TOURNAMENTS

N. ALON

TEL AVIV + IAS
Ranking a Tournament

A Tournament is an oriented complete graph

\[ g(n) = \max k \text{ such that all tournaments on } n \text{ vertices contains an acyclic subgraph with } \geq \frac{1}{2} \binom{n}{2} + \frac{1}{2} k \text{ edges.} \]

Erdős and Moon (65): \[ \Omega(n) \leq g(n) \leq O(n^{3/2} (\log n)^{1/2}) \]

\[ \Omega(n^{3/2}) \leq g(n) \leq O(n^{3/2}) \]

Spencer (70)

Spencer (80)

W. F. de la Vega (83)
THUS, \exists \text{ TOURNAMENTS THAT DO NOT ADMIT A "GOOD" RANKING.}

PROBLEM: EXPLICIT CONSTRUCTIONS?

(\text{ERDÖS+MOON(65)}, \text{SPENCER(85)})
For the mathematician, even combinatorialist, not familiar with probabilistic methods a much weaker method can be quite impressive: There exist tournaments which cannot be ranked so that more than 51% of the games are in order. Construction of specific tournaments with this property appears to be quite difficult and quite possibly, in this author's opinion, impossible.
**DEF:** For a prime \( p \equiv 3 \pmod{4} \), the quadratic residue tournament \( T_p \) has \( V(T_p) = \mathbb{Z}_p \) with \( i \to j \iff i-j \equiv 0 \pmod{p} \) (i.e., a quadratic residue).

**Lemma:** If \( T_p = (V,E) \), \( A, B \subseteq V \), and \( A \cap B = \emptyset \) then

\[
|e(A,B) - e(B,A)| \leq |A|^\frac{1}{2} |B|^\frac{1}{2} \cdot p^\frac{1}{2} \cdot n^\frac{1}{2}.
\]

\[
|\{i \to j \in E : i \in A, j \in B\}|
\]
**PF (SKETCH):** If \( C = (C_{u,v})_{u,v \in V} \)

\[
C_{u,v} = \begin{cases} 
1 & \text{if } u \rightarrow v \\
-1 & \text{if } v \rightarrow u \\
0 & \text{if } v = u 
\end{cases}
\]

Then \( C^T C = \mu I - J \).

Hence, the eigenvalues of \( C^T C \) are 0 and \( \mu \) and thus:

\[
| \varepsilon (A, B) - \varepsilon (B, A) | = | X_A^t C X_B |
\leq \| X_A \|_2 \| C X_B \|_2 = | A |^{1/2} (X_B^t C^T C X_B)^{1/2}
\leq | A |^{1/2} \mu^{1/2} | B |^{1/2}.
\]

\( \blacksquare \)
If $T_p = (V,E)$ is the quadratic residue tournament, and $v_{i_1}, v_{i_2}, \ldots, v_{i_p}$ is a permutation of its vertices, then roughly half the edges are directed in each direction.

$\Rightarrow T_p$ contains no acyclic subgraph with more than $\frac{1}{2} \binom{n}{2} + c n^{3/2} \log n$ edges.
A recent application [A(05), Charbit, Thomassen, Ye (06)]: The problem of computing the max. no. of edges in an acyclic subgraph of a given input tournament is NP-hard.

- settles a conjecture of Bang-Jensen + Thomassen (92)
- NP hardness under randomized reduction proved by Ailon, Charikar, Newman (05).
- Even approximating the maximum to within an additive error of $n^{2-\epsilon}$ is NP-hard.
REDUCTION FROM MAXIMUM ACYCLIC SUBGRAPH
IN A GENERAL DIGRAPH $F$:
A "BLOW UP" OF $F$ WITH "$T_F$ ON TOP"
GIVES RESULT.
TOURNAMENTS, BOXES AND VOTING PARADOXES

N. ALON, G. BRIGHTWELL, N. KIERSTEAD, A. KOSTOCHKA & P. WINKLER
Fellowships
Each member of a committee of $2^r-1$ has a linear order on candidate $x$ is better than $y$ if $\geq r$ members rank $x$ higher.

