Coverable functions

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Coverable functions

- Let us recall that given a Boolean function f, we denote by:
  - $cnf(f)$ - minimum number of clauses needed to represent f by a CNF.
  - $ess(f)$ - maximum number of pairwise disjoint essential sets of implicates of f.
- A function f is coverable, if $cnf(f)=ess(f)$.

Talk outline

- We already know from the previous talk, that not every function is coverable.
- We shall show, that quadratic, acyclic, quasi-acyclic, and CQ Horn functions are coverable.
- Before that we shall show, that in case of Horn functions we can restrict our attention to only pure Horn functions.

Negative implicates

- Let f be a Horn function.
- Let $\mathcal{X}$ be an exclusive set of implicates of f, such that no two clauses in $\mathcal{E} = \mathcal{I}(f) \setminus \mathcal{R}(\mathcal{X})$ are resolvable.
- Then there exists an integer k, and pairwise disjoint essential sets $\mathcal{Q}_1, \ldots, \mathcal{Q}_k \subseteq \mathcal{E}$, such that for every CNF $\mathcal{C}$ representing f:
  - $|\mathcal{C} \cap \mathcal{Q}_j| = 1, j = 1, \ldots, k$
  - $\mathcal{C}$ does not contain other elements of $\mathcal{E}$.

Negative implicates

- We can use this proposition to negative implicates, if we put:
  - $\mathcal{X}$ = pure Horn implicates of f, and
  - $\mathcal{E}$ = negative implicates of f.
- Now we can observe that:

$$ess(f) = ess(\mathcal{X}) + k$$

- Therefore we can restrict our attention to only pure Horn functions.

CNF Graph

- For a Horn CNF φ let $G_\varphi = (N, A_\varphi)$ be the digraph defined as:
  - N is the set of variables of φ.
  - (x, y) belongs to A_φ, if there is a clause C in φ, which contains $\overline{x}$ and y.
- G_f, where f is the function represented by φ, is transitive closure of G_φ.

Quadratic functions

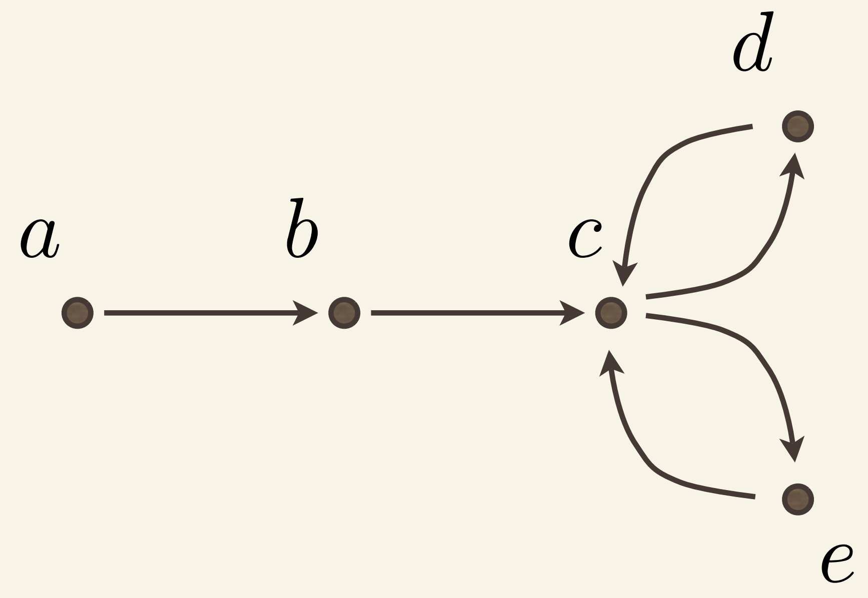
- A quadratic function is function, which can be represented by a CNF φ, in which every clause consists of at most two literals.
- Minimization algorithm for pure Horn quadratic functions:
  - Make φ prime and irredundant.
  - Construct CNF graph G_{φ}.
  - Find strong components of G_{φ}.
  - Replace strong components by cycles.

Example

- Let us consider the following CNF:

$$
\begin{aligned}
& (\overline{a} \vee b) \wedge (\overline{b} \vee c) \wedge (\overline{c} \vee d) \\
\wedge \quad & (\overline{d} \vee c) \wedge (\overline{c} \vee e) \wedge (\overline{e} \vee c)
\end{aligned}
$$

- CNF graph follows:

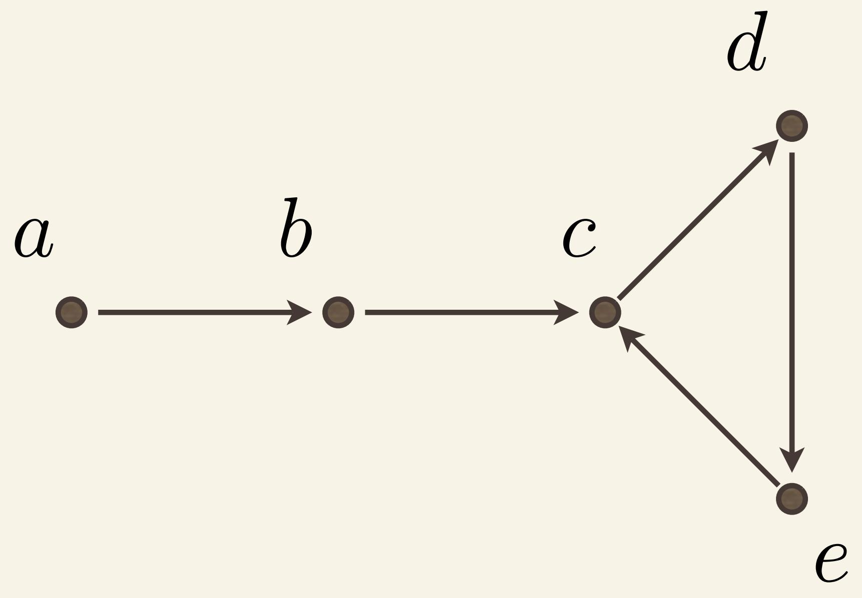


Example

- A shortest CNF:

