in (B_2 \cup B_2^a)$ for each $x_2 \in X_2$ and $g(x_1, x_2^0) \in (B_1 \cup \{a\})$ for each $x_1 \in X_1$. Let us note that $(B_1 \cup \{a\}) \cap (B_2 \cup B_2^a) = \{a\}$. Hence, $g(x_1^0, x_2^0) = a$ and the situation $(x_1^0, x_2^0) \in X$ is a Nash equilibrium in the game (g, u), by the definitions of a and B_2^a. $\square$

Now we will show that Theorem 1 does not generalize the case $n = 3$. The concept of tightness is naturally extended to this case. Yet, for 3-person game forms tightness is no longer necessary [19] nor sufficient [18, 19] for Nash-solvability. We reproduce the corresponding two examples here.

8.4 n-person game forms and their effectivity functions

Let $I = \{1, \ldots, n\}$ be a set of players and $A = \{a_1, \ldots, a_p\}$ be a set of outcomes. Subsets $K \subseteq I$ and $B \subseteq A$ are called *coalitions* and *blocks*, respectively. Furthermore, let X_i be a (finite) set of strategies of a player $i \in I$. The n-tuples of strategies $x = (x_i \in X_i, i \in I) \in X = \prod_{i \in I} X_i$ are called situations. A *game form* is a mapping $g : X \to A$. If each player $i \in I$ chooses a strategy $x_i \in X_i$ then a situation x and the corresponding outcome $g(x)$ appear. A game form g is realized by an n-dimensional table (by a matrix for $n = 2$) whose entries are the outcomes $a \in A$. Typically, the mapping g is not injective, that is, the same outcome can appear in several distinct situations.

Given I and A, an *effectivity function* (EFF) is defined as a mapping $\mathcal{E} : 2^I \times 2^A \to \{0, 1\}$. Its values $E(K, B)$ are interpreted as follows. If $\mathcal{E}(K, B) = 1$ (respectively, $\mathcal{E}(K, B) = 0$) then we say that the coalition $K \subseteq I$ is effective (respectively, not effective) for the block $B \subseteq B$, meaning that K can (respectively, cannot) guarantee that an outcome from B will appear. Since $2^I \times 2^A = 2^{I \cup A}$, we can say that $\mathcal{E}$ is a Boolean function whose set of arguments $I \cup A$ is a mixture of the players and outcomes.

Let us recall that, given an arbitrary Boolean function F, its dual F^d is defined by the formula $F^d(x) = \overline{F}(\overline{x})$. In other words, to get F^d we negate F itself and every its variable. Furthermore, F is called *self-dual* if $F^d = F$.

Let us reformulate the above two definitions for EFFs. Given an EFF $\mathcal{E}$, its dual $\mathcal{E}^d$ is defined by formula $\mathcal{E}(K,B) + \mathcal{E}^d(I \setminus K, A \setminus B) = 1$. Respectively, $\mathcal{E}$ is self-dual if $\mathcal{E} = \mathcal{E}^d$, that is, if $\mathcal{E}(K,B) + \mathcal{E}(I \setminus K, A \setminus B) = 1$ for each $K \subseteq I$ and $B \subseteq A$. We can rewrite this formula as $\mathcal{E}(K,B) = 0$ iff $\mathcal{E}(I \setminus K, A \setminus B) = 1$. In the literature the self-dual EFFs sometimes are called *maximal* [34, 33, 35].

To each game form $g : X \to A$ we assign an EFF $\mathcal{E}_g : 2^I \times 2^A \to \{0,1\}$ as follows. Given a coalition $K \subseteq I$ and block $B \subseteq A$, the EFF $\mathcal{E}_g(K,B)$ takes value 1 if and only if K has a strategy that guarantees that an outcome of B will appear independently on the strategy of the complementary coalition $I \setminus K$; in other words, $\mathcal{E}_g(K,B) = 1$ if and only if there exists an $x_K = (x_i, i \in K)$ such that $g(x_K, x_{I \setminus K}) \in B$ for each $x_{I \setminus K} = (x_i, i \notin K)$. Let us note that $\mathcal{E}(K,B) = 1$ if $K = I$ and $B \neq \emptyset$ but $\mathcal{E}(I, \emptyset) = 0$; furthermore, $\mathcal{E}(K,B) = 0$ if $K = \emptyset$ and $B \neq A$, yet, by convention $\mathcal{E}(\emptyset, A) = 1$.

