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On effectivity functions of game forms

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On effectivity functions of game forms

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Abstract. To each game form g an effectivity function (EFF) E_g can be naturally assigned. An EFF E will be called *formal* (respectively, formal-minor) if $E = E_g$ (respectively, $E \leq E_g$) for a game form g.
(i) An EFF is formal if and only if is superadditive and monotone.
(ii) An EFF is formal-minor if and only if it is weakly superadditive.
Theorem (ii) looks more sophisticated, yet, it is simpler and instrumental in the proof of (i). In addition, (ii) has important applications in social choice, game, and even graph theories. Constructive proofs of (i) were given by Moulin, in 1983, and by Peleg, in 1998. (Peleg's proof works also in case of an infinite set of outcomes.) Both constructions are elegant, yet, the set of strategies X_i of each player $i \in I$ in g might be doubly exponential in size of the input EFF E. In this paper, we suggest a third construction such that $|X_i|$ is only linear in the size of E.
One can verify in polynomial time whether an EFF is formal (or superadditive); in contrast, verification of whether an EFF is formal-minor (or weakly superadditive) is a CoNP-complete decision problem.
Keywords: effectivity function, monotone, superadditive, weakly superadditive, self-dual, maximal; game form, tight, totally tight

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1 Introduction

The effectivity function (EFF) is an important concept of voting theory that describes the distribution of power between the voters and candidates. This concept was introduced in the early 80s by Abdou [1, 2], Moulin and Peleg [21], [20] Chapter 7, [22], [23] Chapter 6. We also refer the reader to the book "Effectivity Functions in Social Choice" by Abdou and Keiding [3] for numerous applications of EFFs in the voting and game theories.

An EFF can be viewed as a Boolean function whose set of variables is the mixture of the voters (players) and candidates (outcomes); see Section 2.1.

A game form g can be viewed as a game in normal form in which no payoffs are defined yet and only an outcome $g(x)$ is associated with each strategy profile x. To every game form g an EFF E_g can be naturally assigned; see Section 4.

Some important properties of g depend only on its EFF E_g; for example, the existence of the core or (in case of two players) Nash equilibria for an arbitrary payoff; see [20] Chapter 7, [23] Chapters 6, [3] Chapter 3, and [12, 13] and also [17] Section 4.

It is a natural and important problem to characterize the EFFs related to game forms. Already in [21] it was mentioned that for each game form g its EFF E_g is monotone and superadditive. The inverse statement is true too, yet, it is more difficult.

An EFF E will be called *formal* (respectively, formal-minor) if $E = E_g$ (respectively, $E \leq E_g$) for a game form g. The following two claims hold:

(i) An EFF is formal if and only if is superadditive and monotone;

(ii) An EFF is formal-minor if and only if it is weakly superadditive.

In both cases the EFFs must satisfy some natural "boundary conditions"; see Sections 2.2 and 2.3 for the definitions and more details.

Theorem (ii) looks more sophisticated, yet, it is simpler and instrumental in the proof of (i). In addition, (ii) has important applications in social choice, game, and even graph theories; see [20] Chapter 7 and [4, 5, 6].

Constructive proofs of (i) were given by Moulin, in 1983, and by Peleg, in 1998. (In fact, Peleg proved a slightly more general statement that includes, in particular, the case of infinite sets of outcomes.) Both constructions are interesting and elegant, yet, in both, the set of strategies X_i of each player $i \in I$ in g is doubly exponential in size of the input EFF E. In this paper, we suggest a third construction such that $|X_i|$ is only linear in the size of E.

Furthermore, an EFF E will be called T-formal (TT-formal) if $E = E_g$ for a tight (totally tight (TT)) game form g; see Sections 8 and 9 for definitions. Obviously, the families of TT-formal, T-formal, and formal EFFs are nested, since every TT game form is tight; see Section 9.

Moulin's results readily imply that an EFF is T-formal if and only if it is maximal, superadditive, monotone, and satisfies the boundary conditions. In this paper, we add to this list one more property, which also holds for each TT-formal EFF, and show that the extended list of properties is a characterization of the two-person TT-formal EFFs, leaving the n-person case open.

2 Basic properties

2.1 Effectivity functions as Boolean functions of players and outcomes

Given a set of players (or voters) $I = \{1, \ldots, n\}$ and a set of outcomes (or candidates) $A = \{a_1, \ldots, a_p\}$, subsets $K \subseteq I$ are called *coalitions* and subsets $B \subseteq A$ *blocks*. An *effectivity function* (EFF) is defined as a mapping $\mathcal{E} : 2^I \times 2^A \to \{0, 1\}$. We say that coalition $K \subseteq I$ is effective (respectively, not effective) for block $B \subseteq A$ if $\mathcal{E}(K, B) = 1$ (respectively, $\mathcal{E}(K, B) = 0$).

Since $2^I \times 2^A = 2^{I \cup A}$, we can say that EFF $\mathcal{E}$ is a Boolean function whose set of variables $I \cup A$ (of cardinality $n + p$) is a mixture of the players and outcomes.

An EFF describes the distribution of power of voters and of candidates.

For two EFFs E and E' on the same variables $I \cup A$, obviously, the implication $E' = 1$ whenever $E = 1$ is equivalent with the inequality $E \leq E'$.

The "complementary" function, $\mathcal{V}(K, B) \equiv \mathcal{E}(K, A \setminus B)$, is called the *veto function*; by definition, K is effective for B if and only if K can veto $A \setminus B$. Both names are frequent in the literature [1, 2, 9, 14, 15, 16, 20, 21, 22, 23].

