Bang-bang and Singular Controls

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Bang-Bang Control

A solution to a problem that is linear in the control frequently involves discontinuities in the optimal control

$$\max_u \int_0^T [f_1(t,x) + u f_2(t,x)]\, dt$$

$$x' = g_1(t,x) + u g_2(t,x)$$

$$x(0) = 1$$

$$a \le u(t) \le b$$

Contd.

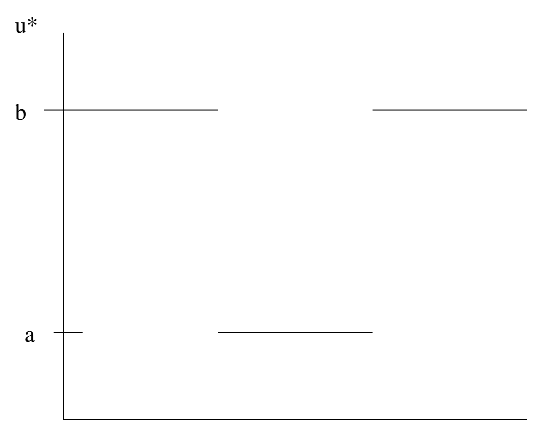
$$\begin{aligned} H &= f_1(t,x) + u f_2(t,x) + \lambda\left(g_1(t,x) + u g_2(t,x)\right) \\ &= u\left(f_2(t,x) + \lambda g_2(t,x)\right) + f_1(t,x) + \lambda g_1(t,x) \\ &= u\phi(t,x,\lambda) + \quad \text{rest} \;, \end{aligned}$$

where ϕ is switching function. Maximize H w.r.t. u at u^*

$$u^*(t) = \begin{cases} a & \text{if } \phi(t,x^*,\lambda) < 0 \\ ? & \text{if } \phi(t,x^*,\lambda) = 0 \\ b & \text{if } \phi(t,x^*,\lambda) > 0 \end{cases}$$

If $\phi(t, x^*, \lambda(t)) = 0$ is not sustained over an interval of time, then the control is bang-bang.

Bang-bang always at the extreme values of the control set.



If $\phi(t, x^*, \lambda(t)) = 0$ over an interval of time, the value of u^* is singular.

The choice of u^* must be obtained from other information than " max H w.r.t. u".

The times when the OC switches from a to b or vice-versa or switches to singular control are called switch times.
(Sometimes difficult to find).

Example 1

$$\max \int_0^T (1-u)x\,dt \quad \text{max sales}$$

$$x' = ux \qquad x(0) = x_0 > 0$$

$x(t)$ stock can be reinvested to expand capacity or sold for revenue and $x(t) > 0$

$u(t)$ fraction of stock to be reinvested

$$0 \le u(t) \le 1$$

$$H = (1-u)x + \lambda ux$$

Hamiltonian

$$H = u(x(\lambda - 1)) + x$$

$$\frac{\partial H}{\partial u} = x(\lambda - 1)$$

$$\lambda' = -\frac{\partial H}{\partial x} = -(u(\lambda - 1) + 1) = u - 1 - \lambda u, \quad \lambda(T) = 0$$

when $\lambda(t) - 1 > 0, \quad u^* = 1 \quad \Rightarrow \lambda' = -\lambda$

when $\lambda(t) - 1 < 0, \quad u^* = 0 \quad \Rightarrow \lambda' = -1$

Can $\frac{\partial H}{\partial u} = 0$ on a subinterval?
Then $\lambda = 1$ and $\lambda' = 0$ on that subinterval.

Contradicts adjoint ODE. No singular case here

Bang-bang

Either $\lambda > 1$ giving $u = 1$ and $\lambda' = -\lambda$ or $\lambda < 1$ giving $u = 0$ and $\lambda' = -1$.

λ is decreasing and $\lambda(T) = 0$.

On $[\hat{t}, T], \quad \lambda < 1 \quad \lambda = T - t$

$u^* = 0, \quad x^* = x^*(\hat{t})$.