An EFF is assigned to a game form (that is, $\mathcal{E} = \mathcal{E}_g$ for some g) if and only if $\mathcal{E}$ is monotone, superadditive, and satisfies the above boundary conditions. This nice characterization was obtained in [34]; see also [33] and [35] for the proof and necessary definitions.

It is easy to see that the equations $\mathcal{E}_g(K,B) = \mathcal{E}(I \setminus K, A \setminus B) = 1$ for no g can hold simultaneously, since otherwise $\mathcal{E}_g(I, \emptyset) = 1$. In other words, the implication $\mathcal{E}_g(K,B) = 1 \Rightarrow \mathcal{E}_g(I \setminus K, A \setminus B) = 0$ holds for each game form, unlike the inverse one $\mathcal{E}_g(K,B) = 0 \Rightarrow \mathcal{E}_g(I \setminus K, A \setminus B) = 1$. If it holds too then the game form g is called *tight*. Let us remark that g is tight if and only if its EFF $\mathcal{E}_g$ is self-dual.

We can reformulate this definition in Boolean terms as follows. Let us assign a Boolean variable a to every outcome $a \in A$. (For simplicity the denote the outcome and the corresponding variable by the same symbol.) For each coalition $K \subseteq I$ we introduce a positive (without negations) DNF

$$F_K = F_K(g) = \bigvee_{x_K = (x_i, i \in K)} \quad \bigwedge_{x_{I \setminus K} = (x_i, i \notin K)} g(x_K, x_{I \setminus K}).$$

Then g is tight if and only if for each $K \subseteq I$ DNFs $F_K(g)$ and $F_{I \setminus K}(g)$ define dual monotone Boolean functions, $F_K^d = F_{I \setminus K}$. Let us remark that duality always holds for $K = \emptyset$ and $K = I$. Indeed, by the above boundary condition, $F_\emptyset(g) = \wedge_{a \in A} a$ and $F_I(g) = \vee_{a \in A} a$. Let us also remark that a two-person game form g is tight if and only if two DNFs $F_1 = F_1(g)$ and $F_2 = F_2(g)$ define dual Boolean functions. For example, in Figure 1 only the third game form is tight and in Figure 2 the last two game forms are tight, while the first two are not.

8.5 Nash-solvable but not tight 3-person game form

Given three players ($|I| = 3$, $I = \{1, 2, 3\}$) each of which has two strategies, $X_i = \{0, 1\}$ for each $i \in I$, and two outcomes ($|A| = 2$, $A = \{a_1, a_2\}$), let us define a $2 \times 2 \times 2$ game form $g : \prod_{i \in I} X_i \to A$ by formula

$g(x_1, x_2, x_3) = a_1$ if $x_1 = x_2 = x_3$ and $g(x_1, x_2, x_3) = a_2$ otherwise.

It is easy to see that every two players, say, $1, 2$, are effective for the outcome a_2. To enforce it they can just choose $x_1 = 0$ and $x_2 = 1$. Yet, they are not effective for a_1. It is also clear that a single player is effective only for the whole set $A = \{a_1, a_2\}$. Thus, we obtain

$F^1 = F_1(g) = F_2(g) = F_3(g) = a_1 a_2$,

$F^2 = F_{\{2,3\}}(g) = F_{\{3,1\}}(g) = F_{\{1,2\}}(g) = a_2$.

Since $(F^1)^d \neq F^2$, we conclude that this game form g is not tight.