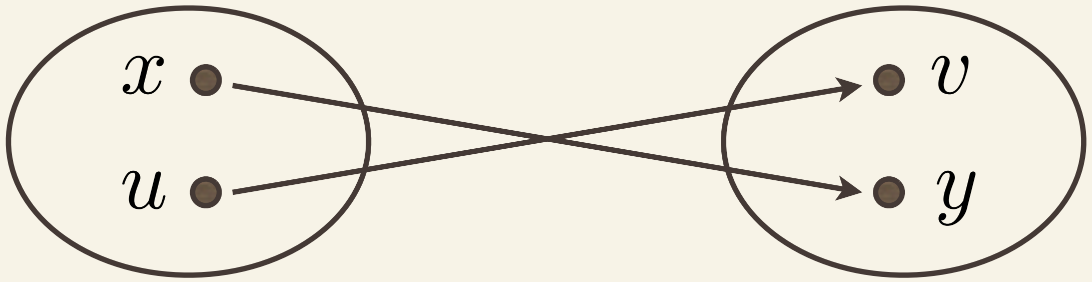
$$(\overline{a} \vee b) \wedge (\overline{b} \vee c) \wedge (\overline{c} \vee d) \wedge (\overline{d} \vee e) \wedge (\overline{e} \vee c)$$

- and its CNF graph:



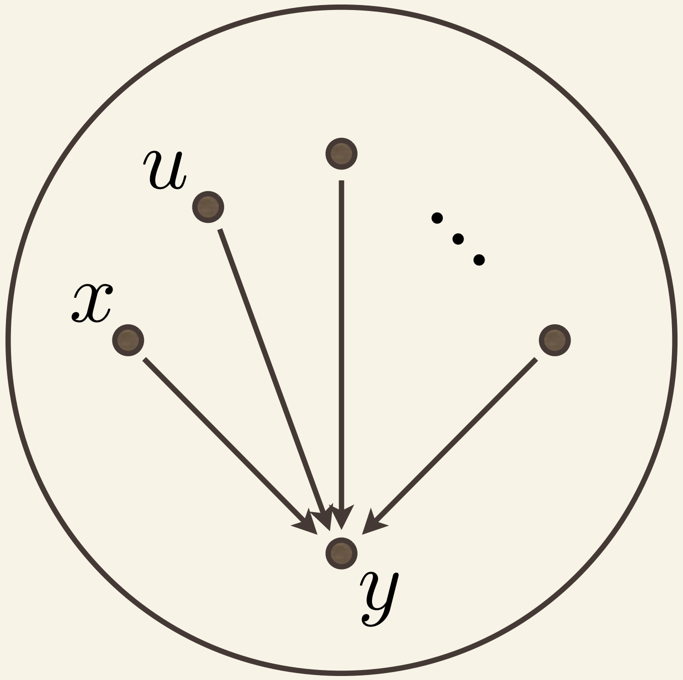
Disjoint essential sets for quadratic functions

- Let us have a clause $(\overline{x} \vee y)$ and let us define essential set $\mathcal{E}$ for this clause.
- If x and y belong to different strong components of G_f, we put $(\overline{u} \vee v)$ into $\mathcal{E}$, if u belongs to the same strong component as x and v belongs to the same strong component as y.



Disjoint essential sets ...

- If x and y belong to the same component of G_f, we put $(\overline{u} \vee y)$ into $\mathcal{E}$ for every u in this component.



- It is easily possible to find vector based definition of these sets as well.
- If the input CNF is minimum, the sets are disjoint.

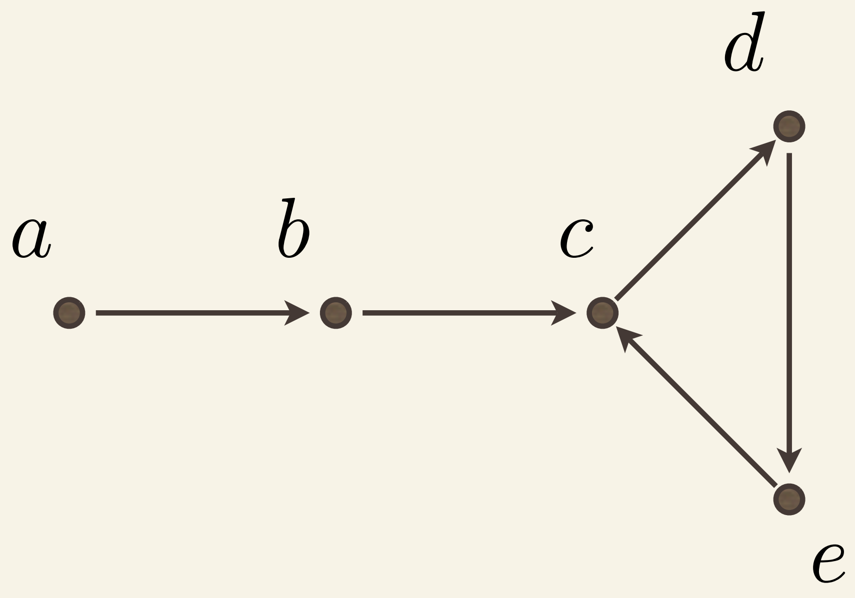
Example

- For our shortest CNF

$$(\overline{a} \vee b) \wedge (\overline{b} \vee c) \wedge (\overline{c} \vee d) \wedge (\overline{d} \vee e) \wedge (\overline{e} \vee c)$$

