RATE-DIVERSITY TRADEOFF OF SPACE-TIME CODES AND OPTIMAL CONSTRUCTIONS

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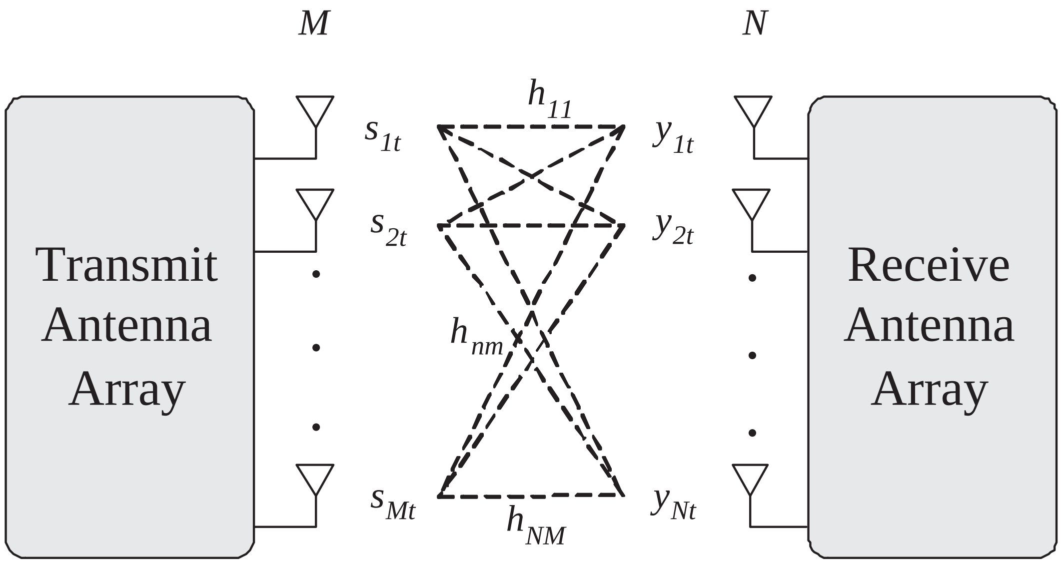
Joint Work

(joint work with Hsiao-feng (Francis) Lu)

Talk Draws From ....

- **H. F. Lu and P. V. Kumar, "Rate-diversity tradeoff of s-t codes with fixed alphabet and optimal constns. for PSK modln,"** ***IEEE Trans. IT*****, Oct. 2003.**
- **——-, "A unified construction of s-t codes with optimal rate-diversity tradeoff," submitted to** *IEEE Trans. IT*, **Jan. 2003.**
- **———, "S-T trellis codes with optimal rate-diversity tradeoff," submitted to** *IEEE Trans. IT*, **Feb. 2003.**
- **———, "Optimal constructions of space-time codes over multiple fading blocks," submitted to** *IEEE Trans. IT*, **Aug. 2003.**
- **——-,"Generalized Unified Construction of Space-Time Codes with Optimal Rate-Diversity Tradeoff," submitted to** ***ISIT 2004, Chicago*****.**
- **H. F. Lu, "Design and performance analysis of space-time codes," Ph.D. dissertation, University of Southern California, Aug. 2003.**
- **Presented in part at** ***ISIT 2003*****,** ***Oberwolfach 2003*****; to appear in** ***Proc. Globecom 2003*** **.**

Space-Time Channel Model



- $N_t = M$ transmit antennas
- $N_r = N$ receive antennas
- Quasi-static Rayleigh fading channel ⇒ channel fixed for T symbol times
- Assume $T \geq N_t$

Why Multiple Antennas ? – Two Viewpoints

No. of Antennas	Pairwise Error Probability (PEP)	≈ Channel Capacity (large SNR)
$\mathrm{N_t} = \mathrm{N_r} = 1$	$\mathrm{PEP} \propto \mathrm{SNR}^{-1}$	$\mathrm{C} = \log(\mathrm{SNR})$
$N_r > 1$ $N_t = 1$	$\mathrm{PEP} \propto \mathrm{SNR}^{-N_r}$	$\mathrm{C} = \log(\mathrm{SNR})$
$N_r > 1$ $N_t > 1$	$\mathrm{PEP} \propto \mathrm{SNR}^{-N_r N_t}$	$C = \min\{N_r, N_t\} \log(\mathrm{SNR})$
	$N_t N_r =$ diversity gain Increased Reliability	$\min\{N_t, N_r\} =$ multiplexing gain Increased Capacity

Qn. Can the increased reliability and capacity be realized simultaneously ?
Ans. No, there is a tradeoff (but again, 2 viewpoints, one old, one new.....)