2.2 Boundary conditions

The complete ($K = I,\ B = A$) and empty ($K = \emptyset,\ B = \emptyset$) coalitions and blocks will be called *boundary* and play a special role. From now on, we assume that the following *boundary conditions* hold for all considered EFFs:

$E(K, \emptyset) = 0$ and $E(K, A) = 1\ \forall\ K \subseteq I$;

$E(I, B) = 1$ unless $B = \emptyset$; $E(\emptyset, B) = 0$ unless $B = A$;

$E(I, \emptyset) = 0,\ \ E(\emptyset, A) = 1.$

In fact, the value of $E(\emptyset, A)$ is irrelevant. However, in Section 8 we will define self-duality (maximality) of an EFF by the equation

$$E(K, B) + E(I \setminus K, A \setminus B) \equiv 1 \text{ for all } K \subseteq I,\ B \subseteq A.$$

Thus, formally, since $E(I, \emptyset) = 0$, we have to set $E(\emptyset, A) = 1$, otherwise self-duality will never hold.

2.3 Monotonicity and the minimum monotone majorant of an effectivity function

An EFF is called *monotone* if the following implication holds:

$$\mathcal{E}(K,B)=1,\ K\subseteq K'\subseteq I,\ B\subseteq B'\subseteq A \ \Rightarrow\ \mathcal{E}(K',B')=1.$$

It is easy to see that the above definition is in agreement with the standard concept of monotonicity of Boolean functions.

A (monotone) Boolean function can be given by the set of its (minimal) true vectors. Respectively, a (*monotone*) EFF E can be given by the list $\{(K_j,B_j);\ j\in J\}$ of all (inclusion-minimal) pairs $K_j\subseteq I$ and $B_j\subseteq A$ such that $E(K_j,B_j)=1$. Let us remark that $\mathcal{K}_E=\{K_j;\ j\in J\}$ and $\mathcal{B}_E=\{B_j;\ j\in J\}$ are multi-hypergraphs whose edges, labeled by the indices $j\in J$, might be not pairwise distinct.

It is also clear that for each EFF E there is a unique minimum monotone EFF E^M such that $E^M\geq E$. This EFF is defined by formula:

$E^M(K^M,B^M)=1$ iff $E(K,B)=1$ for some $K\subseteq K^M\subseteq I$, $B\subseteq B^M\subseteq A$

and is called the minimum monotone majorant of E.

3 Superadditive and weakly superadditive EFFs

3.1 Superadditivity

An EFF E is called *2-superadditive* if the following implication holds:

$$\mathcal{E}(K_1,B_1)=\mathcal{E}(K_2,B_2)=1,\ K_1\cap K_2=\emptyset\ \Rightarrow\ \mathcal{E}(K_1\cup K_2,B_1\cap B_2)=1.$$

More generally, an EFF E is called *k-superadditive* if, for every set of indices J of cardinality $|J|=k\geq 2$, the following implication holds:

if $E(K_j,B_j)=1\ \forall j\in J$ and coalitions $\{K_j;\ j\in J\}$ are pairwise disjoint
(that is, $K_{j'}\cap K_{j''}=\emptyset\ \forall j',j''\in J$ such that $j'\neq j''$) then

$$E(\bigcup_{j\in J}K_j,\bigcap_{j\in J}B_j)=1.$$

In particular, $\bigcap_{j\in J}B_j\neq\emptyset$, since otherwise the boundary condition $E(K,\emptyset)=0$ would fail. By induction on k, it is easy to show that 2-superadditivity implies k-superadditivity for all $k\geq 2$. An EFF satisfying these properties is called *superadditive*.

3.2 Weak superadditivity

Furthermore, an EFF E is called *weakly superadditive* if for every set of indices J the following implication holds:

if $E(K_j, B_j) = 1 \; \forall \, j \in J$ and coalitions $\{K_j;\ j \in J\}$ are pairwise disjoint then

$$\bigcap_{j \in J} B_j \neq \emptyset.$$

Let us remark that weak superadditivity (in contrast to superadditivity) cannot be reduced to the case $|J| = 2$. For example, an EFF E such that

$E(\{1\}, \{a_2, a_3\}) = E(\{2\}, \{a_3, a_1\}) = E(\{3\}, \{a_1, a_2\}) = 1$

is not weakly superadditive, since otherwise $E((\{1, 2, 3\}, \emptyset) = 1$, yet, EFF E might be weakly **2**-superadditive.

Finally, let us note that superadditivity implies weak superadditivity; indeed, otherwise boundary conditions $E(K, \emptyset) = 0$ would not hold. However, the inverse implication fails. For example, an EFF E such that

$E(\{1\}, \{a_2, a_3\}) = E(\{2\}, \{a_3, a_1\}) = 1$, while $E(\{1, 2\}, \{a_3\}) = 0$

is not superadditive but might be weakly superadditive.

3.3 On complexity of verifying (weak) superadditivity

It is a CoNP-complete problem to verify whether a *monotone* EFF E is weakly superadditive; see [5] Theorem 12, Lemma 28, and Remarks 10 and 29.

In contrast, one can easily verify in cubic time whether a (monotone) EFF $E = \{(K_j, B_j);\ j \in J\}$ is superadditive. Indeed, as we know, superadditivity of E is equivalent with its 2-superadditivity and the latter can be verified in cubic time just according to the definition.

3.4 On (weak) superadditivity of a minorant of an EFF

Proposition 1 *If an EFF E is weakly superadditive and $E' \leq E$ then EFF E' is weakly superadditive, too.*

Proof. Let J be a set of indices and $E'(K_j, B_j) = 1$ for each $j \in J$, where coalitions $\{K_j;\ j \in J\}$ are pairwise disjoint. Then $E(K_j, B_j) = 1$ for each $j \in J$, too, since $E \geq E'$. Hence, $\bigcap_{j \in J} B_j \neq \emptyset$, since E is weakly superadditive. Thus, E' is weakly superadditive, too. □

However, the above arguments do not extend to superadditivity, since

$$E\left(\bigcup_{j\in J} K_j, \bigcap_{j\in J} B_j\right) = 1 \text{ and } E' \le E \;\not\Rightarrow\; E'\left(\bigcup_{j\in J} K_j, \bigcap_{j\in J} B_j\right) = 1.$$

For example, let us consider EFFs E and E' such that

$$E(\{1\},\{a_2,a_3\}) = E(\{2\},\{a_3,a_1\}) = E'(\{1\},\{a_2,a_3\}) = E'(\{2\},\{a_3,a_1\}) = 1;$$
$$1 = E(\{1,2\},\{a_3\}) > E'(\{1,2\},\{a_3\}) = 0.$$

Obviously, EFF E' is not superadditive, while EFF E might be superadditive and inequality $E' < E$ might hold. Moreover, both E and E' can be monotone.