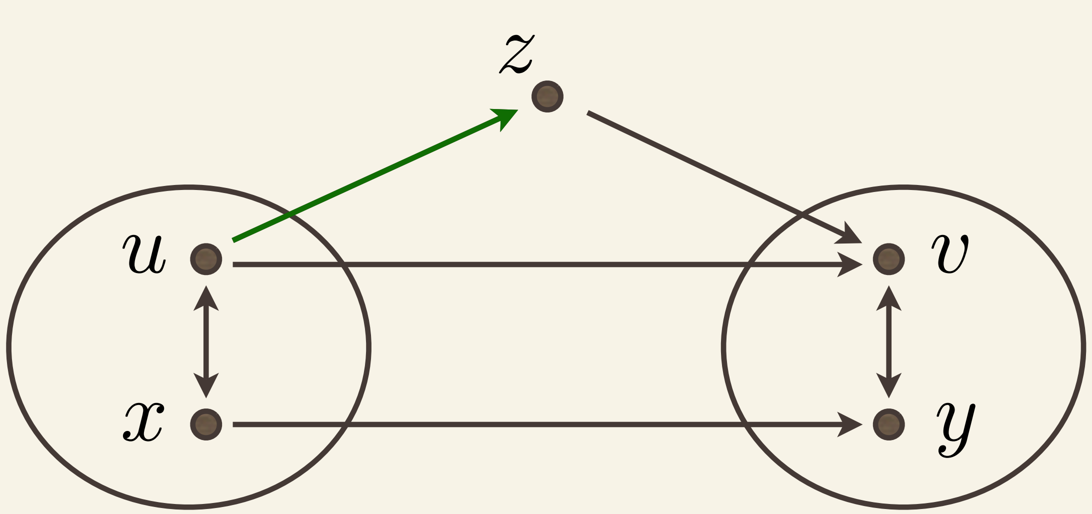
- we would have:

$$
\begin{aligned}
(\overline{a} \vee b) &\quad\longrightarrow\quad \{(\overline{a} \vee b)\} \\
(\overline{b} \vee c) &\quad\longrightarrow\quad \{(\overline{b} \vee c)\} \\
(\overline{c} \vee d) &\quad\longrightarrow\quad \{(\overline{c} \vee d), (\overline{e} \vee d)\} \\
(\overline{d} \vee e) &\quad\longrightarrow\quad \{(\overline{d} \vee e), (\overline{c} \vee e)\} \\
(\overline{e} \vee c) &\quad\longrightarrow\quad \{(\overline{e} \vee c), (\overline{d} \vee c)\}
\end{aligned}
$$



Essentiality of defined sets I

- At first let us assume, that x and y belong to different strong components of G_f.
- We have u in the same SC as x, v in the same SC as y, and $(\overline{u} \vee v) = \mathcal{R}(\overline{u} \vee z, \overline{z} \vee v)$ for some z.
- If z does not belong to the same SC as x or y, then $(\overline{x} \vee y)$ is redundant.
- Therefore one of parent clauses belongs to $\mathcal{E}$.



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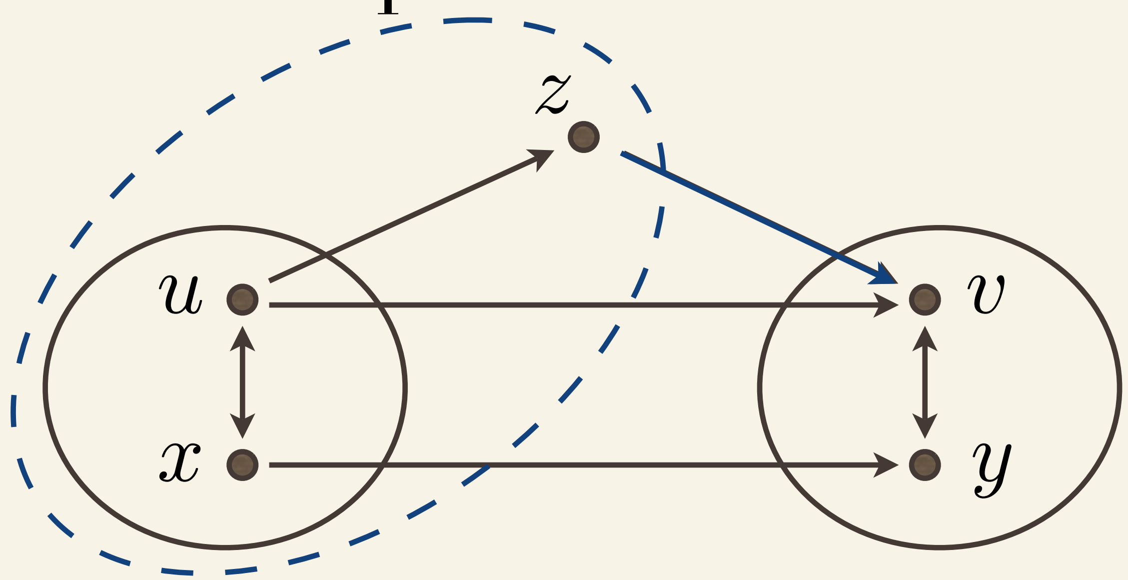