But First, the Received Signal Model

- The received signal is

$$Y = \alpha H S + W,$$

where $\alpha = \sqrt{\frac{\text{SNR}}{N_t}}$,

–

$$H = \begin{bmatrix} h_{11} & \cdots & h_{1N_t} \\ \cdots & \cdots & \cdots \\ h_{N_r 1} & \cdots & h_{N_r N_t} \end{bmatrix} \quad W = \begin{bmatrix} w_{11} & \cdots & w_{1N_t} \\ \cdots & \cdots & \cdots \\ w_{N_r 1} & \cdots & w_{N_r N_t} \end{bmatrix}$$

– h_{ij} and w_{ij} are i.i.d., circularly symmetric $\mathcal{CN}(0,1)$, with common pdf ,

$$p(u) = \frac{1}{\pi} e^{-|u|^2}$$

– S is drawn from space-time code $\mathcal{S}$ and $\mathbb{E}(\sum |S_{n,t}|^2) \leq N_t T$

$$\mathcal{S} = \left\{ S = \begin{bmatrix} s_{1,1} & \cdots & s_{1,T} \\ \vdots & \ddots & \vdots \\ s_{N_t,1} & \cdots & s_{N_t,T} \end{bmatrix} : s_{i,j} \in \mathcal{A} \right\}$$

First Viewpoint

- Suppose the elements of signal matrix S are drawn from *fixed signal constellation* $\mathcal{A}$, say 16-QAM, 8-PSK etc.
- Reliability is judged via worst-case PEP

Relevant Performance Measures:

- The average number of symbols from $\mathcal{A}$ transmitted per time slot:

$$\gamma := \frac{\log_{|\mathcal{A}|} |\mathcal{S}|}{T}$$

  γ is the Signaling Rate

- The PEP has(large SNR) upper bound:

$$P(S_1 \rightarrow S_2) \leq \frac{\text{Const.}}{(\text{SNR})^{N_r d}},$$

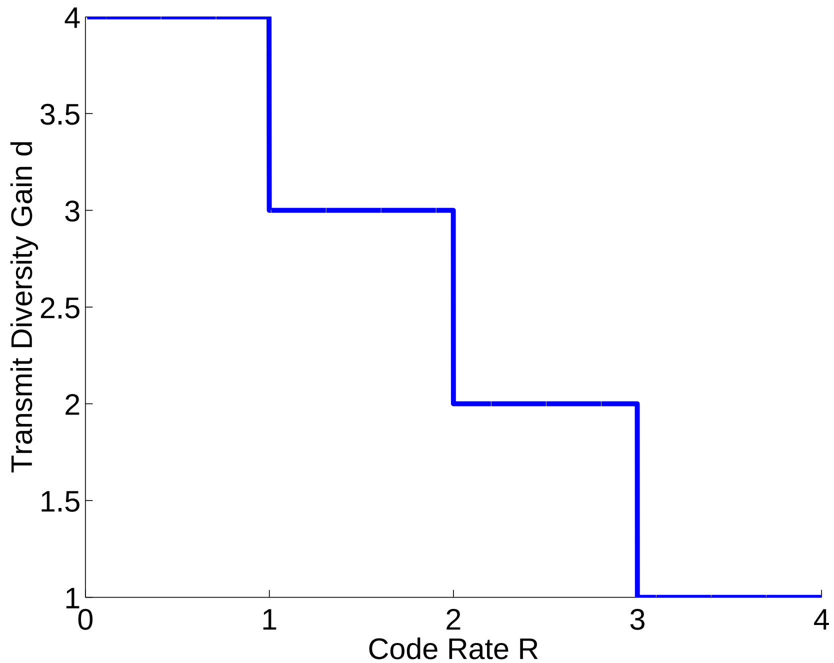
  where the transmit diversity d, $1 \leq d \leq N_t$ is given by

$$d = \min \{ \text{rank}(S_1 - S_2) \mid S_1 \neq S_2, \quad S_1, S_2 \in \mathcal{S} \}$$

Diversity-Rate Tradeoff

Singleton Bd identifies tradeoff between signaling rate γ and transmit diversity d :

$$d \le N_t - \gamma + 1$$



d versus γ ($= R$)
(4 Transmit Antennas)

Some Previous Work

Numerous constructions in the literature, not many address signal constellations considered here: PAM, QAM, and 2^K-PSK and rate-diversity tradeoff.

- **V. Tarokh, N. Seshadri, and A. R. Calderbank, "Space-time codes for high data rate wireless communi-cation: performance criterion and code construction", *IEEE Trans. Inform. Theory*, Mar. 1998.**
- **A. R. Hammons Jr. and H. El Gamal, "On the theory of space-time codes for PSK modulation," *IEEE Trans. Inform. Theory*, Mar. 2000.**
- **Y. Liu, M. P. Fitz, and O. Y. Takeshita, "A rank criterion for QAM space-time codes, " *IEEE Trans. Inform. Theory*, Dec. 2002.**
- **H. El Gamal and A. R. Hammons Jr., "On the design of algebraic space-time codes for MIMO block fading channels, " *IEEE Trans. Inform. Theory*, Jan. 2003.**
- **R. Calderbank, S. Diggavi, N. Al-Dahir, "Space-Time Signaling Based on Kerdock and Delsarte-Goethals Codes, submitted to *ICC 2004*.**
- **N. J. A. Sloane, "Disjoint Subspaces and Space-Time Codes over PSK Modulation,", *DIMACS, 2003*.**

