Degree-Sequence-Forcing Sets

Stephen G. Hartke

Department of Mathematics
University of Illinois at Urbana-Champaign
hartke@math.uiuc.edu

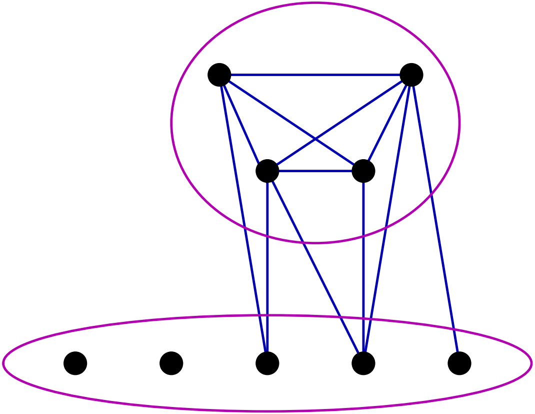
DIMACS/DIMATIA/Rényi Tripartite Conference

April 28, 2006

Joint work with Michael Barrus and Mohit Kumbhat

Split Graphs

Def. A graph G is a split graph if the vertex set can be partitioned into a clique and an independent set.



Split Graphs

Def. A graph G is a split graph if the vertex set can be partitioned into a clique and an independent set.

$$G \text{ is split} \iff G \text{ is } \{2K_2, C_4, C_5\}\text{-free.}$$

[Földes–Hammer, 1976]

Split Graphs

Def. A graph G is a split graph if the vertex set can be partitioned into a clique and an independent set.

$$G \text{ is split} \iff G \text{ is } \{2K_2, C_4, C_5\}\text{-free.}$$

[Földes–Hammer, 1976]

Degree sequence $d(G) = (d_1, \ldots, d_n)$, nonincreasing.

$$G \text{ is split} \iff \sum_{i=1}^{m} d_i = m(m-1) + \sum_{i=m+1}^{n} d_i,$$
$$\text{where } m = \max\{k : d_k \geq k-1\}.$$

[Hammer–Simeone, 1981]

Degree sequence characterization gives a linear time recognition algorithm.

Graph Classes with Both Characterizations

There are several graph classes with both forbidden induced subgraph and degree sequence characterizations.

Split [Hammer–Simeone, 1981]

$$\{2K_2, C_4, C_5\}\text{-free} \iff \sum_{i=1}^{m} d_i = m(m-1) + \sum_{i=m+1}^{n} d_i$$

Graph Classes with Both Characterizations

There are several graph classes with both forbidden induced subgraph and degree sequence characterizations.

Split [Hammer–Simeone, 1981]

$$\{2K_2, C_4, C_5\}\text{-free} \iff \sum_{i=1}^{m} d_i = m(m-1) + \sum_{i=m+1}^{n} d_i$$

Empty

$$\{K_2\}\text{-free} \iff d_1 = \cdots = d_n = 0.$$

Complete

$$\{2K_1\}\text{-free} \iff d_1 = \cdots = d_n = n-1.$$

Graph Classes with Both Characterizations

There are several graph classes with both forbidden induced subgraph and degree sequence characterizations.

Split [Hammer–Simeone, 1981]

$$\{2K_2, C_4, C_5\}\text{-free} \iff \sum_{i=1}^{m} d_i = m(m-1) + \sum_{i=m+1}^{n} d_i$$

Empty

$$\{K_2\}\text{-free} \iff d_1 = \cdots = d_n = 0.$$

Complete

$$\{2K_1\}\text{-free} \iff d_1 = \cdots = d_n = n-1.$$

Threshold [Hammer–Ibaraki–Simeone, 1978]

$$\{2K_2, C_4, P_4\}\text{-free} \iff \sum_{i=1}^{r} d_i = r(r-1) + \sum_{i=r+1}^{n} \min\{r, d_i\} \ldots$$

Graph Classes with Both Characterizations

There are several graph classes with both forbidden induced subgraph and degree sequence characterizations.

Split [Hammer–Simeone, 1981]

$$\{2K_2, C_4, C_5\}\text{-free} \iff \sum_{i=1}^{m} d_i = m(m-1) + \sum_{i=m+1}^{n} d_i$$

Empty

$$\{K_2\}\text{-free} \iff d_1 = \cdots = d_n = 0.$$

Complete

$$\{2K_1\}\text{-free} \iff d_1 = \cdots = d_n = n-1.$$

Threshold [Hammer–Ibaraki–Simeone, 1978]

$$\{2K_2, C_4, P_4\}\text{-free} \iff \sum_{i=1}^{r} d_i = r(r-1) + \sum_{i=r+1}^{n} \min\{r, d_i\} \ldots$$

Pseudosplit [Maffray–Preissmann, 1994]

$$\{2K_2, C_4\}\text{-free} \iff \text{split or } \sum_{i=1}^{q} d_i = q(q+4) + \sum_{i=q+6}^{n} d_i \ldots$$

Degree-Sequence-Forcing Sets

Let $\mathcal{F}$ be a set of graphs.