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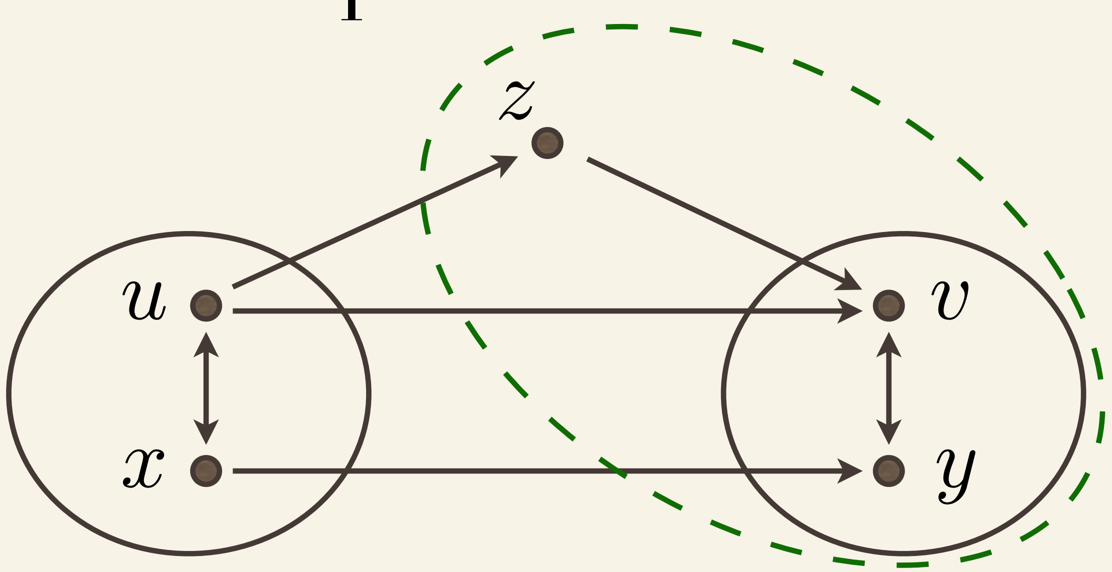
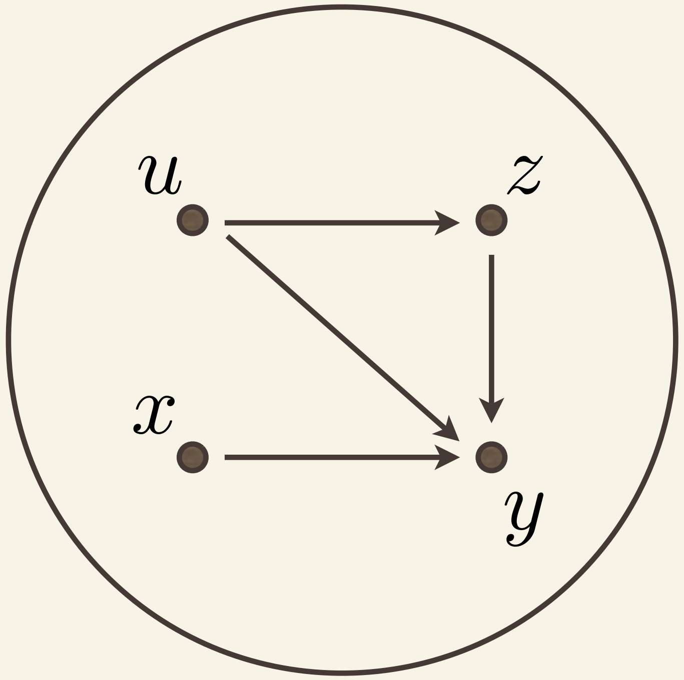


Essentiality II

- Now let us assume, that x and y belong to the same strong component of G_f.
- We have u in this strong component and z, for which $(\overline{u} \vee y) = \mathcal{R}(\overline{u} \vee z, \overline{z} \vee y)$.
- It follows, that z belong to the same strong component as well.



Acyclic functions

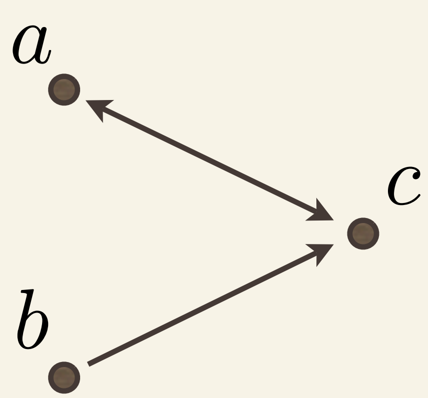
- A function f is acyclic, if its CNF graph is acyclic.
- Prime and irredundant CNF is the only minimum representation of an acyclic function.
- Given the only prime and irredundant acyclic CNF φ, we define for each clause $C \in \varphi$ an essential set $\mathcal{E}_C = \{C\}$.
- This set is essential due to similar reasons as in the case of quadratic functions.
- Vector based definition is also possible.

Quasi-acyclic functions

- A function f is quasi-acyclic, if every two variables x and y, which belong to the same strong component of G_f, are logically equivalent.
- Definition of essential sets is a combination of cases of quadratic and acyclic function.

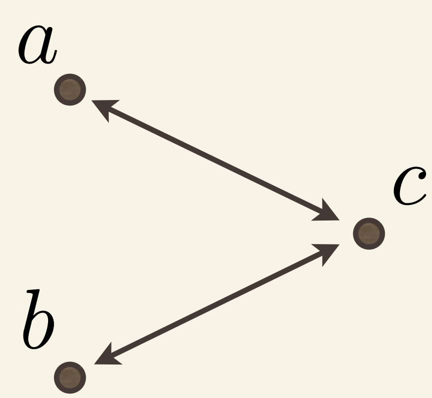
CQ functions

- A Horn CNF φ is CQ, if in every clause $C \in \varphi$ at most one subgoal belongs to the same strong component as its head.
- A Horn function f is CQ, if it can be represented by a CQ CNF.



$(\overline{a} \vee \overline{b} \vee c) \wedge (\overline{c} \vee b)$

is CQ



$(\overline{a} \vee \overline{b} \vee c) \wedge (\overline{c} \vee b) \wedge (\overline{c} \vee a)$

is CQ

CQ and essential sets

- Any prime CNF representation of a CQ function is a CQ CNF.
- In order to be able to define disjoint essential sets, we have to investigate structure of minimum CQ CNFs and minimization algorithm for CQ functions.

Decomposition lemma

Let us have:

- a function f,
- a chain of exclusive subsets $\emptyset = \mathcal{X}_0 \subseteq \mathcal{X}_1 \subseteq \cdots \subseteq \mathcal{X}_t$ in which $\mathcal{R}(\mathcal{X}_t) = \mathcal{I}(f)$,
- minimal subsets $\mathcal{C}_i^* \subseteq \mathcal{X}_i \setminus \mathcal{X}_{i-1}, i = 1, \ldots, t$, such that $\mathcal{R}(\mathcal{X}_{i-1} \cup \mathcal{C}_i^*) = \mathcal{R}(\mathcal{X}_i)$.

