Deterministic Calibration and Nash Equilibrium

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With Dean Foster (tomorrow at 10:30!)

The Central Question

Can we view a Nash Equilibrium as a convergent point of a "sensible" learning process?

Few general results, short of exhaustive search.

Some Convergence Results

- Extensive list of special cases
  - zero sum
  - 2 player, 2 action
  - potential games
- "Exhaustive search" results:
  - Hypothesis Testing [Foster & Young]
  - Distributed Search [Hart & MasColell]

Outline

- Background on learning in games
- Online Prediction:
  - the calibration task
  - Theorem 1: There exists a deterministic weakly calibrated algorithm.
- Game Theory:
  - Learning using the "public" algorithm
  - Game Theory: If players use the same weakly calibrated algorithm, the joint frequency of play converges into the set of convex combinations of NE.

A Learning Process

1. Players make predictions of other players using:
   - the joint history of play
   - private utility functions
2. Players then take best responses.

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- Fictitious play is the most well studied example
- What constitutes "good" predictions?

Calibration

"Suppose in a long (conceptually infinite) sequence of weather forecasts, we look at all those days for which the forecast probability of precipitation was, say, close to some given value p and ... determine the long run proportion f of such days on which the forecast event (rain) in fact occurred. ... if f=p the forecaster may be termed well calibrated." Dawid [1982]

not stringent condition: 0 1 0 1 0 1 0 ...

Is calibration always possible?

- Yes, with private randomization [Foster & Vohra]
- But how does the forecaster announce his predictions?
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- Theorem: [Foster & Vohra] If players make calibrated predictions in the learning process, then convergence is to correlated equilibria.

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The Online Prediction Setting

- Finite Outcome Space $\{1, 2, ...,m\}$
- Sequence of Outcomes $X_1, X_2...$ where X_t is of the form $(0,0,1,0,0)$
- The empirical frequency at time t is $\frac{1}{T}\sum_{t=1}^{T} X_t$
- A forecasting method F maps histories to probability forecasts. At time t,

$$f_t = F(X_1, X_2, \ldots X_{t-1})$$

Calibration Error

- $I\{f_t \approx p\}$ is the indicator function which is 1 iff f_t is ε-close to p, and 0 otherwise.
- For a sequence X, the calibration error is
- $$R_T(p, X) = \frac{1}{T}\sum_{t=1}^{T} I\{f_t \approx p\}(X_t - f_t)$$
- This is an "internal" regret (directional).
- F is calibrated if $\forall X, \forall p,\ R_T(p, X) \to 0$
- Deterministic F are not calibrated [Oakes; Dawid]
- Randomized F can be [Foster & Vohra]

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"Weak" Calibration Error

- Replace $I\{f_t \approx p\}$ with a continuous (bounded) "test function" w
- For X, the weak calibration error w.r.t. w is
- $$R_T(w, X) = \frac{1}{T}\sum_{t=1}^{T} w(f_t)(X_t - f_t)$$
- Again, a notion of internal regret
- F is **weakly** calibrated if $\forall X, \forall w, \; R_T(w, X) \to 0$
- Definition using weak convergence of measures

A Deterministic Algorithm Exists

- Theorem 1: There exists a deterministic, weakly calibrated F.
- The proof is constructive.
- How do we use this algorithm to calibrated in the standard sense?

Corollary: Randomized Rounding of F

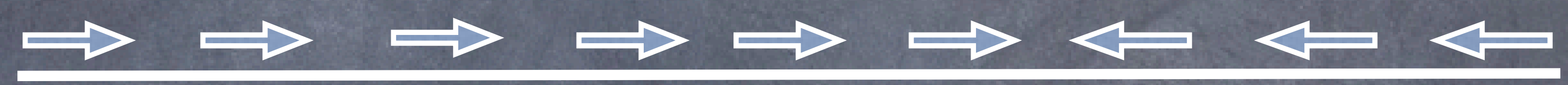
Say deterministic F makes predictions:

0.8606, 0.2387, 0.5751, 0.40051 ...