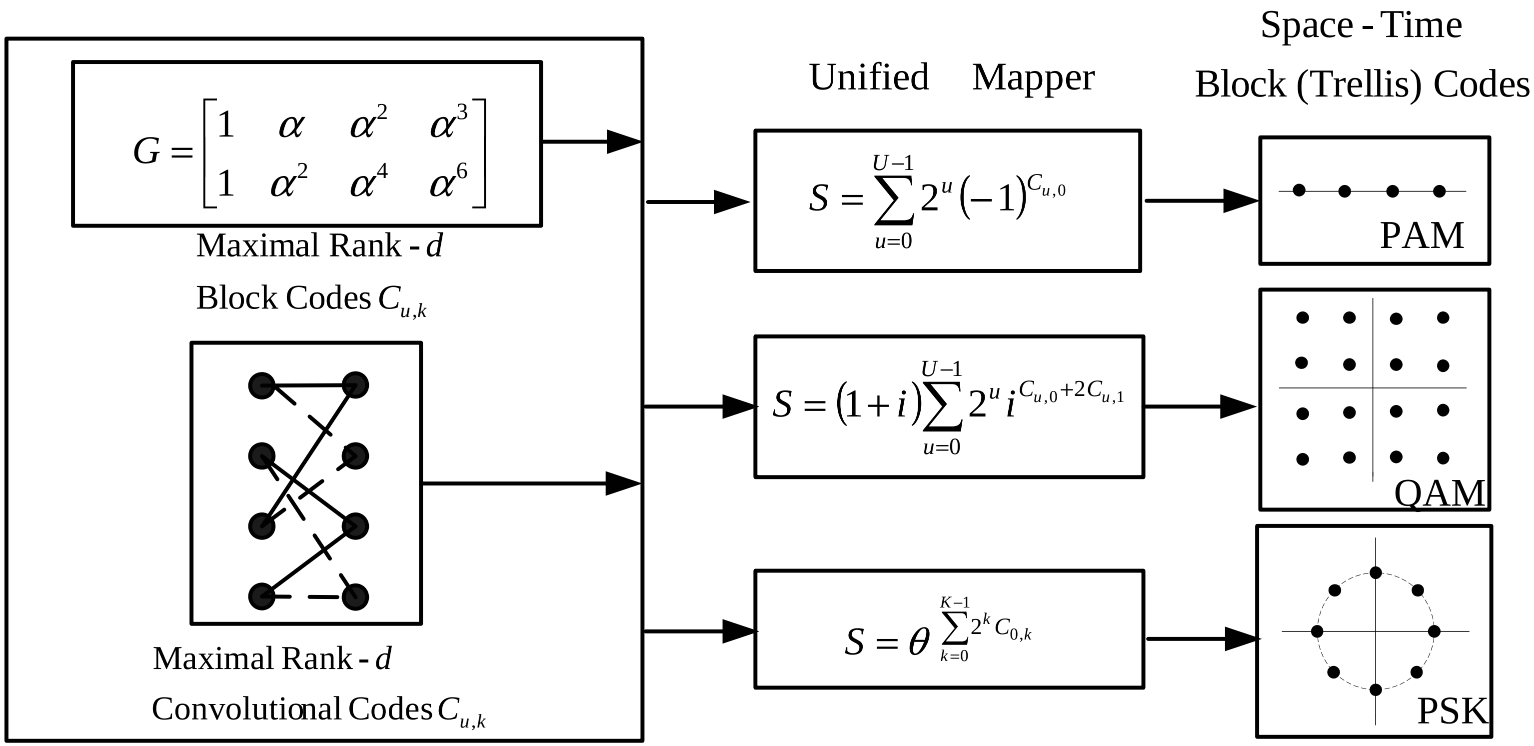
A Unified Construction Achieving Rate-Diversity Tradeoff

- Our construction aimed at practical constellations such as PAM, QAM and 2^K-PSK
- **Input:** Maximal rank-d binary block or convolutional code $\mathcal{C}$ (such codes are readily constructed)
- **Output:** Signal matrix S

$$(C_{0,0}, C_{0,1}, \cdots, C_{U-1,K-1}) \mapsto S = \sum_{u=0}^{U-1} 2^u \omega^{\sum_{k=0}^{K-1} 2^k C_{u,k}},$$

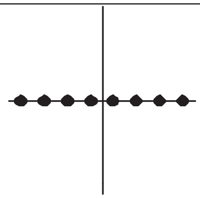
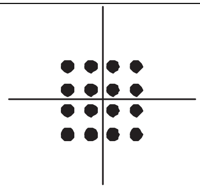
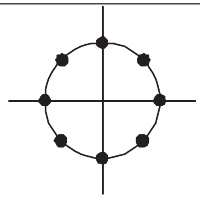
- $C_{u,k}$ are codewords drawn (with possible repetition) from $\mathcal{C}$
- $\omega = \exp(\frac{\imath 2\pi}{2^K})$
- Space-time code achieves transmit diversity d and has rate $\gamma = N_t - d + 1$
- By choosing $\mathcal{C}$ appropriately, every point on the rate-diversity tradeoff can be achieved