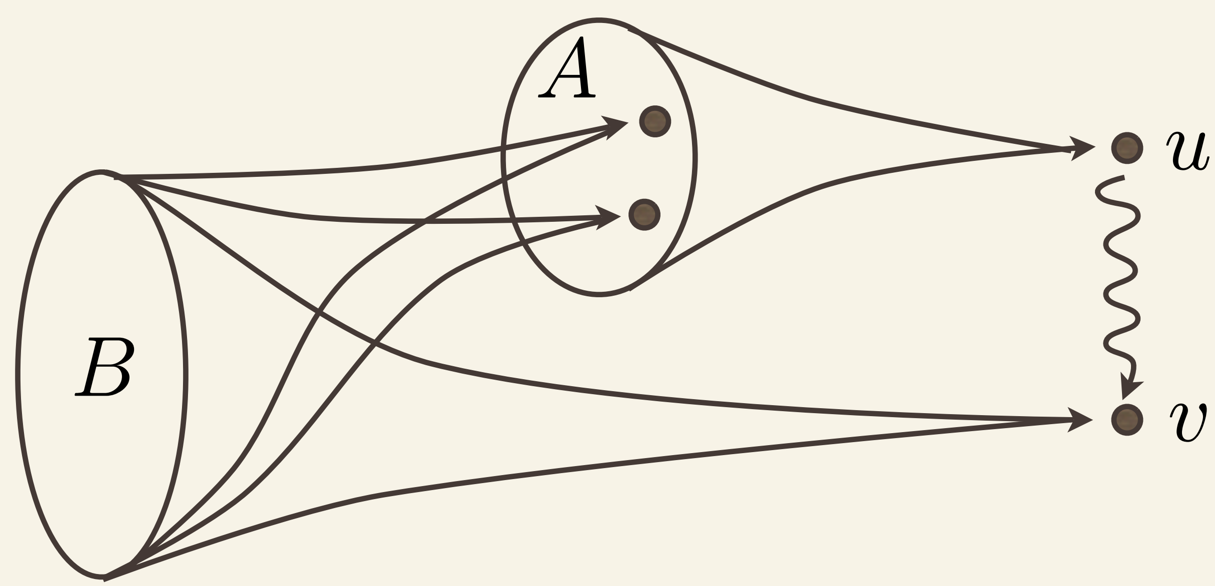
Then:

- $\mathcal{C}^* = \bigcup_{i=1}^{t} \mathcal{C}_i^*$ is a minimal representation of f.

If we can find these sets effectively and solve corresponding subproblems effectively, we are done.

Clause graph

- Let φ be a pure Horn CNF representing a function f, we define clause graph $D_\varphi = (V_\varphi, E_\varphi)$ as follows:
  - $V_\varphi = \varphi$
  - $(A \vee u, B \vee v) \in E_\varphi$ if and only if:
    - v can be reached from u by a path in G_φ, and
    - for every $a \in A$, $(B \vee a)$ is an implicate of f.



Properties of clause graphs

- By $D_f = (V_f, E_f)$ we denote $D_{\mathcal{I}(f)}$.
- By $Cone_H(u)$, where H is a digraph and u one of its vertices, we denote the set of vertices, from which there is a path to u in H.
- If $C = \mathcal{R}(C_1, C_2)$, then $(C_1, C) \in E_f$ and $(C_2, C) \in E_f$
- Therefore $Cone_{D_f}(C)$ is an exclusive set.
- If K is a strong component of D_f containing C, then $Cone_{D_f}(C) \setminus K$ is again an exclusive set.
- Although the size of D_f may be exponentially larger than φ, it is sufficient to work with D_φ, which can be constructed in polynomial time.

Back to decomposition lemma

- Let $K_1, \dots, K_t$ be strong components of D_f in topological order, and
- let us define $\mathcal{X}_i = \bigcup_{j=1}^{i} K_j, i = 1, ..., t.$
- Every $\mathcal{X}_i, i = 1, ..., t$ is an exclusive set and we can use it in decomposition lemma.
- Representation given by $\mathcal{X}_i \cap \varphi$ is sufficient for our needs.
- Now we only have to solve partial problem for each strong component K_i of D_f.