The class of $\mathcal{F}$-free graphs is the set of graphs that do not contain any member of $\mathcal{F}$ as an induced subgraph.

Def. $\mathcal{F}$ is degree-sequence-forcing (DSF) if the class of $\mathcal{F}$-free graphs has a degree sequence characterization.

Degree-Sequence-Forcing Sets

Let $\mathcal{F}$ be a set of graphs.

The class of $\mathcal{F}$-free graphs is the set of graphs that do not contain any member of $\mathcal{F}$ as an induced subgraph.

Def. $\mathcal{F}$ is degree-sequence-forcing (DSF) if the class of $\mathcal{F}$-free graphs has a degree sequence characterization.

Ques: Can we characterize the DSF sets?

Degree-Sequence-Forcing Sets

Let $\mathcal{F}$ be a set of graphs.

The class of $\mathcal{F}$-free graphs is the set of graphs that do not contain any member of $\mathcal{F}$ as an induced subgraph.

Def. $\mathcal{F}$ is degree-sequence-forcing (DSF) if the class of $\mathcal{F}$-free graphs has a degree sequence characterization.

Ques: Can we characterize the DSF sets?

Are there infinitely many DSF sets?

Degree-Sequence-Forcing Sets

Let $\mathcal{F}$ be a set of graphs.

The class of $\mathcal{F}$-free graphs is the set of graphs that do not contain any member of $\mathcal{F}$ as an induced subgraph.

Def. $\mathcal{F}$ is degree-sequence-forcing (DSF) if the class of $\mathcal{F}$-free graphs has a degree sequence characterization.

Ques: Can we characterize the DSF sets?

Are there infinitely many DSF k-sets?

Complements

Let $\overline{\mathcal{F}} = \{\overline{H} : H \in \mathcal{F}\}$

Prop. $\mathcal{F}$ is DSF set iff $\overline{\mathcal{F}}$ is.

Complements

Let $\overline{\mathcal{F}} = \{\overline{H} : H \in \mathcal{F}\}$

Prop. $\mathcal{F}$ is DSF set iff $\overline{\mathcal{F}}$ is.

Proof:

Given a graphic sequence d, d has an $\overline{\mathcal{F}}$-free realization

$\iff$ $\overline{d}$ has an $\mathcal{F}$-free realization

$$\overline{d} = (n-1-d_n, n-1-d_{n-1}, \ldots, n-1-d_1)$$

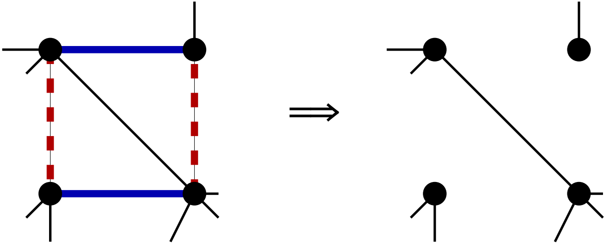
$\iff$ all realizations of $\overline{d}$ are $\mathcal{F}$-free

$\iff$ all realizations of d are $\overline{\mathcal{F}}$-free.

$\square$

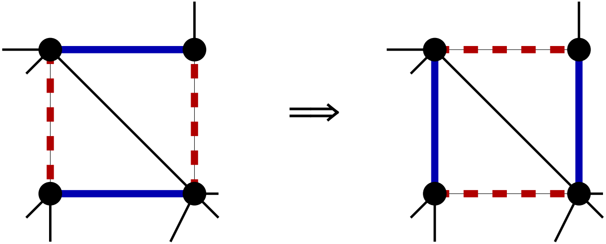
2-Switches

Def. A 2-switch is



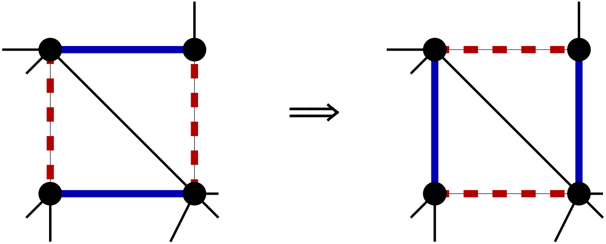
2-Switches

Def. A 2-switch is



2-Switches

Def. A 2-switch is

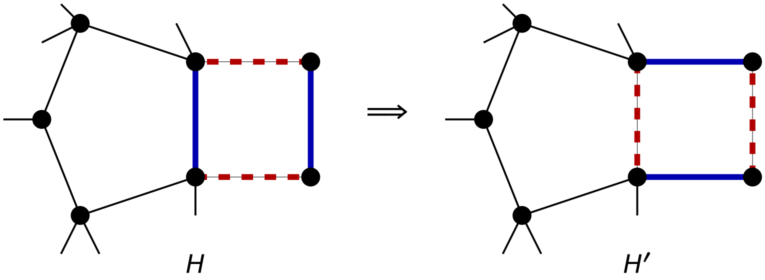


Thm. If G and H have the same degree sequence, then G can be obtained through several 2-switches applied to H.