3.5 On superadditivity and weak superadditivity of the minimum monotone majorant of an EFF

It is clear that superadditivity of an EFF E does not imply even weak 2-superadditivity of a majorant $E' \ge E$. Indeed, let us consider, for example, the "absolutely minimal" EFF E defined by formula: $E(K,B) = 1$ if and only if $B = A$. (Recall that $E(\emptyset, A) = 1$, by the boundary conditions.) Obviously, E is superadditive and inequality $E \le E'$ holds for every EFF E'.

However, both superadditivity and weak superadditivity of an EFF E are inherited by the minimum monotone majorant $E' = E^M$ of E.

Proposition 2 *If EFF E is (weakly) superadditive then its minimum monotone majorant E^M is (weakly) superadditive, too.*

Proof. Let J be a set of indices and $E^M(K_j^M, B_j^M) = 1$ for each $j \in J$, where coalitions $\{K_j^M;\ j \in J\}$ are pairwise disjoint. Then, by definition of E^M, equality $E(K_j, B_j) = 1$ holds for some $K_j \subseteq K_j^M$, $B_j \subseteq B_j^M$, and $j \in J$. In particular, these coalitions $\{K_j;\ j \in J\}$ are pairwise disjoint, too.

If E is weakly superadditive then $\bigcap_{j\in J} B_j \ne \emptyset$. Hence, $\bigcap_{j\in J} B_j^M \ne \emptyset$ and, thus, E^M is weakly superadditive, too.

If E is superadditive then $E(\bigcup_{j\in J} K_j, \bigcap_{j\in J} B_j) = 1$. Hence, by containments $K_j \subseteq K_j^M$ and $B_j \subseteq B_j^M$ for $j \in J$, by monotonicity of E^M, and by inequality $E^M \ge E$, we conclude that $E^M(\bigcup_{j\in J} K_j^M, \bigcap_{j\in J} B_j^M) = 1$ and, thus, E^M is superadditive, too. □

Yet, the inverse implication holds only for weak superadditivity.

Proposition 3 *An EFF E is weakly superadditive whenever its minimum monotone majorant E^M is weakly superadditive.*

Proof. Let J be a set of indices and $E(K_j, B_j) = 1$ for each $j \in J$, where coalitions $\{K_j;\ j \in J\}$ are pairwise disjoint. Then, $E^M(K_j, B_j) = 1$, too, by inequality $E^M \geq E$. Hence, $\bigcap_{j \in J} B_j \neq \emptyset$, by weak superadditivity of E^M. Thus, EFF E is weakly superadditive, too. □

Corollary 1 *An EFF E is weakly superadditive if and only if its minimum monotone majorant E^M is weakly superadditive.*

Proof. It follows immediately from Propositions 2 and 3. □

However, Proposition 3 does not extend to the case of superadditivity. For example, an EFF E such that

$E(\{1\}, \{a_3\}) = E(\{2\}, \{a_3\}) = E(\{1\}, \{a_2, a_3\}) = E(\{2\}, \{a_3, a_1\}) = 1$,
and $E(\{1, 2\}, \{a_3\}) = 0$.

is not superadditive, while E^M might be superadditive.

4 Game forms and their effectivity functions

Let X_i be a finite set of strategies of the player $i \in I$ and $X = \prod_{i \in I} X_i$. A *game form* is defined as a mapping $g : X \to A$ that assigns an outcome $a \in A$ to each strategy profile $x = (x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n = X$. We will assume that mapping g is surjective, that is, $g(X) = A$; yet typically, g is not injective, that is, the same outcome might be assigned to several distinct strategy profiles.

A game form can be viewed as a game in normal form in which payoffs are not specified yet. Given a game form g, let us introduce an EFF E_g as follows:

$E_g(K, B) = 1$ for a coalition $K \subseteq I$ and block $B \subseteq A$ if and only if there is a strategy $x_K = \{x_i;\ i \in K\}$ of coalition K such that the outcome $g(x_K, x_{I \setminus K})$ is in B for every strategy $x_{I \setminus K} = \{x_i;\ i \notin K\}$ of the complementary coalition.

Remark 1 *The EFF E_g was introduced in [21], where it is called α-EFF of g and, respectively, notation α-E_g is applied. The EFF β-E_g is also defined in [21]. Yet, we find it more convenient to substitute E_g and E_g^d for α-E_g and β-E_g, where the dual EFF E_g^d will be introduced in Section 8.*

Let us recall that the boundary values $E_g(\emptyset, B)$ are not defined yet. By the boundary conditions, we set $E_g(\emptyset, A) = 1$ and $E_g(\emptyset, B) = 0$ whenever $B \neq A$.

Let us also notice that $E_g(I, \emptyset) = 0$ and $E_g(I, B) = 1$ for all non-empty $B \subseteq A$, since g is surjective. Thus, all boundary conditions hold for EFF E_g.

Proposition 4 *EFF E_g is monotone and superadditive for every game form g.*

This statement was shown already by Abdou [1, 2], Moulin and Peleg [21].

Proof. First, let us consider monotonicity. If $E_g(K, B) = 1$ then, by definition, coalition K has a strategy $x_K = \{x_i;\ i \in K\}$ enforcing B. Furthermore, if $K \subseteq K'$ and $B \subseteq B'$ then K' has a strategy $x_{K'} = \{x_i;\ i \in K'\}$ enforcing B'. Indeed, $g(x) \in B \subseteq B'$ whenever coalitionists of K play in accordance with x_K, while players of $K' \setminus K$ apply arbitrary strategies. In this case, $E(K', B') = 1$, too. Hence, E_g is monotone.