Let us show that g is Nash-solvable. Indeed, if all three players prefer a_1 to a_2 then, clearly, two situations $(x \in X | x_1 = x_2 = x_3 = 0)$ and $(x \in X | x_1 = x_2 = x_3 = 1)$ are both Nash equilibria. If a player, say 1, prefers a_2 to a_1 then the situation $(x \in X | x_1 = 1, x_2 = x_3 = 0)$ is a Nash equilibrium. Indeed, in this case $g(x) = a_2$ and no player, neither 2 nor 3, can switch it to a_1. Although player 1 could do this (just substituting $x_1 = 0$ for $x_1 = 1$), yet, he is not interested, since he prefers a_2 to a_1.

8.6 Tight but not Nash-solvable 3-person game form

Given three players ($|I| = 3$, $I = \{1, 2, 3\}$) each of which has six strategies,

$$X_i = \{x_i = (x'_i, x''_i) \mid x'_i \in \{0, 1\}, x''_i \in \{0, 1, 2\}\}; i \in I,$$

and three outcomes ($|A| = 3$, $A = \{a_1, a_2, a_3\}$), let us define a $6 \times 6 \times 6$ game form $g : \prod_{i \in I} X_i \to A$ as follows:

$$g(x) = g(x_1, x_2, x_3) = g(x'_1, x''_1, x'_2, x''_2, x'_3, x''_3) = a_j, \quad \text{where}$$

$$j - 1 = \begin{cases} (x''_1 + x''_2 + x''_3) \mod 3 \text{ if } x'_1 = x'_2 = x'_3, \\ (x''_1 + x''_2) \mod 3 \text{ if } 1 = x'_1 > x'_2 = 0, \\ (x''_2 + x''_3) \mod 3 \text{ if } 1 = x'_2 > x'_3 = 0, \\ (x''_3 + x''_1) \mod 3 \text{ if } 1 = x'_3 > x'_3 = 0. \end{cases}$$

First let us notice that g is well defined, since the above four conditions, $x'_1 > x'_2$, $x'_2 > x'_3$, $x'_3 > x'_3$, and $x'_1 = x'_2 = x'_3$, do form a partition of X. Indeed, no two of the first three inequalities can hold simultaneously, since $x'_i \in \{0, 1\}$ takes only two values for each $i \in \{1, 2, 3\}$. In fact, these four conditions partition the $6 \times 6 \times 6$ cube X in three $3 \times 3 \times 6$ cuboids corresponding to the three inequalities and two $2 \times 2 \times 2$ cubes corresponding to the equalities.

Now, let us show that g is tight. Indeed, any two players, say, $1, 2 \in I$, are effective for every outcome $a_j \in A$. To guarantee it, they just choose $1 = x'_1 > x'_2 = 0$ to take the control and then force a_j choosing ($x''1$ and x''_2) such that $x''_1 + x''_2 = j - 1(\mod 3)$. On the other hand, single player is effective only for the whole set A. Thus we obtain

$F^1 = F_1(g) = F_2(g) = F_3(g) = a_1a_2a_3$,

$F^2 = F_{2,3}(g) = F_{3,1}(g) = F_{1,2}(g) = a_1 \vee a_2 \vee a_3$.

Since $(F^1)^d = F^2$, we conclude that the considered game form g is tight.

Moreover, for each player $i \in I$ and for each strategy $x_i \in X_i$ the obtained restricted game form $g[x_i]$ of the remaining two players is tight too. Indeed, due to symmetry, without loss of generality, we can choose any strategy. For example, let us fix $x_1 = (x'_1, x''_1) = (1, 2)$. Then in the obtained game form $g[x_1]$ player 2 can enforce any outcome $a_j \in A$. To do so he should just choose $x'_2 = 0$ to get $1 = x'_1 > x'_2 = 0$ and take the control. Then he should choose $x''_2 = j \mod 3$, since in this case $(x''_1 + x''_2) \mod 3 = 2 + x''_2) \mod 3 = j - 1$ which results in a_j.

