Nash-solvability and Boolean duality

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Let $I = \{1, \ldots, n\}$ and $A = \{a_1, \ldots, a_p\}$ be sets of *players (voters)* and *outcomes (candidates)*, respectively.

A *utility function* is a mapping $u : I \times A \to \mathbb{R}$; the value $u(i, a)$ is interpreted as the profit of player $i \in I$ in case when outcome $a \in A$ is realized.

Furthermore, let X_i be a (finite) set of *strategies* of player $i \in I$.

A *game form* is a mapping $g : X \to A$, where $X = X_1 \times \ldots \times X_n$ is the set of strategy profiles (situations) $x = (x_1, \ldots, x_n)$.

A *game in normal form* is a pair (g, u).

A *Nash equilibrium* is a strategy profile $x \in X$ such that $u(i, x) \geq u(i, x')$ for every $i \in I$ and each $x' \in X$ such that $x'_j = x_j$ for all $j \in I \setminus \{i\}$. In other words, no player $i \in I$ can make profit by substituting a new strategy x'_i for x_i if all other players $j \in I \setminus \{i\}$ keep the same strategies.

Game form g is called *Nash-solvable* if the corresponding game (g, u) has a Nash equilibrium for every utility function u.

Given a game form g, let us assign a Boolean variable to each outcome $a \in A$ and for every coalition $K \subseteq I$ define a Boolean function F_K by the following monotone DNF: $F_K = \vee_{x_K} \wedge_{x_{I \setminus K}} g(x_K, x_{I \setminus K})$, where $x_K = (x_i; i \in K)$ and $x_{I \setminus K} = (x_i; i \notin K)$ are strategies of coalitions K and $I \setminus K$ respectively.

Theorem. A two-person ($n = 2$) game forms is Nash-solvable if and only if Boolean functions F_1 and F_2 are dual, $F_1^d = F_2$.

In particular, this result implies that, for $n = 2$, Nash-solvability for arbitrary u is equivalent to Nash-solvability for zero-sum u (that is, $u(1, a) + u(2, a) = 0$ for all $a \in A$) that take only values $+1$ (win) and -1 (lose).

Interestingly, for $n > 2$ duality $F_K^d = F_{I \setminus K}$ for all (or some) coalitions $K \subseteq I$ is not necessary nor sufficient for Nash-solvability.

In this talk we consider applications of the above old theorem to positional game forms with perfect information modeled by digraphs. In case of acyclic digraphs, Nash-solvability is well-known. This result is referred to as Zermelo (1912), von Neumann (1944), and Kuhn (1953) Theorem. Yet, if digraph G contains directed cycles (dicycles) then not much is known.

Every terminal (dead-end) position of G is, by definition, an outcome. Also dicycles are outcomes. We consider two cases:

(i) Each dicycle is a separate outcome.

Then, we obtain necessary and sufficient conditions of Nash-solvability for $n = 2$ assuming also that digraph $G = (V, E)$ is bi-directed: $(u, v) \in E$ iff $(v, u) \in E$; Boros, VG, Makino, and Shao, RRR-30-2007.

(ii) All dicycles form one outcome.

Nash-solvability in this case is an open problem. Yet, for $n = 2$ it easily follows from the above theorem. This observation is due to Gimbert and Sorensen, 2008.