Now, let us prove superadditivity. Let $E(K_1, B_1) = E(K_2, B_2) = 1$ and $K_1 \cap K_2 = \emptyset$. By definition of E_g, coalition K_j has a strategy x_{K_j} enforcing B_j, where $j = 1$ or 2. Since coalitions K_1 and K_2 are disjoint, they can apply these strategies x_{K_1} and x_{K_2} simultaneously. Obviously, the resulting strategy x_K of the union $K = K_1 \cup K_2$ enforces the intersection $B = B_1 \cap B_2$. □

5 Main theorems

It is natural to ask whether the inverse is true too. Positive answer was given in 1983 by Moulin [20], Theorem 1 of Chapter 7.

Theorem 1 *An EFF is formal if and only if it is monotone and superadditive.*

In 1998, Peleg [24] proved a slightly more general claim. In particular, his proof works for infinite sets of outcomes A. Both proofs are constructive. Yet, the number $|X_i|$ of strategies of a player $i \in I$ is doubly exponential in size of the (**monotone**) input EFF E. In this paper, we suggest a third construction in which $|X_i|$ is only linear in size of E for every player $i \in I$; more precisely,

$$|X_i| = |A| + deg(i, \mathcal{K}_E) \leq |A| + |J| = p + m.$$

Here the monotone EFF $E = \{(K_j, B_j);\ j \in J\}$ is given as in Section 3, $\mathcal{K}_E = \{K_j;\ j \in J\}$ is the corresponding multi-hypergraph of the coalitions, and $deg(i, \mathcal{K}_E) = \#\{j \in J \mid i \in K_j\}$ is the degree of player i in $\mathcal{K}_j$.

The following statement will be instrumental in our proof of Theorem 1 and it is also of independent interest.

Theorem 2 *An EFF is formal-minor if and only if it is weakly superadditive.*

In fact, we can immediately extend this statement as follows.

Theorem 3 *The next four properties of an EFF E are equivalent:*

(i) E is formal minor; (ii) E is weakly superadditive;

(iii) E^M is formal-minor; (iv) E^M is weakly superadditive.

Proof. Equivalence of (i) and (ii) (as well as of (iii) and (iv), in particular) is claimed by Theorem 2. Furthermore, (i) and (iii) are equivalent, too, by the definition of the minimum monotone majorant E^M and monotonicity of E_g. □

Let us remark that Proposition 3 follows from Theorem 3.

We will prove Theorems 1 and 2 in the next two subsections.

In accordance with Section 3.3, it can be verified in polynomial time whether a (monotone) EFF is formal or whether it is superadditive; in contrast, to verify whether a monotone EFF is formal-minor or whether it is weakly superadditive is a CoNP-complete decision problem.

6 Main proofs

6.1 Proof of Theorem 2

Obviously, an EFF E is formal minor if and only if E^M is. Since E^M is monotone, it can be conveniently specified by the list $(K_j, B_j), j \in J$, of all inclusion-minimal pairs such that $E^M(K_j, B_j) = 1$.

Clearly, $E = E^M$ whenever EFF E is monotone; otherwise the input size of E might be much larger: E is specified by the list of **all** (not only inclusion-minimal) pairs $(K_j, B_j), j \in J'$, such that $E(K_j, B_j) = 1$. Yet, we can easily reduce this list J' to J by leaving only inclusion-minimal pairs and eliminating all other. This reduction, obviously, results in E^M. Thus, without loss of generality, we can assume that $E = E^M$, or in other words, that the input EFF E is monotone and given by the list $(K_j, B_j), j \in J$.

Given a monotone weakly superadditive EFF E, we want to construct a game form g such that $E \leq E_g$. To each player $i \in I$ let us give a set of strategies $X_i = \{x_i^j \mid i \in K_j\}$. In other words, given $i \in I$ and $j \in J$, strategy x_i^j is unique whenever $i \in K_j$ and it is not defined otherwise. Thus, $|X_i| = deg(i, \mathcal{K})$, where $\mathcal{K}$ is the multi-hypergraph of coalitions $\mathcal{K} = \{K_j, j \in J\}$.

Given $j \in J$, a (unique) strategy $x_{K_j} = \{x_i^j; i \in K_j\}$ of coalition K_j is called *proper*. If for each such strategy and each strategy $x_{I \setminus K_j}$ of the complementary coalition, inclusion $g(x_{K_j}, x_{I \setminus K_j}) \in B_j$ holds then game form g will be called *proper*, too.

Let us show that the above condition is not contradictory whenever EFF E is weakly superadditive. Indeed, if a strategy profile $x = (x_1, \ldots, x_n)$ is proper with respect to several coalitions $\{K_j, j \in J' \subseteq J\}$ then, obviously, these coalitions are pairwise disjoint and, hence, $\bigcap_{j \in J'} B_j \neq \emptyset$.

For each strategy profile $x \in X$ let us choose an outcome a from this intersection and fix $g(x) = a$. If x is proper for no $j \in J$ then choose $g(x) \in A$ arbitrarily. This construction defines a proper game form $g : X \to A$. The desired inequality $E \leq E_g$ obviously holds for each proper game form g. Indeed, let $E(K, B) = 1$; then $E(K_j, B_j) = 1$ for some $j \in J$; then $g(x_{K_j}, x_{I \setminus K_j}) \in B_j$ for every $x_{I \setminus K_j}$ whenever x_{K_j} is the proper strategy of K_j. □

			a_1	a_2				
			a_2	a_3				
			a_3	a_4				
a_1	a_2	a_4	a_1	a_2	a_1	a_2	a_4	a_1
a_1	a_3	a_4	a_1	a_3	a_1	a_3	a_4	a_1
			a_1	a_4	a_1	a_2	a_3	a_4
			a_2	a_2	a_4	a_1	a_2	a_3
			a_3	a_3	a_3	a_4	a_1	a_2
			a_1	a_4	a_2	a_3	a_4	a_1

Table 1: Two-person EFF E_g.