Special Instances of the Unified Construction



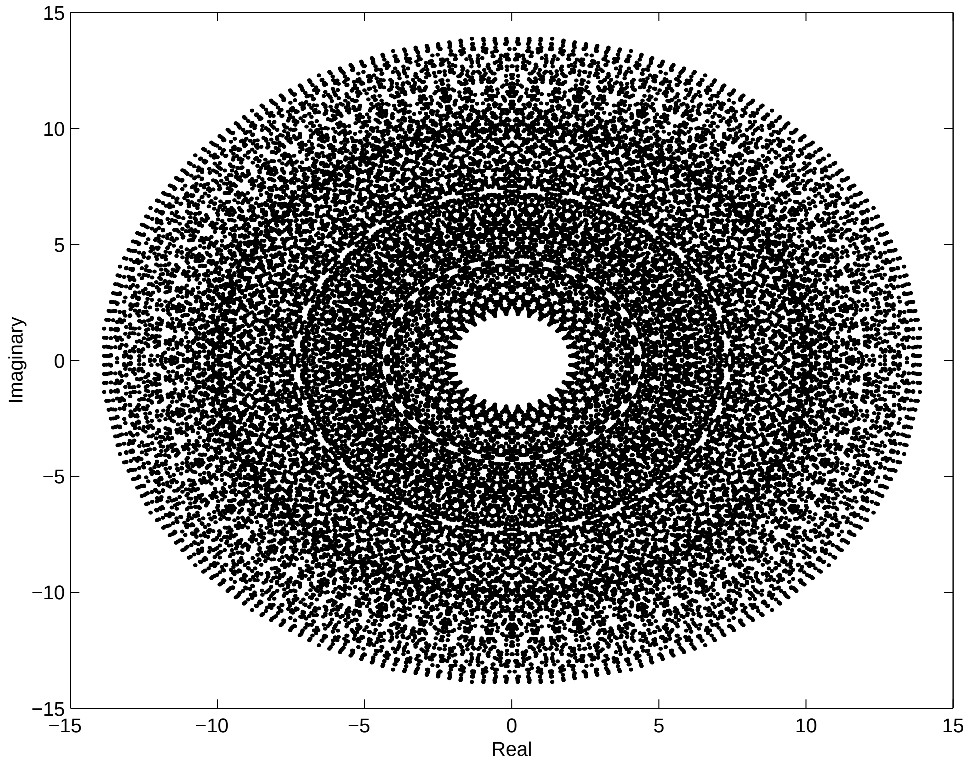
Unified Construction Maps Binary Codes to Space-Time Codes

Signal Expressions for Some Example Constellations

	Constellation	Corresponding Expression
8-PAM		$S = (-1)^{C_{00}} + 2(-1)^{C_{01}} + 4(-1)^{C_{02}}$
16-QAM		$(1+\imath)\ \left[\imath^{C_{00}+2C_{01}} + 2\imath^{C_{10}+2C_{11}}\right]$
8-PSK		$S = \theta^{C_{00}+2C_{01}+4C_{02}}, \quad \theta = e^{\frac{\imath 2\pi}{8}}$

Other Possible Constellations :-)

Signal constellation of size = 32, 768 $\quad (U = 3,\ K = 5,\ \theta = \exp\left(\imath\, \frac{2\pi}{32}\right))$



Explaining Why the Construction Works - a Simple Lemma

Lemma Let $\theta = \exp\{\imath \frac{2\pi}{2^K}\}$. Then

$$\frac{\theta^a - \theta^b}{1-\theta} = a \oplus b \quad (\text{mod } (1-\theta)),$$

where $\oplus$ indicates addition modulo 2.

Proof

$$\begin{aligned} \frac{\theta^a - \theta^b}{1-\theta} &= \frac{[1-(1-\theta)]^a - [1-(1-\theta)]^b}{1-\theta} \\ &= (b-a) \quad (\text{mod } (1-\theta)) \\ &= a \oplus b \quad (\text{mod } 2). \end{aligned}$$

This follows because $(1-\theta)$ divides 2 :

$$\begin{aligned} 2 &= (1-(-1)) = (1-(\theta^{2^{K-1}})) = (1+\theta^{2^{K-2}})(1-\theta^{2^{K-2}}) \\ &= (1+\theta^{2^{K-2}})(1+\theta^{2^{K-3}})(1+\theta^{2^{K-4}})\cdots(1+\theta)(1-\theta). \end{aligned}$$

Explaining the Construction in 16-QAM Case

$$\frac{S' - S}{(1+\imath)(1-\imath)} = \frac{[\imath^{C'_{00}+2C'_{01}} + 2\imath^{C'_{10}+2C'_{11}}] - [\imath^{C_{00}+2C_{01}} + 2\imath^{C_{10}+2C_{11}}]}{(1-\imath)}$$
$$= [C'_{00} \oplus C_{00}] \pmod 2$$

Thus

$$\underline{u}^{\dagger}\left[\frac{S'-S}{(1+\imath)(1-\imath)}\right] = \underline{0}^T \iff [\underline{u}^{\dagger} \pmod{(1-\imath)}]\left[C'_{00} \oplus C_{00}\right] = \underline{0}^T.$$

So if $C'_{00} - C_{00}$ has full rank, so does $S' - S$. More generally,

$$\text{Rank}_{\mathcal{C}}\ (S' - S) \geq \text{Binary Rank}\ (C'_{00} \oplus C_{00}).$$

Similar arguments apply when $C'_{00} = C_{00}$, but $C'_{01} \neq C_{01}$ etc.