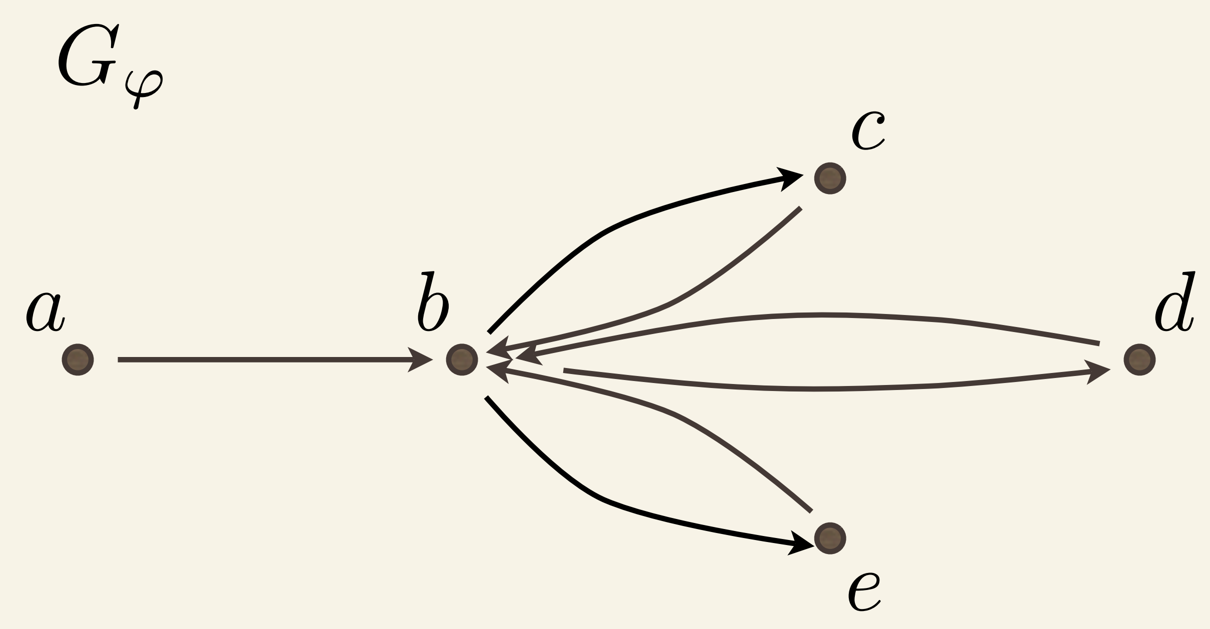
Strong components

- We say, that an implicate $(A \vee u)$ of f is of
  - type 0, if no element of A belong to the same strong component of G_f as u, and it is of
  - type 1, if one element of A belongs to the same strong component of G_f as u.
- If K is a strong component of D_f and f is CQ, then all clauses belonging to K are of the same type.
- Therefore we can assign this type to K as well.
- If K is of type 0, we can leave the clauses in $K \cap \varphi$ as they are, primality and irredundancy of φ is sufficient in this case.

Type 1 (example)

- We shall demonstrate what we can do with strong components of type 1 on the following example:

$$\begin{aligned} \varphi &= (\overline{b} \vee c) \wedge (\overline{b} \vee e) \wedge (\overline{a} \vee \overline{c} \vee b) \\ &\wedge (\overline{a} \vee \overline{e} \vee b) \wedge (\overline{a} \vee \overline{d} \vee b) \wedge (\overline{a} \vee \overline{b} \vee d) \end{aligned}$$



Type 1 (example)

- D_φ has two strong components:

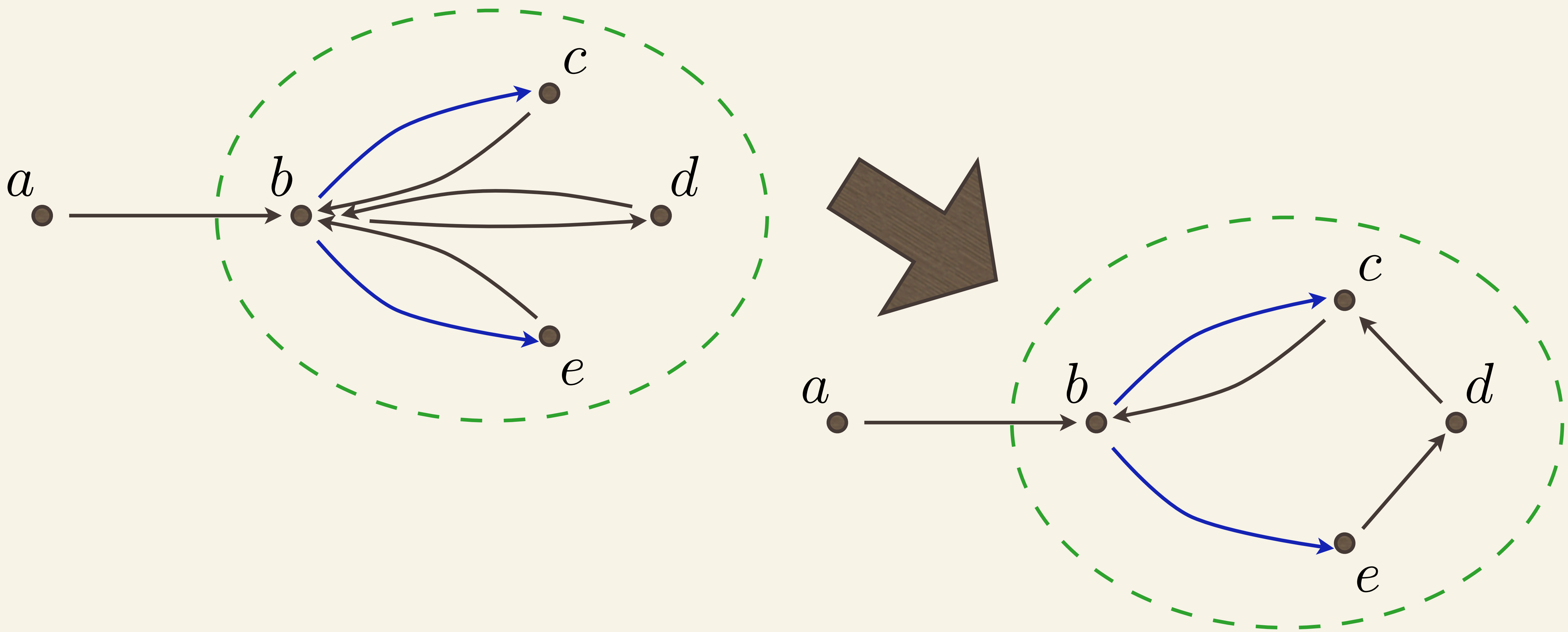
  $K_1 = \{(\bar{b} \vee c), (\bar{b} \vee e)\}$

  $K_2 = \{(\bar{a} \vee \bar{c} \vee b), (\bar{a} \vee \bar{e} \vee b), (\bar{a} \vee \bar{d} \vee b), (\bar{a} \vee \bar{b} \vee d)\}$

- K_1 is itself minimum (primality and irredundancy are sufficient for it).