Let us consider an example given by the upper left corner (the first two rows and columns) of Table 1. In this example $I = \{1, 2\}$, $A = \{a_1, a_2, a_3, a_4\}$, and EFF E is given by the list:

$E(1, \{a_1, a_2, a_4\}) = E(1, \{a_1, a_3, a_4\}) =$
$E(2, \{a_1, a_2, a_3\}) = E(2, \{a_2, a_3, a_4\}) = 1.$

Each of the four entries of the desired game form must be an outcome of the corresponding intersection. The obtained EFF E_g is given by the list:

$E_g(1, \{a_1, a_2\}) = E_g(1, \{a_1, a_3\}) = E_g(2, \{a_1\}) = E_g(2, \{a_2, a_3\}) = 1.$

Of course, $E \leq E_g$, however, $E \neq E_g$. Similar observations were made by Moulin; see [20] Theorem 1 of Chapter 7, pp. 166-168.

Remark 2 *Let $\mathcal{K} = \{K_j, j \in J\}$ and $\mathcal{X} = \{x_{K_j}, j \in J\}$ be families of coalitions and their strategies. If the coalitions of $\mathcal{K}$ are pairwise disjoint (vice versa, pairwise intersect) then the corresponding faces in the direct product $X = \prod_{i \in I} X_i$ intersect (vice versa, might be pairwise disjoint). This observation, which is instrumental in the above proof of Theorem 2, was mentioned in 1978 [18] and illustrated for $n = 3$ and $\mathcal{K} = \{\{1, 2\}, \{2, 3\}, \{3, 1\}\}$.*

6.2 Proof of Theorem 1

Now we assume that EFF $E = \{(K_j, B_j);\ j \in J\}$ is monotone and superadditive and want to construct a game form g such that $E = E_g$. In the previous section, we already got a game form g' such that $E \leq E_{g'}$. To enforce the equality, we will have to extend g' to g as follows. To each player $i \in I$, in addition to the **proper** strategies $X'_i = \{x_i^j;\ i \in K_j\}$, we will add $p = |A|$ **backup** strategies $X''_i = \{x_i^b;\ b \in \{0, 1, \ldots, p-1\}\}$. Thus, $X_i = X'_i \cup X''_i$ for all $i \in I$ and $X = \prod_{i \in I} X_i = \prod_{i \in I}(X'_i \cup X''_i)$.

Thus, each strategy profile $x \in X$ defines a unique partition $I = K' \cup K''$, where $K' = K'(x)$ and $K'' = K''(x)$ are the coalitions of all "proper" and "backup" players, respectively, that is, $x_i \in X'_i$ for $i \in K'$ and $x_i \in X''_i$ for $i \in K''$. To obtain the desired game form $g : X \to A$ (such that $E_g = E$), we will define $g(x)$ successively for $|K''(x)| = k(x) = k = 0, 1, \ldots, n$.

Two extreme cases, $k = 0$ and $k = n$ are simple. If $k(x) = 0$, that is, in x all players choose proper strategies , then $g(x) = g'(x)$ is defined as in the previous section. If $k(x) = n$, that is, in x all players choose backup strategies $x_i \in X''_i = \{x_i^{b_i};\ b_i \in \{0, 1, \dots, p-1\}\}$, then

$$g(x) = a_r \in A = \{a_1, \dots, a_p\}, \text{ where } r - 1 = \sum_{i=1}^{n} b_i \mod p. \quad (1)$$

Table 3 and the lower right 4×4 corner of Table 1 provide two examples, with $n = p = 3$ and $n = 2, p = 4$, respectively.

Now, we plan to define $g(x)$ for $k(x) \in \{1, \dots, n-1\}$.

First, we have to extend the concepts of a proper coalition, strategy, and game form defined in the previous section. Given a strategy profile $x \in X$, let us consider partition $I = K'(x) \cup K''(x)$, where players of K' and K'' choose in x their proper and backup strategies, respectively. A coalition K_j is called *proper* if $x_i = x_i^j$ for each $i \in K_j$. By this definition, $K_j \subseteq K'(x)$, that is, each proper coalition is a subcoalition of $K'(x)$. The obtained strategy $x_{K_j} = \{x_i^j;\ i \in K_j\}$ of coalition K_j is called *proper*, too. If for each such strategy and every strategy $x_{I \setminus K_j}$ of the complementary coalition, inclusion $g(x_{K_j}, x_{I \setminus K_j}) \in B_j$ holds then game form g will be also called *proper*. As before, these conditions are not contradictory whenever EFF E is (weakly) superadditive. Indeed, if several coalitions $\{K_j;\ j \in J' \subseteq J\}$ are proper with respect to a given strategy profile $x = (x_1, \dots, x_n)$ then, obviously, these coalitions are pairwise disjoint and, hence, $B(x) = \bigcap_{j \in J'} B_j \neq \emptyset$.

Two strategy profiles $x', x'' \in X$ will be called equivalent if the corresponding partitions coincide, or in other words, if $K'(x') = K'(x'') = K$ and, moreover, $x'_i = x''_i$ for every $i \in K$. Obviously, these classes partition X.

Given $x \in X$, let $|K''(x)| = k(x) = k$ and $|B(x)| = q(x) = q$; furthermore, let for simplicity $K''(x) = \{1, \dots, k\} \subseteq I$ and $B(x) = \{a_1, \dots, a_q\} \subseteq A$.

We generalize formula (1) for arbitrary integral $q \leq p$ and $k \leq n$ as follows:

$$g(x) = a_r \in B(x) = \{a_1, \dots, a_q\}, \text{ where } r - 1 = \sum_{i=1}^{k} b_i \mod q. \quad (2)$$

whenever in the given profile $x \in X$ each player $i \in K''(x)$ chooses a backup strategy $x_i = b_i \in \{0, 1, \dots, p-1\}$.