Example of 16-QAM Construction

Example Suppose $C'_{00} = C'_{01} = C'_{10} = C'_{11} = [0]$ and

$$C_{00} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad C_{01} = C_{10} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

then

$$\frac{S' - S}{(1+\imath)(1-\imath)} = \frac{[\imath^{C_{00}+2C_{01}} + 2\imath^{C_{10}+2C_{11}}] - [\imath^{C'_{00}+2C'_{01}} + 2\imath^{C'_{10}+2C'_{11}}]}{(1-\imath)}$$

$$= \begin{bmatrix} 1+2\imath & 0 & 0 & 1+2\imath \\ 0 & 2\imath & 2\imath & 0 \\ 0 & 2\imath & 2\imath-1 & 2\imath-1 \\ 0 & 2\imath & 1+2\imath & 1+2\imath \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \pmod{1-\imath}$$

$$= C'_{00} \oplus C_{00} \pmod{1-\imath}$$

Thus $\text{Rank}_{\mathcal{C}}\ (S' - S) \geq$ Binary Rank $\left[C'_{00} \oplus C_{00}\right] = 2.$

Maximal Rank-d Binary Block Code: Construction

Theorem 1. *For any* $1 \le M \le T < \infty$ *and* $1 \le d \le M$, *set* $\gamma = M - d + 1$ *and define the set of code polynomials by*

$$\mathcal{F} = \left\{ f(x) \;:\; f(x) = \sum_{i=0}^{\gamma-1} f_i x^{2^i},\; f_i \in \mathbb{F}_{2^T} \right\}.$$

The collection $\mathcal{C}$ *of* $2^{\gamma T}$ $(M \times T)$ *code matrices*

$$\mathcal{C} = \left\{ \begin{bmatrix} \underline{f(1)}^T \\ \underline{f(\alpha)}^T \\ \vdots \\ \underline{f(\alpha^{M-1})}^T \end{bmatrix} \;:\; f(x) = \sum_{i=0}^{\gamma-1} f_i x^{2^i},\; f_i \in \mathbb{F}_{2^T} \right\},$$

where by $\underline{f(\alpha^i)}$ *we mean the representation of the element* $f(\alpha^i)$ *as a binary* $(T \times 1)$ *column vector, forms a maximal rank-*d *binary block code.*

This construction is possibly known earlier; appears in both Lu & Kumar and Lusina, Gabidulin & Bossert, *IEEE Trans IT*, Oct. 2003

Example Maximal Rank-$d = 2$ Binary Code

- $N_t = T = 4, \quad \gamma \leq N_T - d + 1 = 3$
- α primitive in $\mathbb{F}_{16}$, $\alpha^4 + \alpha + 1 = 0$
- $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\} = \{1, \alpha, \alpha^2, \alpha^3\}$ basis for $\mathbb{F}_{16}$ over $\mathbb{F}_2$
- $f(x)$ of form $a_1 x + a_2 x^2 + a_4 x^4$ so rate $= 3$

Example: $f(x) = \alpha^3 x + x^4$ and $g(x) = 0$ and

$$C_f = \begin{bmatrix} f(\gamma_1) \\ f(\gamma_2) \\ f(\gamma_3) \\ f(\gamma_4) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad C_g = \begin{bmatrix} g(\gamma_1) \\ g(\gamma_2) \\ g(\gamma_3) \\ g(\gamma_4) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then

$$\Delta C = C_f \oplus C_g = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ has rank } \geq 2 = d \; .$$

Why This Works

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} \begin{bmatrix} f(\gamma_1) \\ f(\gamma_2) \\ f(\gamma_3) \\ f(\gamma_4) \end{bmatrix} = 0 \Leftrightarrow \sum_i a_i f(\gamma_i) = 0 \Leftrightarrow f(\sum_i \gamma_i) = 0$$

since $f(x) = \alpha^3 x + x^4$ is an *additive* polynomial.