Undesirable: A choice of winners is a non-winner which is better than all winners.
MAJORITY TOURNAMENTS

For 2²⁻¹ linear orders on \([n] = \{1, 2, \ldots, n\}\),
the \(R\)-majority tournament is the tournament on \([n]\) in which
\(i \rightarrow j \iff i\) is above \(j\) in \(R\) orders.

Example:

\[L_1; \quad 1 \rightarrow 2 \rightarrow 3\]
\[L_2; \quad 2 \rightarrow 3 \rightarrow 1\]
\[L_3; \quad 3 \rightarrow 1 \rightarrow 2\]
which tournaments are majority tournaments

mc garvey (53): every tournament is a \( k \)-majority tournament for some \( k \),
\( k \leq n^2 \) [\( n = \text{no. of vertices} \)].

stearns (59): \( k \leq o(n) \) suffices

erdős & moser (64): \( O(n/\log n) \) orders suffice (and this is tight).
Schütte, Erdös (63):

∀x ∈ A tournament T = (V, E) with no dominating set of size ≤ t. That is, ∀v ∈ V, |v'| = t ∈ evv that beats each v ∈ V.

Hence, large committees may have problems even with many fellowships. What about small committees?
CONJECTURE (KIRSTEAD & TROTTER):

\[ \forall k \in F(\omega) (\leq \infty) \text{ such that every } \omega \text{-majority tournament has a dominating set of size } \leq F(\omega) \] 

[that is; \( F(\omega) \) fellowships suffice for a committee of size \( 2^{\omega-1} \)].

DEF: \( F(\omega) = \text{smallest } F \text{ such that every } \omega \text{-majority tournament has a dominating set of size } \leq F. \)
A geometric result: for \( x, y \in \mathbb{R}^d \)

\[
\text{Box}(x, y) = \text{smallest box with faces parallel to the coordinates hyperplanes that contains } x \text{ and } y.
\]

\[
\text{Box}(x, y) = \{ z \in \mathbb{R}^d : z_i \text{ is between } x_i \text{ and } y_i \; \forall i \}. \]

Thm (Bárány & Lehel 87): \( \forall d \exists C = c(d) \text{ \forall finite } V \subseteq \mathbb{R}^d \)

\( \exists V' \subseteq V, |V'| \leq c(d) \text{ such that } V \subseteq \bigcup_{x, y \in V'} \text{Box}(x, y). \)

Moreover, \( c(d) \leq (2d^2 + 1)d \cdot 2^d \)

E.g., \( c(2) = 4 \)
PROP: \( F(2) \leq C(22\cdot-1) \)

PF: GIVEN 22\cdot-1 LINEAR ORDERS \( L_1, \ldots, L_{22\cdot-1} \) ON \([n]\), DEFINE \( V = \{ p^{(1)}, p^{(2)}, \ldots, p^{(n)} \} \subseteq R^{22\cdot-1} \)

\( p^{(i)} \) = RANK OF \( i \) IN \( L_j \)

TAKE \( V' \subseteq V \), \( |V'| \leq C(22\cdot-1) \) WITH \( V \subseteq U \) BOX \( (x, y) \).

\( \{ i \mid p^{(i)} \notin V' \} \) DOMINATES \( V' \)

Indeed, IF \( p^{(r)} \in \text{Box}(p^{(i)}, p^{(j)}) \), THEN EITHER \( i \) OR \( j \) IS ABOVE \( r \) IN \( > \) \( \leq \) ORDERS.
COROLLARY:

FOR A COMMITTEE OF 3

\[ F(2) \leq C(3) \leq (2 \cdot 3^2 + 1)^3 \cdot 2^3 = 13,123 \cdot 2^3 \]

FELLOWSHIPS SUFFICE!

CAN WE DO BETTER?
Can actually show:

- \( \Omega(b^2 / \log n) \leq f(n) \leq O(b^2 \log n) \)
- \( f(z) = 3 \)

[i.e., for a committee of 3, 3 fellowships suffice].
OPEN: \[ F(1) = 2, \quad F(2) = 2, \quad 4 \leq F(3) < 360 \]

[How many fellowships does a committee of 5 need?]

\[ \Omega \left( \frac{h}{\log h} \right) \leq F(h) \leq O \left( h \log h \right) \]

MORAL: BIGGER COMMITTEES NEED BIGGER BUDGET!