Forests

Prop. Every DSF set contains a forest.

Proof:
Take the graph $F \in \mathcal{F}$ with the fewest number of cycles.



H' has fewer cycles than any graph in $\mathcal{F}$.

Contradiction if F had a cycle. □

Singleton DSF Sets

Prop. $\{K_1\}$, $\{K_2\}$ and $\{2K_1\}$ are the only singleton DSF sets.

Proof: $\mathcal{F} = \{F\}$

F is a forest.

Singleton DSF Sets

Prop. $\{K_1\}$, $\{K_2\}$ and $\{2K_1\}$ are the only singleton DSF sets.

Proof: $\mathcal{F} = \{F\}$

F is a forest.

$\overline{F}$ is a forest.

Singleton DSF Sets

Prop. $\{K_1\}$, $\{K_2\}$ and $\{2K_1\}$ are the only singleton DSF sets.

Proof: $\mathcal{F} = \{F\}$

F is a forest.

$\overline{F}$ is a forest.

an extremal result: F has at most 4 vertices.

Singleton DSF Sets

Prop. $\{K_1\}$, $\{K_2\}$ and $\{2K_1\}$ are the only singleton DSF sets.

Proof: $\mathcal{F} = \{F\}$

F is a forest.

$\overline{F}$ is a forest.

an extremal result: F has at most 4 vertices.

...and then you check the small cases. □

More Required Classes

Prop. Every DSF set contains a graph from each of the following classes:

1. forests of stars,
2. disjoint union of complete graphs,
3. complete bipartite graphs.

and a graph from each of the complements.

DSF Pairs

Thm. $\mathcal{F} = \{F_1, F_2\}$ is a DSF pair if and only if $\mathcal{F}$ is one of the following sets:

1. $\{A, B\}$, where A is one of K_1, K_2, or $2K_1$, and B is arbitrary;
2. $\{P_3, K_3\}$, $\{P_3, K_3 + K_1\}$, $\{P_3, K_3 + K_2\}$, $\{P_3, 2K_2\}$, $\{P_3, K_2 + K_1\}$;
3. $\{K_2 + K_1, 3K_1\}$, $\{K_2 + K_1, K_{1,3}\}$, $\{K_2 + K_1, K_{2,3}\}$, $\{K_2 + K_1, C_4\}$;
4. $\{K_3, 3K_1\}$;
5. $\{2K_2, C_4\}$.

DSF Pairs

Thm. $\mathcal{F} = \{F_1, F_2\}$ is a DSF pair if and only if $\mathcal{F}$ is one of the following sets:

1. $\{A, B\}$, where A is one of K_1, K_2, or $2K_1$, and B is arbitrary;
2. $\{P_3, K_3\}$, $\{P_3, K_3 + K_1\}$, $\{P_3, K_3 + K_2\}$, $\{P_3, 2K_2\}$, $\{P_3, K_2 + K_1\}$;
3. $\{K_2 + K_1, 3K_1\}$, $\{K_2 + K_1, K_{1,3}\}$, $\{K_2 + K_1, K_{2,3}\}$, $\{K_2 + K_1, C_4\}$;
4. $\{K_3, 3K_1\}$;

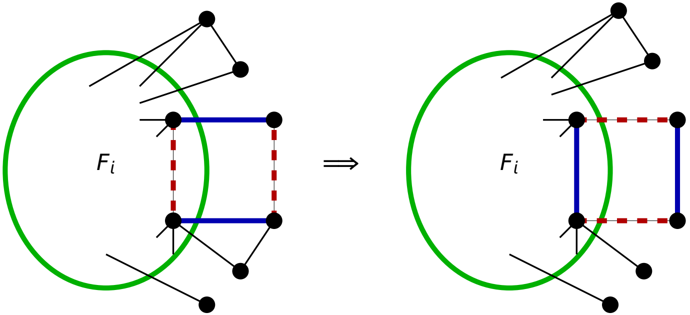
   Uninteresting: these classes are unigraphs

5. $\{2K_2, C_4\}$.