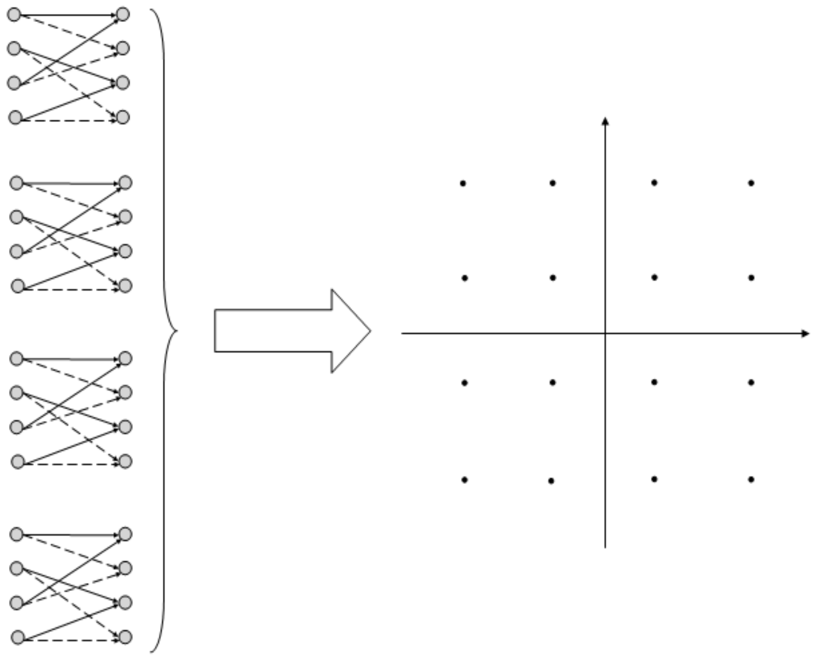
Thus

$$|\text{ left nullspace of } C \;| \leq \quad \#\text{ of zeros of } f \quad \leq \; 2^2$$

so that

$$\text{rank}(C) \; \geq 2 \; = \; N_t - \gamma + 1 \; = \; 4 - 3 + 1.$$

Trellis Codes Achieving Rate-Diversity Tradeoff



- Trellis codes derived by replacing block codes by convolutional codes
- the resulting codes have a product trellis description

Maximal Rank-d Convolutional Code Generation: Example

- $N_t = 4$, Rate $\gamma = 3$, diversity gain $d \leq N_t - \gamma + 1 = 2$
- Let $f(D) = D^4 + D + 1$, a primitive polynomial of degree 4 over $\mathbb{F}_2$.

Set polynomial generator matrix

$$G(D) = \begin{bmatrix} 1 & D & D^2 & D^3 \\ 1 & D^2 & D^4 & D^6 \\ 1 & D^4 & D^8 & D^{12} \end{bmatrix} \pmod{D^4 + D + 1}$$

$$= \begin{bmatrix} 1 & D & D^2 & D^3 \\ 1 & D^2 & D+1 & D^3 + D^2 \\ 1 & D+1 & D^2+1 & D^3 + D^2 + D + 1 \end{bmatrix}$$

Why Does This Work ?

Consider left nullspace of a code matrix:

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ D & D^2 & D^4 \\ D^2 & D^4 & D^8 \\ D^3 & D^6 & D^{12} \end{bmatrix} \begin{bmatrix} u_0(D) \\ u_1(D) \\ u_2(D) \end{bmatrix} = 0 \pmod{D^4 + D + 1}$$

can be rewritten

$$\sum_{i=0}^{2} u_i(D)[a(D)]^{2^i} = 0 \pmod{D^4 + D + 1}$$

where $a(D) = \sum_{i=0}^{3} a_i D^i$.

Since a polynomial of degree ≤ 4 can have at most 4 zeros, the left nullspace of the matrix is at most 2, so that the code matrix is of rank $\geq 4 - 2 = 2$.

Coding across L-Blocks

$$Y = [Y_1 \; Y_2 \; \cdots \; Y_L] = \sqrt{\frac{\rho}{N_t}} [H_1 \; H_2 \; \cdots \; H_L] \begin{bmatrix} S_1 & & \\ & \ddots & \\ & & S_L \end{bmatrix} + [W_1 \; W_2 \; \cdots \; W_L] \, ,$$

- Transmit diversity gain (El Gamal & Hammons):