Type 1 (example)

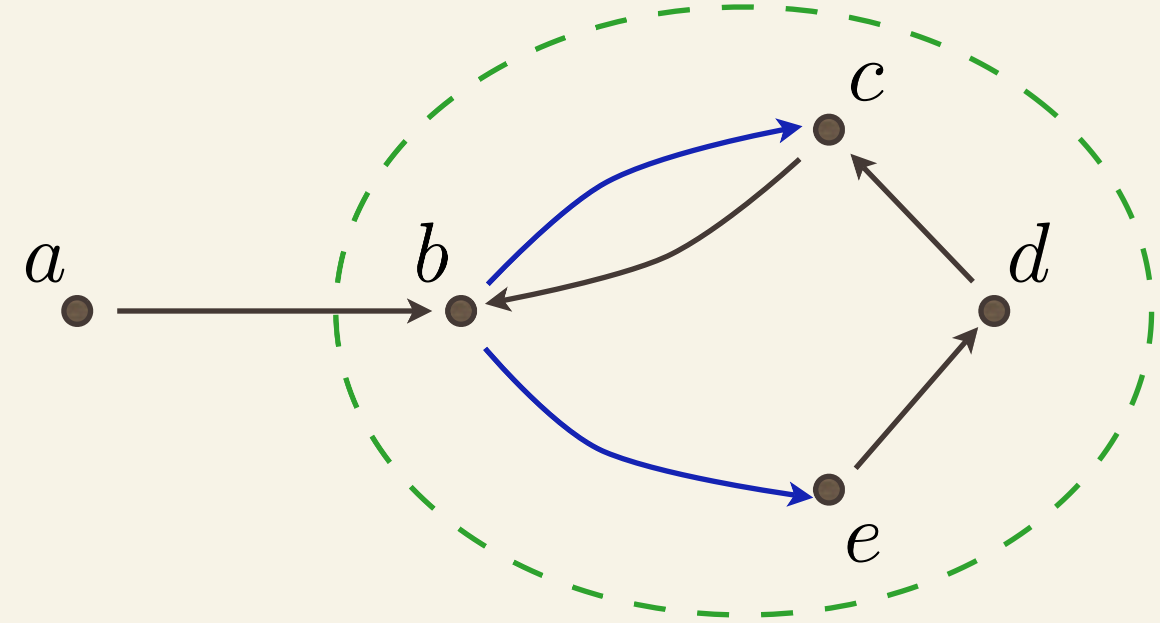
- We can find smaller representation of K_2 by finding a smaller representation of strong component of G_φ containing b, c, d, and e, but blue arcs generated by clauses in K_1 cannot change.



Type 1 (example)

- By this we get an equivalent minimum CNF:

$$\begin{aligned}\varphi' &= (\bar{b} \vee c) \wedge (\bar{b} \vee e) \wedge (\bar{a} \vee \bar{e} \vee d) \\ &\wedge (\bar{a} \vee \bar{d} \vee e) \wedge (\bar{a} \vee \bar{e} \vee b)\end{aligned}$$

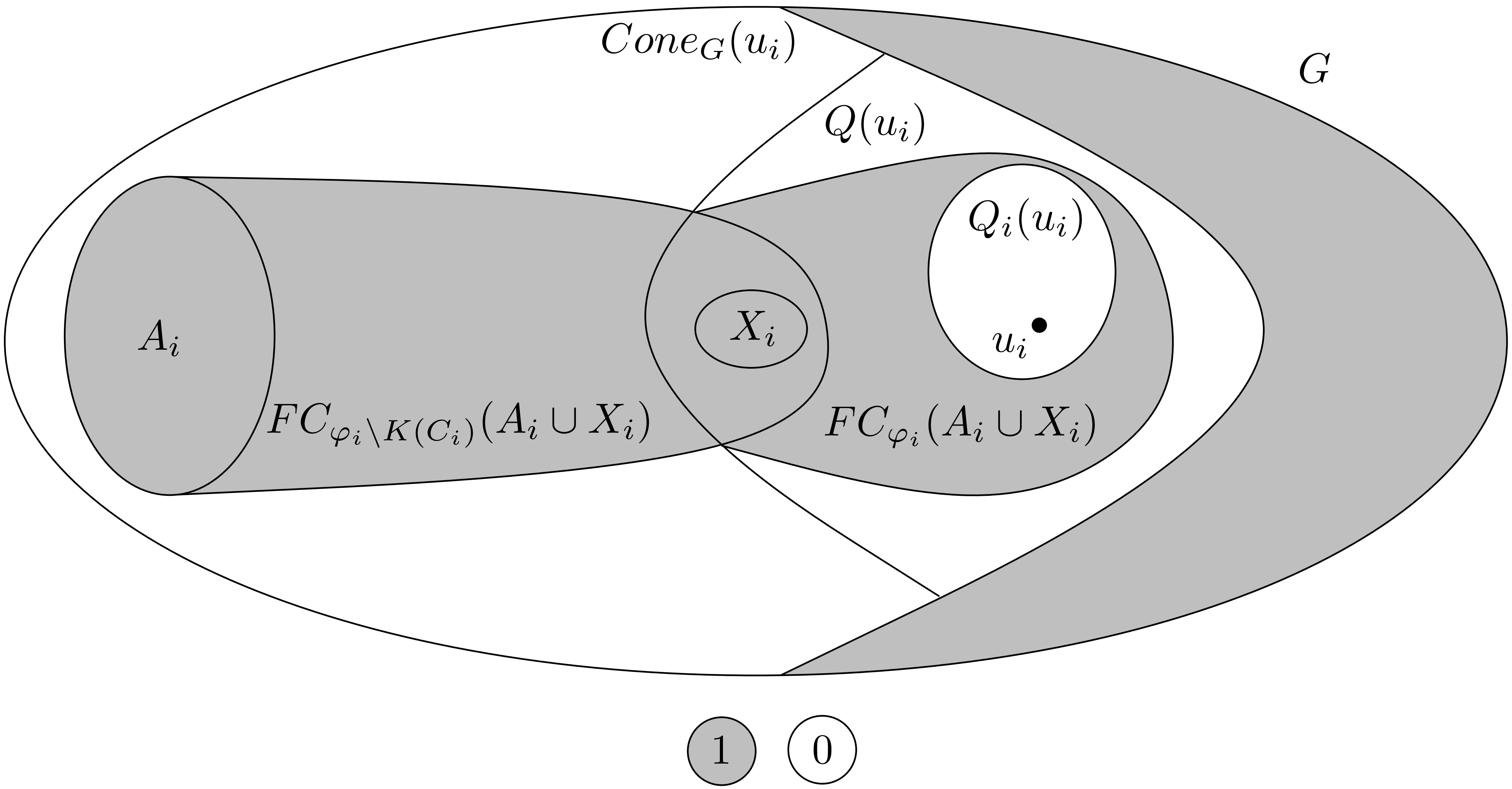


- Smallest representation of a strong component with some fixed arcs can be found in polynomial time.