Respectively, player 3 is effective only for the whole set A and we obtain:

$F_2 = F_2(g) = a_1 \vee a_2 \vee a_3,\ F_3 = F_3(g) = a_1a_2a_3$ and $F_2^d = F_3$.

Yet, g is not Nash-solvable. To show this let us choose a utility function u that realizes so-called "Condorcet" preference profile

$u(1, a_1) > u(1, a_2) > u(1, a_3)$,
$u(2, a_2) > u(2, a_3) > u(2, a_1)$,
$u(3, a_3) > u(3, a_1) > u(3, a_2)$.

and show that the obtained normal form game (g, u) has no Nash equilibrium.

Let $x = (x_1, x_2, x_3) = (x'_1, x''_1, x'_2, x''_2, x'_3, x''_3)$ be an arbitrary situation.

Case 1: $x'_1 = x'_2 = x'_3$. In this case, by definition, $g(x) = a_j$, where $j = 1 + ((x''_1 + x''_2 + x''_3) \mod 3)$, and it is clear that each player, by changing the strategy, can get each outcome of A. Hence, x is not a Nash equilibrium.

Case 2: equalities $x'_1 = x'_2 = x'_3$ do not hold. In this case, without loss of generality, we can assume that $1 = x'_1 > x'_2 = 0$. Then, by definition, $g(x) = a_j$, where $j = 1 + ((x''_1 + x''_2) \mod 3)$. In this situation the strategy of player 3 is irrelevant and (s)he cannot change the outcome by choosing another strategy. However, each player 2 or 3 can obtain any given outcome of A. Let us note that the present outcome $a_j = g(x)$ may be the best for one of these two players but not for both. Hence, x is not a Nash equilibrium, since this latter player can change the strategy and get a better outcome.

8.7 Nash-solvability of a 3-person game form is not uniquely defined by its effectivity function

By Theorem 1, a 2-person game form g is Nash-solvable if and only if it is tight, that is, the corresponding EFF $\mathcal{E}_g$ is self-dual. In Sections 8.5 and 8.6 we demonstrated that Theorem 1 does not extend the case of 3-person game forms, for which tightness is no longer necessary

(Section 8.5) nor sufficient (Section 8.6) for Nash-solvability. Of course, this is also true for n-person game forms with $n \geq 3$, since one can get each such game form from a 3-person one by simply introducing $n-3$ dummy-players.

Here we extend these negative results and show that, in principle, Nash-solvability of a 3-person game form g is not uniquely defined by its EFF $\mathcal{E}_g$. Namely, we construct two 3-person game forms g and g' such that g is Nash-solvable, while g' is not, although $\mathcal{E}_g = \mathcal{E}_{g'}$. We take g' from Section 8.5 and define g by the following 3-dimensional table.

$$\begin{array}{ccc} a_2a_1a_2 & a_2a_2a_1 & a_2a_1a_2 \\ a_1a_2a_1 & a_2a_1a_2 & a_1a_2a_2 \\ a_2a_2a_2 & a_1a_2a_1 & a_2a_1a_2 \end{array}$$

Thus, g and g' have the same 3 players and 2 outcomes. Yet, in g each player $i \in I$ has 3 (instead of 2) strategies, $X_i = \{0,1,2\}$; furthermore, $g(x) = g(x_1,x_2,x_3) = a_2$ when $x_1 + x_2 + x_3$ is even and also in three "odd" situations $x \in \{(1,2,0),(0,0,1),(2,1,2)\}$; otherwise $g(x) = a_1$. It is easy to verify that g and g' have the same EFF given in Section [?]. Indeed, each two players are effective for a_2, while one player can only trivially guarantee $A = \{a_1, a_2\}$.