$$d \; := \; \sum_{\ell=1}^{L} rank(S_\ell - S'_\ell),$$

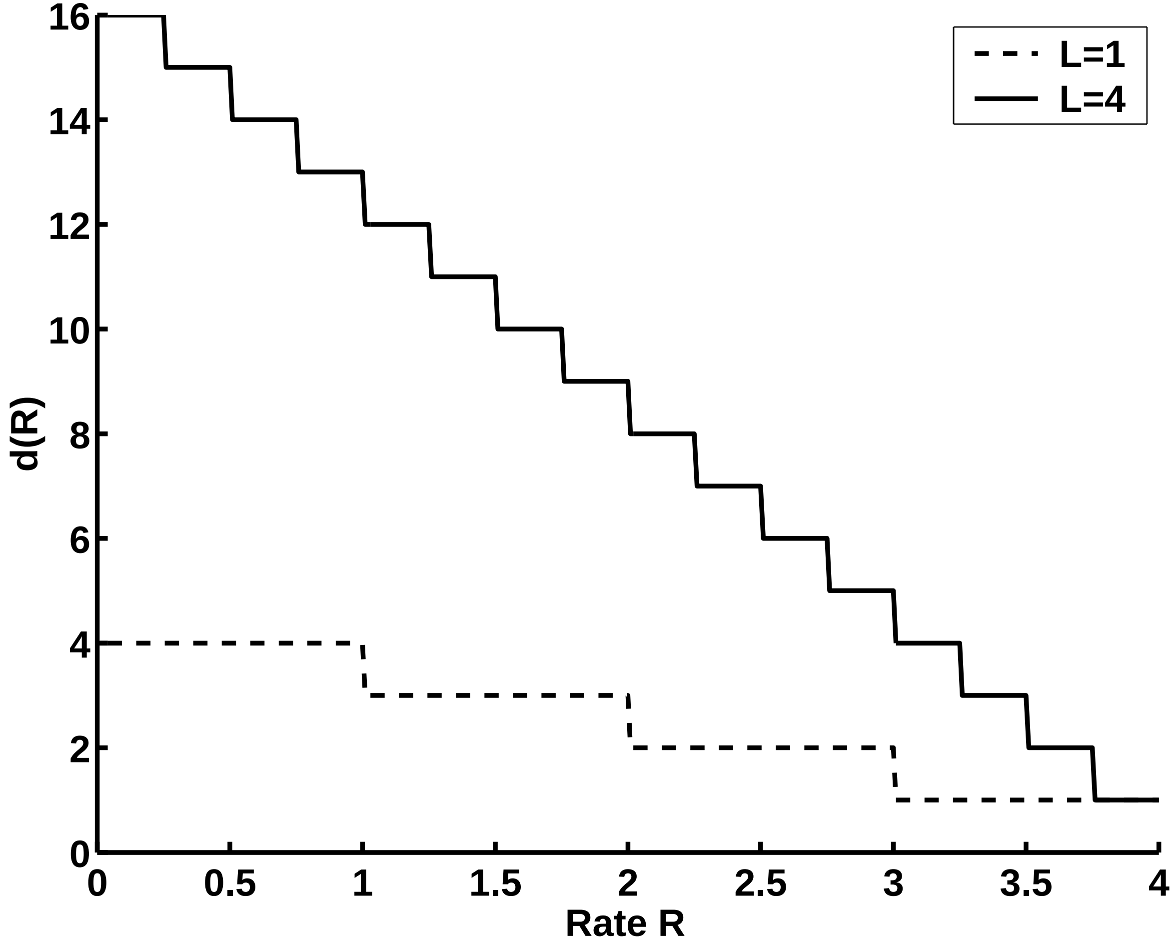
- New tradeoff:

$$\gamma \leq \frac{N_t L - d + 1}{L}$$

- Here we achieve tradeoff for case $T \geq LN_t$ by noting that

$$\text{rank} \begin{bmatrix} S_1 & & \\ & \ddots & \\ & & S_L \end{bmatrix} \geq \text{rank} \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_L \end{bmatrix}$$ which we know how to handle .

Tradeoff in Case of Transmission Over Multiple Blocks



Decoding

- **Given code structure:**

$$(C_{0,0}, C_{0,1}, \cdots, C_{U-1,K-1}) \;\mapsto\; S = \sum_{u=0}^{U-1} 2^u \omega^{\sum_{k=0}^{K-1} 2^k C_{u,k}},$$

it is hoped that multilevel decoding and/or iterative decoding techniques can be used to help decode the code

- **QAM Case: If $A_j,\ 1 \le j \le 2^\gamma$ are all binary codewords, signal can be placed in linear dispersion form thereby enabling sphere decoding :**

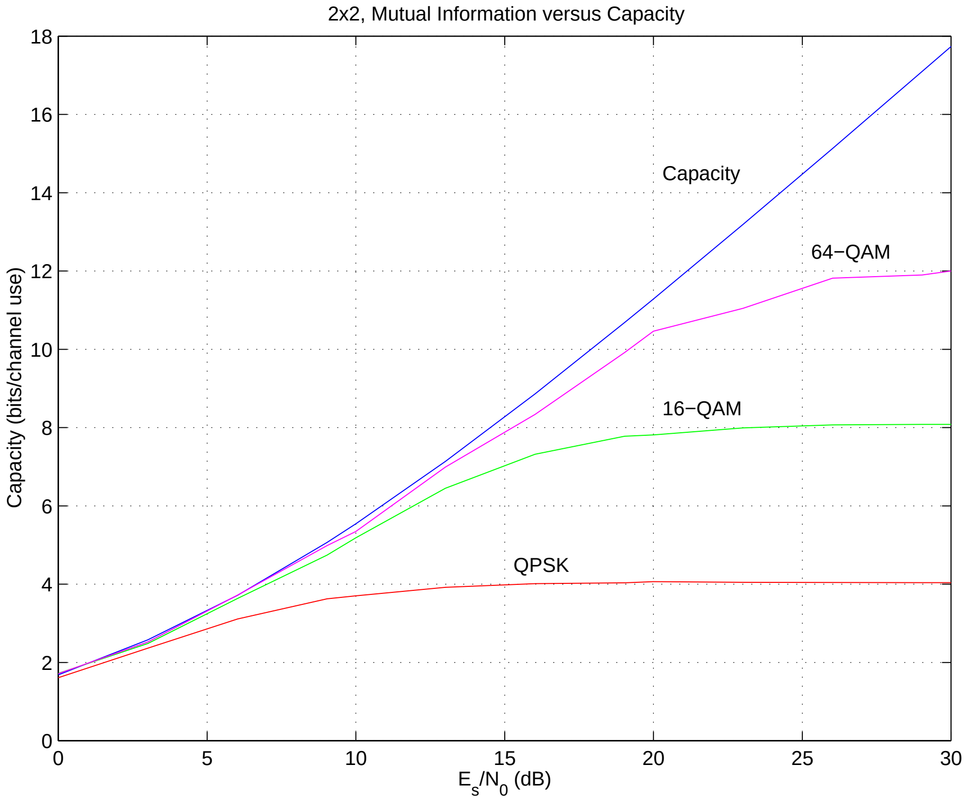
$$S \;=\; \sum_{j=1}^{2^\gamma} (u_j + \imath v_j)(-1)^{A_j}.$$