- Randomly round the forecast onto ε-spaced grid $\{p\}$
- Corollary 1: Almost surely, in the limit, $\forall X \, \forall p$
- $$|R_T(p,X)| < 2\varepsilon$$
- F makes "public" predictions

The Algorithm: Forecast the Fixed Point

- For a grid of points {p}, each point has associated 'vector' error $R_T(p)$



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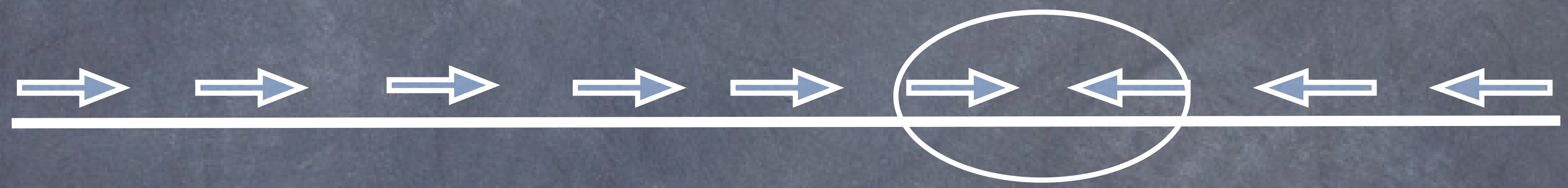

- This defines a mapping: $p \dashrightarrow p+R_T(p)$
- Forecast any fixed point of this interpolated function

The Proof

- Use Blackwell's Approachability Theorem
- "Simpler" proof than most
  - geometric property satisfies approachability condition
- Take grid size to 0 (convergence rate exponential in dim)
- Some comments:
  - this generalizes to an internal regret algorithm for the online convex programming problem

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Game Theory: The Setting

- Players: $1, 2, \ldots n$
- Action set for player i: A_i
- Action spaces: $A = \Pi A_j$ and $A_{-i} = \Pi_{j \neq i} A_j$
- Utility function for player i: $u_i : A \rightarrow [0, 1]$
- If p is a distribution over A, then p_{-i} is a distribution over A_{-i}

Definitions of Equilibrium

p is a Nash equilibrium if

1. p is a product distribution
2. If a_i has positive probability under p, then a_i is a best response to p_{-i}
3.

p is a correlated equilibrium without condition 1 and a natural modification to 2.

A Learning Process

Predictions:

- Players know joint history $X_1, X_2, \ldots X_{t-1}$ where each $X_k \in A$
- Players make forecast f_t which is a distribution over A_{-i}

Actions:

- Player i takes a best response to f_t

Calibration and Learning

Suppose players make calibrated predictions

Theorem: [Foster & Vohra] The frequency of empirical play converges into the set of correlated equilibria.

$$d\left(\frac{1}{T}\sum_{t=1}^{T} X_t, CE\right) \to 0$$

The "Public" Learning Process

Public Predictions:

- At time t, the joint forecast is $f_t = F(X_1, X_2, \ldots X_{t-1})$
- f_t is a distribution over A

Players' Actions:

- Player i marginalizes to get $(f_t)_{-i}$
- Player i then takes "continuous" ε-best response to $(f_t)_{-i}$ (which is a mixed strategy)
- Using randomized rounding is one scheme

Interpretation of the Process

- Players use the same algorithm (based on F).
- Predictions are guaranteed to be calibrated, regardless of how the other players act.
- Algorithm uses observable information and is uncoupled.
- Additional ‘consistency’ in the predictions.

Theorem 2

Assuming:

- F is any weakly calibrated
- the best response functions slowly sharpen

Then:

- The joint empirical frequency of play converges into the set of convex combinations, almost surely
- $$d\left(\frac{1}{T}\sum_{t=1}^{T} X_t, \mathrm{Convex}(NE)\right) \to 0$$
- (Merging) The play and predictions are often close to some Nash Equilibrium:

The Proof

Say f used "often":

- Independence: All players act independently conditioned on of f . Call this constant product play p.
- Best Responses: All players take best responses to f.
- Deterministic Calibration: f is calibrated -> $f=p$
  - why we get NE not just CE

Convergence Rates and Practicality

- Come to Dean's Talk: 10:30
- Doubly exponential in #players
- What do we really need to check?
  - much less, in general
  - even less in structure games
- We do really want to check ourselves on opponents decisions?