It is also easy to verify that if $g(x) = a_1$ then each player can switch to a_2 by choosing another strategy and if $g(x) = a_2$ then at least two of three players can switch to a_1. This observation implies that, unlike g', game form g is not Nash-solvable. Indeed, let us consider a utility function u such that two players prefer a_1 to a_2 and one has the opposite preference. It is clear that situation x cannot be a Nash equilibrium in both cases, $g(x) = a_1$ or $g(x) = a_2$.

Now, let us take g' from Section 8.6 and define g by the following 3-dimensional table

$$\begin{array}{ccc} a_1a_1a_1 & a_1a_2a_3 & a_1a_xa_x \\ a_2a_2a_2 & a_1a_2a_3 & a_xa_2a_x \\ a_3a_3a_3 & a_1a_2a_3 & a_xa_xa_3 \end{array}$$

Thus, g and g' have the same 3 players and 3 outcomes. We assume that the outcomes labeled by a_x can take arbitrary (perhaps, different) values in $A = \{a_1, a_2, a_3\}$. Yet, in g each player $i \in I$ has 3 (instead of 6) strategies.

It is easy to verify that g and g' have the same EFF given in Section 8.6. Indeed, each of two players is effective for every outcome, while one player can only trivially guarantee the whole set $A = \{a_1, a_2, a_3\}$.

It is also easy to verify that g is Nash-solvable. Indeed, without loss of generality we can assume that $u(1,a_1) \geq u(1,a_2) \geq u(1,a_3)$. Then "the upper left" situation x is a Nash equilibrium. Indeed, $g(x) = a_1$ and it is easy to see that a_1 remains whenever player 2 or 3 chooses any other strategy. Unlike them, player 1 by changing the strategy can get both a_2 or a_3. Yet, (s)he is not interested, since a_1 is the best outcome for 1.

The above two examples show that among two game forms with the same EFF one may be Nash-solvable, while the other one not. Let us also note that the EFF is self-dual in the second example, while in the first one it is not.

Remark 9 *It is an interesting general question which properties of game forms (and other structures) are uniquely defined by the corresponding EFFs. For example, the core of a cooperative game $C(\mathcal{E}, u)$ by the definition depends only on the EFF $\mathcal{E}$ and utility function u; see e.g. [34, 33, 35]. By Theorem 1, a 2-person game form g is Nash-solvable if and only if its EFF $\mathcal{E}_g$ is self-dual. However, this result does not generalize the case of 3-person game forms. In [23], the class of veto voting schemes is considered for which the result of elections is uniquely defined by the corresponding effectivity (equivalently, veto) function. Somewhat surprisingly, not only game structures but also quite different objects may have properties uniquely defined by some EFFs. For example, in [6, 8], an EFF $\mathcal{E}_G$ is assigned to each graph G and it is shown that such properties of G as perfectness or kernel-solvability depend only on $\mathcal{E}_G$.*

9 Appendix 2. Proof of Proposition 12

First, let us notice that all six decision problems, (b11), (b12), (b21), (b22), (b) and (a), of Proposition 12 are in co-NP. Indeed, given a digraph $\overrightarrow{G} = (V, \overrightarrow{E})$, positions $j', j'' \in V$, and partitions $P : V = V_1 \cup V_2$ and $Q : V = V^1 \cup V^2$, all conditions, (i),(ii),(iii), (b11), (b12), (b21), (b22), and hence, (a) and (b), of Section 1.6 can be easily verified in linear time.

Now we have to prove that each of the six decision problems, (b11), (b12), (b21), (b22), (b), and (a), is NP-hard. First, we consider (b21) and (b), then, by a trivial modification, extend the result to (b11), (b22), and (b12), and finally, show that NP-hardness of (b12) implies NP-hardness of (a). Let us polynomially reduce (b21) from the following NP-hard problem on verifying Boolean inequalities.