   Interesting: these are the pseudosplit graphs

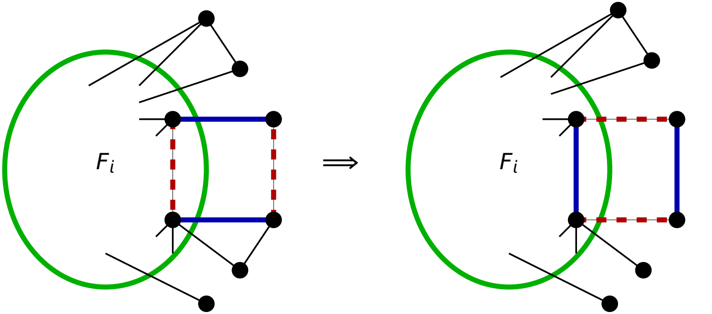
Small Counterexamples

Prop. Let $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ not be a DSF set. Then there exists an "$\mathcal{F}$-breaking pair" (H, H').



Small Counterexamples

Prop. Let $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ not be a DSF set. Then there exists an "$\mathcal{F}$-breaking pair" (H, H').



There exists an $\mathcal{F}$-breaking pair on at most $\max\{n(F_i)\} + 2$ vertices.

Number of Vertices and Edges

Let $\mathcal{F} = \{F_1, ..., F_k\}$ be a set, where $n(F_1) \leq ... \leq n(F_k)$.

If $n(F_1) + 2 < n(F_2)$ and $\{F_1\}$ is not a DSF set,

then $\mathcal{F}$ is not a DSF set.

Proof: $\{F_1\}$-breaking pair (H, H') cannot contain $F_2, ..., F_k$.

Number of Vertices and Edges

Let $\mathcal{F} = \{F_1, \ldots, F_k\}$ be a set, where $n(F_1) \leq \ldots \leq n(F_k)$.

If $n(F_1) + 2 < n(F_2)$ and $\{F_1\}$ is not a DSF set,

then $\mathcal{F}$ is not a DSF set.

Proof: $\{F_1\}$-breaking pair (H, H') cannot contain $F_2, \ldots, F_k$.

Cor. If $\mathcal{F}$ is a DSF set, then $n(F_{i+1}) \leq n(F_i)$.

Number of Vertices and Edges

Let $\mathcal{F} = \{F_1, \ldots, F_k\}$ be a set, where $n(F_1) \leq \ldots \leq n(F_k)$.

If $n(F_1) + 2 < n(F_2)$ and $\{F_1\}$ is not a DSF set,

then $\mathcal{F}$ is not a DSF set.

Proof: $\{F_1\}$-breaking pair (H, H') cannot contain $F_2, \ldots, F_k$.

Cor. If $\mathcal{F}$ is a DSF set, then $n(F_{i+1}) \leq n(F_i)$.

A similar statement can be proved stating that the graphs $F_1, \ldots, F_k$ can't differ by more than $2n(F_i)$ edges.

Number of DSF k-sets

$\mathcal{F}$ must contain both a forest and the complement of a forest.

Number of DSF k-sets

$\mathcal{F}$ must contain both a forest and the complement of a forest.

Thm. If $\mathcal{F}$ is a DSF set, then

$$n(F_1) \leq 2k + \frac{1}{2} + \sqrt{12k^2 - 10k + \frac{1}{4}} \leq 6k.$$

DSF Triples

(Partial) **Thm.** $\mathcal{F} = \{F_1, F_2, F_3\}$ is a DSF triple if $\mathcal{F}$ is one of the following sets:

1. $\{F_1, F_2\}$ is a DSF pair not $\{2K_2, C_4\}$, and F_3 is any graph;

2. $\{F_1, F_2\} = \{2K_2, C_4\}$; and F_3 induces $2K_2$ or C_4, or F_3 is one of the following:

   C_5 $\quad$ P_4 $\quad$ K_n $\quad$ $K_n - e$

   $K_{1,3}$ $\quad$ $K_{1,3} + K_1$ $\quad$ paw $\quad$ $\ldots$

   and complements

3. $\{F_1, F_2\}$ is not a DSF pair: bound is $n(F_1) \leq 15$.

   e.g., $\{3K_1, 2K_2, paw\}$

Future Work

1. Improve the bound on the number of DSF k-sets.
2. Find the degree characterizations of DSF sets.
3. Complete the classification of DSF triples.
4. Can the process of checking the finite cases be automated?

Degree-Sequence-Forcing Sets

Stephen G. Hartke

Department of Mathematics
University of Illinois at Urbana-Champaign
hartke@math.uiuc.edu

DIMACS/DIMATIA/Rényi Tripartite Conference

April 28, 2006

Joint work with Michael Barrus and Mohit Kumbhat