Hamiltonian

Switch time $T-1$

$$\text{If} \quad T-1 \leq 0, \quad T \leq 1$$

$$u^* \equiv 0$$

$$x^* = x_0$$

If $T > 1$

$u^* = 1$ $\qquad\qquad$ $u^* = 0$

0 $\qquad\qquad$ $T-1$ $\qquad\qquad$ T

$x^* = x_0 e^t$ $\qquad\qquad$ $x^* = x_0 e^{T-1}$

Example 2

$$\min_{-1\le u\le 1}\int_0^1 (2-5t)u(t)\,dt$$

$$x' = 2x + 4te^{2t}u$$

$$x(0) = 0 \quad x(1) = e^2$$

$$H = (2-5t)u + \lambda\left(2x + 4te^{2t}u\right)$$

$$\lambda' = -\frac{\partial H}{\partial x} = -2\lambda$$

$\lambda = \lambda_0 e^{-2t}$ no transversality condition

$$H = \left(2 - 5t + 4\lambda t e^{-2t}\right) u + 2\lambda x$$

$$H = (2 - 5t + 4\lambda_0 t)\, u + 2\lambda x$$

$$H = (2 + 4\lambda_0 t - 5t)\, u + 2\lambda x$$

$(2 + 4\lambda_0 t - 5t)$ is switching function and it will switch from $+$ to $-$ at most once.

at $t = 0, \quad 2 + (4\lambda_0 - 5)(0) > 0,$

Initially $\frac{\partial H}{\partial u} > 0$, $u^* = -1$ on $[0, \hat{t})$

one case

If $u^* \equiv -1$ on $[0, 1]$

$$x' \text{ DE } \;\&\; x(0) = 0$$
$$\Rightarrow x = -2t^2 e^{2t}.$$

That solution does not satisfy $x(1) = e^2$.

There must be one switch.

On $(\hat{t}, 1], 2 + (4\lambda_0 - 5)t < 0, \quad u^* = 1$

$$x' \text{ DE } \ \& \ x(1) = e^2 \Rightarrow x = e^{2t}(2t^2 - 1).$$

x^* must be continuous at $\hat{t}$

$$e^{2\hat{t}}\left(2\hat{t}^2 - 1\right) = -2\hat{t}^2 e^{2\hat{t}} \Rightarrow \hat{t} = \frac{1}{2}$$

Switching function $= 0$ at $\hat{t}$

$$2 + (4\lambda_0 - 5)\hat{t} = 0 \Rightarrow \lambda_0 = \frac{1}{4}$$

Optimal solution

$$u^* = -1 \qquad\qquad u^* = 1$$

$$0 \qquad\qquad 1/2 \qquad\qquad 1$$

$$x^* = -2t^2 e^{2t} \qquad\qquad x^* = e^{2t}(2t^2 - 1)$$

Basic resource Model

(Clark p. 95) “Fishery”

$$\max \int_0^T e^{-\delta t}(pqx - c)E\, dt$$

$$\frac{dx}{dt} = F(x) - qEx$$

E control (effort), p price and q “catchability”

$$0 \le E(t) \le E_{\max}$$

max discounted profit

revenue - cost

$$H = e^{-\delta t}(pqx - c)E + \lambda(F(x) - qEx)$$
$$= \left[e^{-\delta t}(pqx - c) - \lambda qx\right] E + \lambda F(x).$$

Singular control occurs when the coefficient of control E is zero over a time interval (switching function is zero).

when $e^{-\delta t}(pqx - c) - \lambda qx > 0 \quad E^* = E_{\max}$

$e^{-\delta t}(pqx - c) - \lambda qx < 0 \quad E^* = 0$

$e^{-\delta t}(pqx - c) - \lambda qx = 0$ singular case

Singular Case

Suppose $\frac{\partial H}{\partial u} = 0$

$$e^{-\delta t}(pqx - c) - \lambda qx = 0$$

on a time interval. Solve for λ

$$\lambda = e^{-\delta t}\left(p - \frac{c}{qx}\right)$$

$$\frac{d\lambda}{dt} = e^{-\delta t}(-\delta)\left(p - \frac{c}{qx}\right) + e^{-\delta t}\frac{c}{qx^2}\frac{dx}{dt}$$

$$= e^{-\delta t}\left[-\delta\left(p - \frac{c}{qx}\right) + \frac{c}{qx^2}(F(x) - qEx)\right]$$

From necessary conditions

$$\frac{d\lambda}{dt} = -\frac{\partial H}{\partial x} = -\left[e^{-\delta t} pqE + \lambda(F'(x)qE)\right]$$
$$= -\left[e^{-\delta t} pqE + e^{-\delta t}\left(p - \frac{c}{qx}\right)(F'(x) - qE)\right]$$

after substituting in λ. Set 2 expressions for $\frac{d\lambda}{dt}$ equal (cancel $e^{-\delta t}$)

$$-\delta\left(p - \frac{c}{qx}\right) + \frac{c}{qx^2}(F(x) - qEx)$$
$$= -pqE + \left(p - \frac{c}{qx}\right)(qE - F'(x))$$