Several examples are given in Tables 2 and 3, where $p = 4$ or $p = 5$, $q = 3$, $k = 2$ and p = q = k = 3, respectively.

a_1	a_2	a_3	a_1
a_1	a_1	a_2	a_3
a_3	a_1	a_1	a_2
a_2	a_3	a_1	a_1

a_1	a_2	a_3	a_1	a_2
a_2	a_1	a_2	a_3	a_1
a_1	a_2	a_1	a_2	a_3
a_3	a_1	a_2	a_1	a_2
a_2	a_3	a_1	a_2	a_1

Table 2: $q = 3$, $k = 2$, $p = 4$ and $p = 5$.

a_1	a_2	a_3		a_2	a_3	a_1		a_3	a_1	a_2
a_3	a_1	a_2		a_1	a_2	a_3		a_2	a_3	a_1
a_2	a_3	a_1		a_3	a_1	a_2		a_1	a_2	a_3

Table 3: $p = q = k = 3$.

By the above definition, for every $x \in X$, there are exactly $p^{k(x)}$ strategy profiles equivalent with x. Let us define function (game form) g on these profiles in accordance with (2).

In particular, $g(x) = g'(x)$ when $K''(x) = \emptyset$ and $g(x)$ is defined by (1) when $K'(x) = \emptyset$. Table 1 represents an example in which $n = 2$ and $p = 4$.

By construction, each strategy x_K is effective for the block $B(x_K) = \cap_{j \in J'} B_j$, where $J' = J(x_K) \subseteq J$ is defined as follows: x_K is a proper strategy of K_j if and only if $j \in J'$. In particular, $K_j \subseteq K$ for all $j \in J'$.

In general, $E_g(K, B) = 1$ if and only $K \supseteq K_j$ and $B \supseteq B_j$ for a $j \in J$.

In particular, $E_g(K_j, B_j) = 1$ for all $j \in J$, since the proper strategy $x_{K_j} = \{x_i^j;\ i \in K_j\}$ is effective for B_j. Thus, by the above construction, equality $E = E_g$ holds if and only if the input EFF E is monotone and superadditive. □

Remark 3 *In general, the obtained EFF E_g is the minimum monotone and superadditive majorant of the input EFF E.*

Let us also note that the above construction is computationally efficient: for every strategy profile x the corresponding outcome $g(x)$ is determined in polynomial time. Obviously, the same is true in case of Theorem 2 too.

6.3 Theorem 2 results from Theorem 1

We derived Theorems 1 from Theorem 2. In fact, the latter is of independent interest. For example, it is instrumental in the proof of the Berge and Duchet conjecture in [4]; see also [5, 6]. In these papers, Theorem 2 was derived from Theorem 1, since the latter was already published by Moulin.

Remark 4 *In an old joke, a mathematician solved the problem of boiling water in the kettle as follows: "... If water is already in the kettle then out and, by this, the problem is reduced to the previous one".*

An EFF E and its minimum monotone majorant E^M can be weakly superadditive or, respectively, formal-minor only simultaneously. Moreover, $E \leq E_g$ if and only if $E^M \leq E_g$, since E_g and E^M are both monotone. Hence, we can prove Theorem 2 for E^M rather than E. Since EFF E^M is monotone, it is uniquely defined by the set of its minimal "ones" $E^M = \{(K_j, B_j);\ j \in J\}$.

First let us assume that E^M is formal-minor, that is, $E^M \leq E_g$ for a game form g. Furthermore, let $J' \subseteq J$ be a family of pairwise disjoint coalitions, $K'_{j'} \cap K'_{j''} = \emptyset$ for all $j'.j'' \in J'$ such that $j' \neq j''$. Obviously, $E_g(K_j, B_j) = 1$ follows for all $j \in J$, by $E^M \leq E_g$. By Theorem 1, E_g is monotone, superadditive, and satisfies the boundary conditions; see Section 4. Hence, $E_g(\cup_{j\in J'}K_j, \cap_{j\in J'}B_j) = 1$, by superadditivity, and $\cap_{j\in J'}B_j) = 1$, by boundary condition. Thus, EFFs E^M (and E) are weakly superadditive.

Conversely, let E^M be weakly superadditive. Let us define an EFF E' by setting $E'(K, B) = 1$ if and only if $B = A$, or $K = I$ and $B \neq \emptyset$, or there is a non-empty subset $J' \subseteq J$ such that $B \supseteq \cap_{j\in J'}B_j$, $K \supseteq \cup_{j\in J'}K_j$, and the corresponding coalitions, $\{K_j;\ j \in J'\}$ are pairwise disjoint. By this definition, $E^M \leq E'$. Furthermore, it is not difficult to verify that the obtained EFF E' is monotone, superadditive, and satisfies the boundary conditions. Hence, by Theorem 1, $E' = E_g$ for a game form g. Thus, E^M and E are formal-minor.

7 Graphs and their effectivity functions

Given a graph $G = (J, E)$, let us assign a player (outcome) to every its inclusion-maximal clique (independent set) and denote the obtained two sets by I_G and A_G. Then, for every vertex $j \in J$ let us consider the coalition K_j (block B_j) corresponding to all maximal cliques (independent sets) that contain vertex j. The obtained list $\{(K_j, B_j);\ j \in J\}$ defines an EFF E. Let $E_G = E^M$ be the minimum monotone majorant of E; or in other words,

$E_G(K, B) = 1$ if and only if $K_j \subseteq K$ and $B_j \subseteq B$ for a vertex $j \in J$.

The following claim is instrumental in the proof of the Berge and Duchet conjecture in [4]; see also [5, 6].

Lemma 1 *For every graph G the corresponding EFF E_G is formal-minor.*

Proof. Let $J' \subseteq J$ be a set of vertices in G such that the coalitions $\{K_j;\ j \in J'\}$ are pairwise disjoint. Then, obviously, J' is an independent set of G. Indeed, $K_{j'} \cap K_{j''} \neq \emptyset$ if and only if (j', j'') is an edge of G. Let J'' be a maximal independent set that contains J' and $a \in A_G$ be the corresponding outcome. Then, obviously, $a \in \cap_{j\in J'}K_j \neq \emptyset$ □

Thus, there is a game form $g : \prod_{i\in I_G} X_i \to A_G$ such that $E_G \leq E_g$.