Incidentally, $1 \le u_j, v_j \le 2^\gamma$ in binary form satisfy

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{(where each column is binary expansion of a } u_j \text{),}$$

i.e., only one 1 in each row.

QAM Constellation Size and Channel Capacity



Shows a need to vary constellation size as SNR grows (plot from Hochwald/ten Brink)

Diversity-Multiplexing Gain (D-MG) Tradeoff (Zheng/Tse)

Differences between D-MG tradeoff and earlier tradeoff:

- PEP is replaced by the more relevant codeword error probability P_{cwe}
- Code size is allowed to grow with SNR to ensure we are operating close to the ergodic capacity. Zheng and the define multiplexing gain to be r when the source is transmitting $r\log(\mathrm{SNR})$ bits per channel use;
- Principal result of Zheng/Tse:

$$r = (N_t - r)(N_r - r)$$

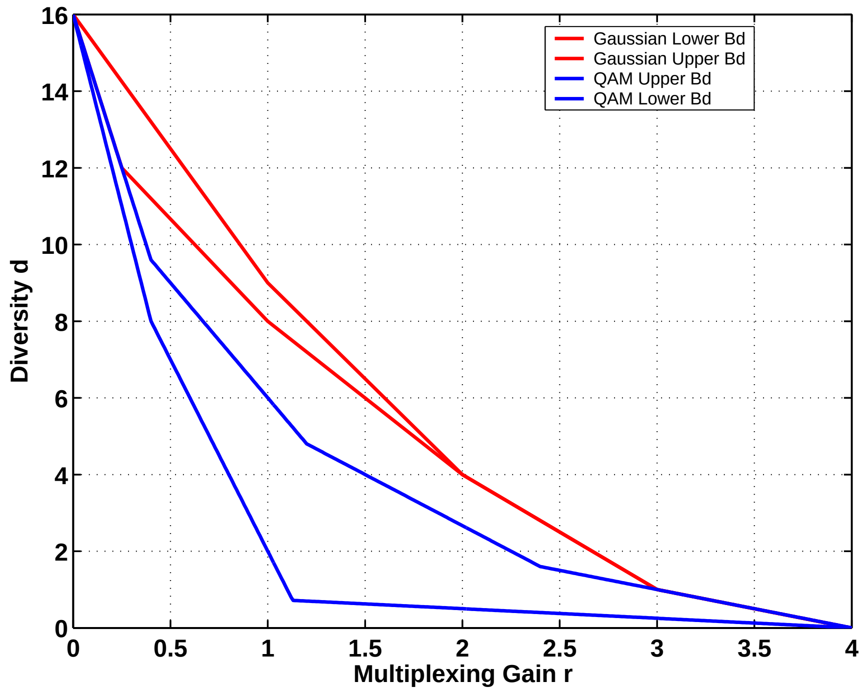
  for integer r. In between points are obtained through straight line interpolation.
- In our construction, we can achieve $r\log(\mathrm{SNR})$ bits per channel use by varying constellation size M^2 and signaling rate γ. Thus, in the QAM case:

$$M^{2\gamma} = \mathrm{SNR}^r \text{ so that}$$

$$Y = \theta HS + W \quad \text{where} \quad \theta = \sqrt{\frac{\mathrm{SNR}}{N_t}\frac{3}{2(M^2-1)}} \doteq \mathrm{SNR}^{1-\frac{r}{\gamma}}$$

Upper/Lower Bounds on D-MG Tradeoff $\quad N_t = N_r = T = 4$

Red = Zheng/Tse (Gaussian) **Blue = our QAM Construction**



$\boldsymbol{d}$ **versus** $\boldsymbol{r}$

$$P_{cwe} \doteq \text{SNR}^{-d} \qquad r \log(\text{SNR}) \ \text{ bits/channel use}$$