Proposition 21 *Given a monotone DNF $D = D_1 \vee \ldots \vee D_N = \bigvee_{n=1}^{N} D_n$ and CNF $C = C_1 \wedge \ldots \wedge C_M = \bigwedge_{m=1}^{M} C_m$ of common variables $\{x_1, \ldots, x_K\} = \{x_k \mid k \in [K] = \{1, \ldots K\}\}$, it is co-NP-complete to verify the inequality $C \leq D$. In contrast, $D \leq C$ can be always checked in linear time.*

Proof First, let us show that inequality $C \geq D$ can be verified in linear time. Indeed, let us choose an $n \in [N] = \{1, \ldots, N\}$, set all variables of D_n to 1, and all other to 0. Then, obviously, $D = 1$. It is also clear that $C \geq D$ does not hold whenever $C_m = 0$ for some $m \in [m] = \{1, \ldots, M\}$ and $n \in [n]$. Otherwise, $C = 1$ whenever $D = 1$, that is, $C \geq D$.

Now let show that verifying inequality $C \leq D$ is co-NP-complete. Obviously, $C \not\leq D$ if and only if $1 = C(x) > D(x) = 0$ for some assignment x. Given x, this inequality can be checked trivially. Hence, verifying $C \leq D$ is in co-NP.

We will show that verifying $C \leq D$ is NP-hard already in the special case when $D = D_0 = x_1y_1 \vee \ldots \vee x_Ny_N$. Indeed, given an arbitrary (non-monotone) CNF C' of variables $x_1, \ldots, x_N$, let us substitute y_n for $\overline{x}_n$ in C' for all $n = 1, \dots, N$ and denote the obtained monotone CNF by C. Obviously, $C \not\leq D_0$ if and only if C' is satisfiable, which is NP-complete to verify. □

To derive Proposition 12 from Proposition 21 we assign a Boolean variable x_j to each position $j \in V$ and consider the following system of Boolean equations:

$$x_j = \bigvee_{j' \in N(j)} x_{j'} \quad \text{for } j \in V_1 \quad \text{and} \quad x_j = \bigwedge_{j'' \in N(j)} x_{j''} \quad \text{for } j \in V_2, \tag{5}$$

where $N(j) \subseteq V$ is the set of all successors of a position $j \in V$.

This system has two trivial solutions: $x_j \equiv 0$ and $x_j \equiv 1$ for all $j \in V$.

Lemma 14 *(a) A pair $(\overrightarrow{G}, P)$ is ergodic if and only if system (5) has only trivial solutions.*

(b) Moreover, $j' \leq j''$ for positions $j', j'' \in V$ if and only if there is no solution of (5) such that $x_{j'} = 1$ and $x_{j''} = 0$.

Proof (a) Indeed, all non-trivial solutions of (5) and all contra-ergodic partitions $Q : V = V^1 \cup V^2$ of $(\overrightarrow{G}, P)$ are in one-to-one correspondence defined by the following simple rule: $j \in V^1$ if and only if $x_j = 1$ (and, respectively, $j \in V^2$ if and only if $x_j = 0$).

(b) Furthermore, $j' \leq j''$ if and only if both inclusions, $j' \in V^1$ and $j'' \in V^2$, hold for no contra-ergodic partition $Q : V = V^1 \cup V^2$. □

Now, we can derive Proposition 12 from Proposition 21 and Lemma 14.

Given a monotone DNF $D = D_1 \vee \ldots \vee D_N = \bigvee_{n=1}^{N} D_n$ and CNF $C = C_1 \wedge \ldots \wedge C_M = \bigwedge_{m=1}^{M} C_m$ of common variables $\{x_1, \ldots, x_K\} = \{x_k \mid k = 1, \ldots K\}$, we will construct a bipartite pair $(\overrightarrow{G}, P)$ as follows. For each $k = 1, \ldots, K$, let us assign to the variable x_k two positions $w_k \in V_1, b_k \in V_2$ and two arcs $(w_k, b_k), (b_k, w_k)$ between them. Furthermore, let us assign a position $c_m \in V_1$ to each implicate $C_m,\ i = 1, \ldots, M$, of CNF C and, respectively, position $d_n \in V_2$ to each implicant $D_n,\ n = 1, \ldots, N$, of DNF D. Then let us introduce an arc (c_m, b_k) (respectively, (d_n, w_k)) if and only if implicate C_m of C (respectively, implicant D_n of D) contains variable x_k. Finally, let us introduce two more positions $c_0 \in V_2, d_0 \in V_1$ and arcs (c_0, c_m) for all $m = 1, \ldots, M$ and (d_0, d_n) for all $n = 1, \ldots, N$.