Although both sets I_G and A_G might be exponential in $|J|$, yet, by construction of Theorem 2, it follows that one can choose a game form g of a "pretty modest" size, namely, $|X_i| \leq |J|$ for all $i \in I_G$.

8 Tight game forms and self-dual EFFs

8.1 Dual and self-dual effectivity functions

To each EFF E let us assign the dual EFF E^d defined by formula:

$E^d(K,B) + E(I \setminus K, A \setminus B) = 1 \ \ \forall \, K \subseteq I, \ B \subseteq A.$

In other words, $E^d(K,B) = 1$ if and only if $E(I \setminus K, A \setminus B) = 0$.

It is not difficult to verify that two EFFs are dual if and only if the corresponding two Boolean functions are dual. (Let us also recall that an EFF is monotone if and only if the corresponding Boolean function is monotone.) Thus, our terminology for EFFs is in agreement with the standard Boolean language.

Respectively, an EFF E is called *self-dual* (or *maximal*) if

$$E(K,B) + E(I \setminus K, A \setminus B) = 1, \ \ \forall \, K \subseteq I, \ B \subseteq A,$$

that is, K is effective for B if and only if $I \setminus K$ is not effective for $A \setminus B$.

It is easy to see that inequality

$$E(K,B) + E(I \setminus K, A \setminus B) \leq 1, \ \ \forall \, K \subseteq I, \ B \subseteq A,$$

holds for every weakly superadditive EFF. Indeed, otherwise

$$E(K,B) = E(I \setminus K, A \setminus B) = 1 \ \text{ and } E(I, \emptyset) = 0,$$

in contradiction with the boundary conditions. In other words, $E(K,B) = 0$ whenever $E(I \setminus K, A \setminus B) = 1$.

An EFF E is self-dual if and only if the inverse implication holds. In other words, the equalities $E(K,B) = E(I \setminus K, A \setminus B) = 0$ might hold for some $K \subseteq I, \ B \subseteq A$ of an EFF E; they cannot hold if and only if EFF E is self-dual.

In particular, the self-dual EFFs are maximal, with respect to the partial order "$\leq$", among the weak superadditive (as well as among superadditive, or formal, or formal-minor) EFFs.

Remark 5 *For this reason, in the literature the term "maximal", rather than "self-dual", is frequent in the literature; see, for example, [20, 23, 3]. However, in this paper we follow Boolean terminology.*

Remark 6 *Let us also recall that, by the boundary conditions, $E(I, \emptyset) = 0$ and $E(\emptyset, A) = 1$, in agreement with self-duality.*

8.2 Tight game forms; T-formal and T-formal-minor EFFs

A game form g is called *tight* if its EFF E_g is self-dual.

Let us recall that EFF E is T-formal (T-formal-minor) if and only if $E = E_g$ (respectively, $E \leq E_g$) for a **tight** game form g. It is not difficult to show that the families of the formal-minor and T-formal-minor EFFs just coincide.

Proposition 5 *An EFF is T-formal-minor if and only if it is formal-minor.*

Proof. Indeed, it is shown in [16] that every game form g can be extended to a tight one; in other words, for each g there is a tight game form g' such that g is a subform of g' and $E_g \leq E_{g'}$. □

Furthermore, just by definition, an EFF is T-formal if and only if it is formal and self-dual. Moreover, the following statement holds.

Theorem 4 *An EFF E is T-formal if and only if it is monotone, superadditive, and self-dual. The next four properties of a self-dual EFF E are equivalent:*

(a) E is T-formal; (b) E is monotone and superadditive;

(c) E is T-formal-minor; (d) E is monotone and weakly superadditive.

Proof. The last claim immediately follows from Theorem 1 and the definition of tightness and results in equivalence of (a) and (b). Furthermore, obviously, (a) implies (c). To show the inverse let us assume indirectly that the strict inequality $E < E_g$ holds for a self-dual EFF E and game for g.

Yet, let us also recall that the inequality $E_g(K, B) + E_g(I \setminus K, A \setminus B) \leq 1$ holds for a game form g and identity $E_g(K, B) + E_g(I \setminus K, A \setminus B) \equiv 1$ holds whenever g is tight. Since $E < E_g$, there is a pair $K \subseteq I, B \subseteq A$ such that $E(K, B) = E(I \setminus K, A \setminus B) = 0$. Then, by duality, $E^d(K, B) = E^d(I \setminus K, A \setminus B) = 1$ and we get a contradiction, since EFF E is self-dual, $E = E^d$.

The same arguments, in slightly different terms, appears already in [20].

Finally, Theorems 1, 2 and the above observations readily imply that (d) is equivalent to (a,b,c), too. □

8.3 On tightness and Nash-solvability

Given sets of players (voters) I and outcomes (candidates) A, the *utility (payoff, preference) function* is introduced by a mapping $u : I \times A \to \mathbb{R}$, where the value $u(i, a)$ is standardly interpreted as a profit of the player $i \in I$ in case the outcome $a \in A$ is realized.

Given also a game form $g : X \to A$, the pair (g, u) is a *game in normal form.*

A strategy profile $x = \{x_i;\ i \in I\} \in \prod_{i \in I} X_i = X$ is called a *Nash equilibrium* in game (g, u) if $u(i, x) \geq u(i, x')$ for each player $i \in I$ and each strategy profile x' obtained from x by substituting a strategy x'_i for x_i. In other words, x is a Nash equilibrium if a player can make no profit in x by choosing another strategy provided all other players keep their old strategies.

A game form g is called Nash-solvable if for each utility function u the obtained game (g, u) has a Nash equilibrium.

Theorem 5 *A two-person game form is Nash-solvable if and only if it is tight.*

This result was obtained in 1975 [12]; see also [13] and [8] Appendix 1, where it is also shown that in case of more than two players tightness is no longer necessary or sufficient for Nash-solvability.