Term involving control E cancel

$$-\delta p + \frac{\delta c}{qx} + \frac{c}{qx^2} F(x) = F'(x) \left(\frac{c}{qx} - p \right)$$

$$\vdots$$

$$F'(x) + \frac{cF(x)}{x(pqx - c)} = \delta$$

optimal state should satisfy this equation when in the singular control case.
(find x^* and use state DE to find u^*).

Simple case

$$F(x) = x(1-x)$$

$p = q = 1, c = 0$ ignoring cost of fishing

Singular case

$$F'(x) + \frac{cF(x)}{x(pqx - c)} = \delta$$

$$1 - 2x + 0 = \delta \Rightarrow x^* = (1-\delta)/2$$

$\Rightarrow (x^*)' = 0$ during singular case

$$x^*(1 - x^*) - E^* x^* = 0 \quad \Rightarrow 1 - x^* - E^* = 0$$

During singular case

$$1 - x^* - E^* = 0$$

$$x^* = \frac{1-\delta}{2}$$

$$1 - \left(\frac{1-\delta}{2}\right) - E^* = 0 \Rightarrow E^* = \frac{1+\delta}{2}$$

If $\delta = 0$, singular case

$$x^* = 1/2$$

$$E^* = 1/2$$

Back to Bioreactor Model

$$x' = Gux - x^2 \quad \text{state (bacteria)}$$

$$x(0) = x_0$$

$$\max \int_0^T (Kx(t) - u(t)\, dt$$

$$0 \le u(t) \le M \quad \text{control (nutrient input)}$$

$$H = Kx - u + \lambda(Gux - x^2)$$

$$\lambda' = -\frac{\partial H}{\partial x} = -(K + \lambda(Gu - 2x))$$

$$\lambda(T) = 0$$

$$H = u(G\lambda x - 1) + kx - \lambda x^2$$

If $G\lambda x^* - 1 = 0$ on a time interval, singular control u^* may satisfy $0 < u^*(t) < M$

$$(\lambda x^*)' = (\lambda x^* - K)x^* \quad \text{using } \lambda, x \text{ DE}$$

$$(\lambda x^*)(T) = 0$$

$$(\lambda x^*)(t) = K\left[1 - \exp\left(-\int_t^T x^*(s)\,ds\right)\right]$$

$$0 \leq (\lambda x^*)(t) < K$$

$(\lambda x^*)(t)$ is a strictly decreasing function. Thus

$$G\lambda x - 1 = 0$$

cannot be maintained on an interval. No singular case here.

Switch

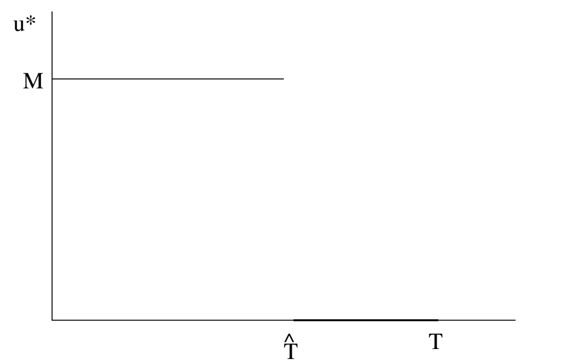
One switch occurs when

$$(\lambda x^*)(t) = \frac{1}{G}$$

$$(\lambda x^*)(t) = K\left[1 - \exp\left(-\int_t^T x^*(s)\,ds\right)\right]$$

Can show if $\frac{1}{G} < K$ and T sufficiently large, there is a switch.

If $KG > 1$
K or G large
Where K is decay rate of contaminant and G is growth rate of bacteria
and T sufficiently large, there is a switch



Otherwise $u^* \equiv 0$ i.e. no nutrient feeding.

Free Terminal Time

The final time T is part of the unknowns.
For example, steer a system from one position to another position in minimum time.

EXTRA condition is the Hamiltonian at the final T is 0.

Exercise

Solve with a partner now

$$\max_u \int_0^2 (2x - 3u)\,dt$$

subject to $x' = u + x$ and $x(0) = 5$ and $0 \leq u(t) \leq 2$.