Van der Waerden Diversions

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Abstract

Van der Waerden's theorem states that every finite coloring of positive integers contains arbitrarily long monochromatic arithmetic progressions (AP). This theorem was generalized in numerous ways, e.g. one being the "density"-type theorem of Szemerédi: every subset of integers that has positive upper density contains an arbitrarily long AP.

In this talk, we survey several other generalizations of van der Waerden's theorem, that have received considerable interest in the last decade.

1. (Rainbow analogues) Every 3-coloring of $[3n]$, such that all the color classes have the same cardinality, contains a *rainbow* $AP(3)$ (joint with Fox, Jungić, Mahdian, and Nešetřil). However, we can construct 4-colorings of $[4n]$, with equinumerous color classes, that contain no rainbow $AP(4)$ (joint with Conlon and Jungić).
2. (generalized van der Waerden triples; joint with Fox) Let a and b be positive integers such that $a \leq b$ and $(a,b) \neq (1,1)$ (the case $(a,b) = (1,1)$ being equivalent to van der Waerden's theorem for $AP(3)$). We prove that there exists a 6-coloring of positive integers that does not contain a monochromatic (a,b)-triple, that is, a triple (x,y,z) such that $y = ax + d$ and $z = bx + 2d$ for some positive integer d. This confirms a conjecture of Landman and Robertson.
3. (joint with Silva) We refute several conjectures of Fox on the existence of color patterns, other than rainbow or monochromatic, along the terms of an arithmetic progression. We construct an equinumerous 2-coloring of $[2n]$ that contains no *color-separating* $AP(6)$ and no *color-alternating* $AP(6)$. We also construct a 2-coloring of positive integers, such that the color discrepancy of every initial interval is bounded by one, and that contains no color-alternating $AP(8)$. Our constructions use Thue-Morse–type sequences.

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