In contrast, for **two-person zero-sum** games tightness remains necessary. More precisely, let $I = \{1, 2\}$, a utility function $u : I \times A \to \mathbb{R}$ is called *zero-sum* if $u(1, a)+u(2, a) = 0$ for each outcome $a \in A$. A game form g is called *zero-sum-solvable* (± 1-solvable) if for every zero-sum (and taking only ± 1-values) utility function u the obtained zero-sum game (g, u) has a saddle point.

Theorem 6 *The following properties of a two-person game form are equivalent:*
(i) Nash-solvability, (ii) zero-sum-solvability, (iii) ± 1*-solvability, (iv) tightness.*

Equivalence of (ii), (iii), and (iv) was demonstrated in 1970 by Edmonds and Fulkerson [10]; see also [11].

9 On totally tight game forms and TT-formal effectivity functions

9.1 Two-person case

Let us start with the case $n = 2$. A two-person game form g is called *totally tight* (TT) if every 2×2 subform of g is tight.

Up to an isomorphism, there are only seven 2×2 game forms:

$$\begin{matrix} a_1a_1 & a_1a_1 & a_1a_1 & a_1a_1 & \quad & a_1a_2 & a_1a_2 & a_1a_2 \\ a_1a_1 & a_1a_2 & a_2a_2 & a_2a_3 & \quad & a_2a_1 & a_2a_3 & a_3a_4 \end{matrix}$$

The first four are tight, while the last three are not. Thus, a 2×2 game form is tight if and only if it has a constant line, row or column.

Let g be a game form with a constant line and let g' be the subform of g obtained by eliminating this line. Obviously, g is TT if and only if g' is TT.

Let us also remark that g might be tight, while g' is not; see [7] for the corresponding examples. However, g is tight whenever g' is tight.

A TT game form with a constant line is called *reducible*.

Somewhat surprisingly, all irreducible TT game form have the same EFF.

Theorem 7 *([7]) Let* $g : X_1 \times X_2 \to A$ *be an irreducible TT two-person game form. Then there are three outcomes* $a_1, a_2, a_3 \in A$ *such that*
$E_g(i, \{a_1, a_2\}) = E_g(i, \{a_2, a_3\}) = E_g(i, \{a_3, a_1\}) = 1,$ *while* $E_g(i, \{a_j\}) = 0,$
for $i \in I = \{1, 2\},\ j \in J = \{1, 2, 3\}.$

It is easy to see that EFF E_g is uniquely defined by the equalities of Theorem 7 and boundary conditions.

We have to remark that a 1×1 game form is TT, too. Yet, formally, this game form g_0 is reducible. The corresponding EFF E_{g_0} is given by equalities $E_{g_0}(1, \{a\}) = E_{g_0}(2, \{a\}) = 1$, where a is a unique outcome, $A = \{a\}$.

We will call this EFF *trivial*, while the EFF of Theorem 7 will be called $(3, 2)$-EFF. Obviously, both EFFs are self-dual and, hence, the corresponding game forms are tight. Since addition of a constant line to a game form respects its tightness, the next statement follows.

Corollary 2 *A totally tight game form is tight.*

The above proof was based on Theorem 7. There is an alternative very short proof based on Theortems 5, 6, and Shapley's condition for solvability of matrix games. If g is TT then every its 2×2 subform g' is tight. Then, obviously, g' is Nash-solvable. (This follows, for example, from Theorems 5 and 6; although "such two guns are too big for a fly that small".) Yet, in 1964, Shapley [25] proved that a matrix has a saddle point whenever every its 2×2 submatrix has one. By Shapley's theorem, game (g, u) has a saddle point for each zero-sum payoff u. Thus, g is tight, by Theorem 6. □

By definition, every TT game form is obtained from an empty or irreducible one by recursively adding constant lines. By this operation, the corresponding EFFs are changed in an obvious way, which we will call an extension by adding constant lines or ACL-extension, for short.

Thus, we obtain a recursive characterization for the EFFs of the TT two-person game forms, or in other words, for the TT-formal two-person EFFs.

Theorem 8 *A two-person EFF E is TT-formal if and only if it is an ACL-extension of the trivial or $(3, 2)$-EFF.*

A recursive characterization of the two-person TT game forms themselves is obtained in [7]. It is based on Theorem 7, yet, a bit surprisingly, is much more complicated than the latter.

9.2 n-person case

Now, let $g : X \to A$ be a n-person game form, where $X = \prod_{i \in I} X_i$ and $I = \{1, \dots, n\}$. Each coalition $K \subseteq I$ such that $K \neq \emptyset$ and $K \neq I$ defines a two-person game form $g_K : X_K \times X_{I \setminus K} \to A$, where

$X_K = \{x_K = \{x_i;\ i \in K\}\}$ and $X_{I \setminus K} = \{x_{I \setminus K} = \{x_i;\ i \notin K\}\}$

are the sets of strategies of two complementary coalitions K and $I \setminus K$.

Game form g is called *totally tight* (TT) if g_K is TT for all K.

An EFF E is called TT-formal (respectively, TT-formal-minor) if $E = E_g$ (respectively, $E \leq E_g$) for a TT game form g.

By definition, every TT-formal (TT-formal-minor) EFF E is T-formal (formal-minor). Thus, we obtain obvious necessary conditions. In particular, E is (i) monotone, (ii) superadditive, and (iii) self-dual (respectively, E and E^M are weakly superadditive).

Furthermore, given an n-person EFF $E : 2^{I \cup A} \to \{0, 1\}$ and a coalition $K \subseteq I$, let us define a two-person EFF E_K which is the restriction of E to K and $I \setminus K$. More precisely, $E_K(K', B) = 1$ if and only if $E(K, B) = 1$ and $K' \subseteq K$. Obviously, for each $K \subseteq I$ EFF E_K is TT-formal (respectively, TT-formal-minor) whenever E is. Thus, we obtain more necessary conditions:

Let us recall that a recursive characterization of the two-person TT-formal EFFs was just obtained in the previous section. Yet, it remains open, whether the obtained necessary conditions are also sufficient for an EFF to be TT-formal. In general, characterizing TT-formal and TT-formal-minor EFFs remains an open problem.

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