By construction, the obtained directed graph is bipartite. Furthermore, by (5), $x_{d_0} = D$ and $x_{c_0} = C$. Hence, by Lemma 14 (b), we have $c_0 \leq d_0$ if and only if $C \leq D$. By Proposition 21, the last condition is co-NP-hard to verify. Thus, setting $j' = c_0$ and $j'' = d_0$, we conclude that problems (b21) and (b) of Proposition 12 are co-NP-complete.

Although, by the above construction, position d_0 is placed in V_1 and c_0 in V_2, it is easy to “replace” one of them or both. To do so, let us just introduce two new positions $c_0' \in V_1,\ d_0' \in V_1$ and arcs $(c_0', c_0),\ (d_0', d_0)$. Then, obviously, $x_{d_0'} = x_{d_0} = D$ and $x_{c_0'} = x_{c_0} = C$, by (5). Thus, Proposition 21 implies parts (b11), (b12), (b21), (b22), and (b) of Proposition 12.

We have to derive part (a), yet. Given an arbitrary bipartite pair $(\overrightarrow{G}, P)$ and two positions $j' \in V_1, j'' \in V_2$, let us introduce the following new arcs: (j, j') for each $j \in V_2$ and (j, j'') for each $j \in V_1$. Obviously, the obtained pair $(\overrightarrow{G}^+, P)$ is bipartite, too. The following statement was given without proof in [25].

Lemma 15 *A partition $Q : V = V^1 \cup V^2$ is contra-ergodic in $(\overrightarrow{G^+}, P)$ if and only if it is a contra-ergodic partition in $(\overrightarrow{G}, P)$ such that $j' \in V^1$ and $j'' \in V^2$.*

Proof "If part". Let $Q : V = V^1 \cup V^2$ be a contra-ergodic partition of $(\overrightarrow{G}, P)$ such that $j' \in V^1$ and $j'' \in V^2$. Then Q is contra-ergodic in $(\overrightarrow{G^+}, P)$, too. Indeed, extending $(\overrightarrow{G}, P)$ to $(\overrightarrow{G}^+, P)$ we obviously respect (i) and (iii). Moreover, (ii) also holds, since the new moves do not enable player 1 to leave V^2 for V^1 (although, she can now move from each $j \in V_1 \cap V^1$ to $j'' \in V^2$); respectively, player 2 still cannot leave V^1 for V^2 (although, he can now move from each $j \in V_2 \cap V^2$ to $j' \in V^1$).

"Only if part". Let partition $Q : V = V^1 \cup V^2$ be contra-ergodic in $(\overrightarrow{G}^+, P)$. Then, it is easy to see that $j' \in V^1$ and $j'' \in V^2$, since otherwise (ii) could not hold for Q. Let us show that Q is a contra-ergodic partition of $(\overrightarrow{G}, P)$, too. Indeed, (i) and (ii) obviously hold. Suppose that (iii) does not. This could happen only if in $(\overrightarrow{G}, P)$ there is a forced move from $j \in V^1 \cap V_1$ to V_2 or from $j \in V^2 \cap V_2$ to V_1, while in $(\overrightarrow{G}^+, P)$ this move is not forced, due to extra arcs. Yet, all these extra arcs are either between V^1 and V^2, or from $j \in (V^1 \cap V_1) \cup (V^2 \cap V_2)$. Thus, (iii) holds for Q, too, and hence, Q is a contra-ergodic partition. □

Obviously, Proposition 12 follows from Proposition 21 and Lemma 15. □

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