Essential sets

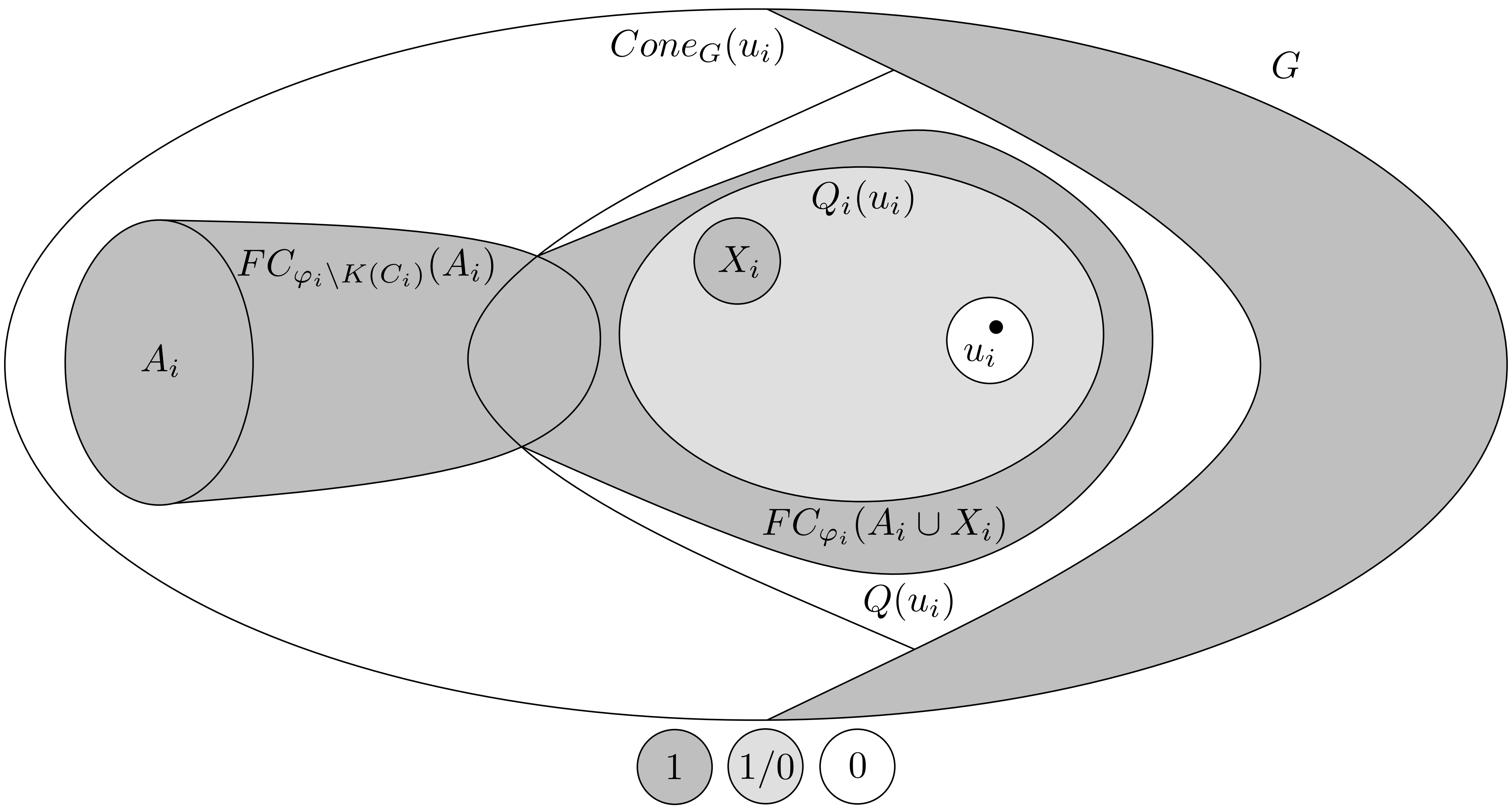
- Based on the minimization algorithm, we can define the essential sets.
- We have to distinguish, whether clause C_i belongs to the strong component $K(C_i)$ of type 0, or 1.
- We give only illustrative pictures of definitions of vectors defining the essential sets to give impression of their complexity.

Type 0



$Cone_G(u_i)$
G
$Q(u_i)$
$Q_i(u_i)$
A_i
X_i
u_i
$FC_{\varphi_i \setminus K(C_i)}(A_i \cup X_i)$
$FC_{\varphi_i}(A_i \cup X_i)$
1
0

Type 1



$Cone_G(u_i)$
G
$Q_i(u_i)$
X_i
$FC_{\varphi_i \setminus K(C_i)}(A_i)$
A_i
u_i
$FC_{\varphi_i}(A_i \cup X_i)$
$Q(u_i)$
1
1/0
0

Conclusions

- There are other classes, about which we can show, that they are coverable. (E.g. interval functions)
- Horn coverable functions form a nontrivial subclass of Horn functions.
- We still do not know, if
  - we can recognize, whether given Horn CNF represent a coverable function,
  - and what is the complexity of minimization of Horn coverable functions.