Sparseness in the implicit equation of rational parametric curves and surfaces*

Ioannis Z. Emiris[1] and Ilias S. Kotsireas[2]

[1] Department of Informatics & Telecommunications, University of Athens, Panepistimiopolis, 15771 Greece, emiris@di.uoa.gr
[2] Department of Computing, Wilfrid Laurier University, 75, University Avenue West, Waterloo N2L 3C5, ON, Canada, ikotsire@wlu.ca, and Ontario Research Centre for Computer Algebra, University of Western Ontario, London N6A 5B7, ON, Canada, Ilias.Kotsireas@orcca.on.ca

Extended Abstract

In [6] we used various tools from toric (or sparse) elimination theory, in order to predict the support of the implicit equation of a parametric curve or (hyper)surface. The problem of switching from a rational parametric representation to an implicit, or algebraic, representation of a curve, surface, or hypersurface lies at the heart of several algorithms in computer-aided design, cf. e.g. [1–3, 5, 8, 10]. Three implicitization algorithms (based on interpolation) are immediately improved by our construction. More specifically, we use information on the support and certain coefficients of the toric (or sparse) resultant. The computed support of the implicit equation depends on the sparseness of the parametric expressions and is much tighter than the one predicted by degree arguments. Our Maple implementation illustrates many cases in which we obtain the exact support. We refer to our method as IPSOS .

In this paper we show how certain coefficients of the implicit equation (sparse resultant) can be predicted as well. This is illustrated with the Fröberg/Dickenstein example which exhibits significant sparseness. example Moreover, we exploit the application of the IPSOS algorithm to the implicitization method of moving lines, which expresses implicit equations in compact determinantal forms. We also show that IPSOS is suitable for *generic implicitization* because the formulation of the algorithm in terms of Newton polytopes and mixed volumes, which exploit the structure in the parametric expressions, depends only on their nonzero terms. On the other hand, we exploit information on the support of the toric (or sparse) resultant by considering the extreme monomials as described in [7, 12].

Our motivation comes mainly from three implicitization algorithms based on interpolation. The first one (see [3]) treats parametric families of curves, surfaces and hypersurfaces. The method has a very wide range of applicability, can handle

* The first author is supported partially by Project 70/4/6452 of the Research Committee of the University of Athens, Greece and the second author by a grant from the Natural Sciences and Engineering Research Council of Canada.

base points, and works both symbolically and numerically, depending on the way one performs the integrations. It may be improved as follows: The method looks for an implicit equation of a particular degree at a time. This implies that any information on the degree of the implicit equation (such as upper bounds) may accelerate execution. More importantly, the method constructs a symmetric singular square matrix and computes a basis of its nullspace. The dimension of this matrix equals the number of possible monomials in the implicit equation, which is in principle $\binom{m+n}{m}$, where the number of parametric equations is n and the algorithm seeks an implicit equation of degree m. The examples show (cf. the table below) that we succeed to constrain the monomials that will appear in the implicit equation, hence diminishing dramatically the size of the matrices. One last improvement concerns the block-Hankel structure of the matrix, but this goes beyond the scope of the current paper. Hankel-like structural properties of implicitization matrices are established in [9].

Our second motivation are algorithms based on perturbed resultant matrices, which yield the implicit equation even in the presence of base points, e.g. [4, 10]. The problem reduces to sparse interpolation, which is substantially accelerated when we can accurately predict the output support. More specifically, the algorithm of Ben-Or and Tiwari requires a number of evaluations which is linear in the bound on the support cardinality [13].

The method of moving lines for implicitization (see [11]) uses a family of lines that follow the curve to set up an overdetermined linear system. A determinant of the matrix of this linear system is the implicit equation of the curve. It is possible to compute a symbolic determinant without developing it, simply by evaluating it at a sufficient number of points. Knowledge of the monomials that appear in the implicit equation speeds up this computation.

The table below shows the results obtained by IPSOS on some examples that we studied in [6], as well as the (very sparse) Fröberg-Dickenstein example. The results are optimal.

Problem	**Input Degree**	**Degree of Implicit Eq.**	**General # monomials**	**# monomials from IPSOS**
Unit Circle	2	2	6	3 (optimal)
Descartes Folium	3	3	10	3 (optimal)
Fröberg-Dickenstein	63	63	20	257 (optimal)
Buchberger	1,2	4	35	2 (optimal)
Busé	3	5	56	4 (optimal)
Bilinear	1,1	2	10	9 (optimal)

The Fröberg-Dickenstein example in IPSOS

We use the Fröberg-Dickenstein example[3] to illustrate IPSOS and how the coefficients of the extremal monomials turn out to belong to $\{-1, +1\}$. Consider

[3] see Proceedings of the 1^{st} Latin-American workshop on Polynomial Systems, July 2003, Buenos Aires, Argentina, A. Dickenstein, I. Emiris, Eds.

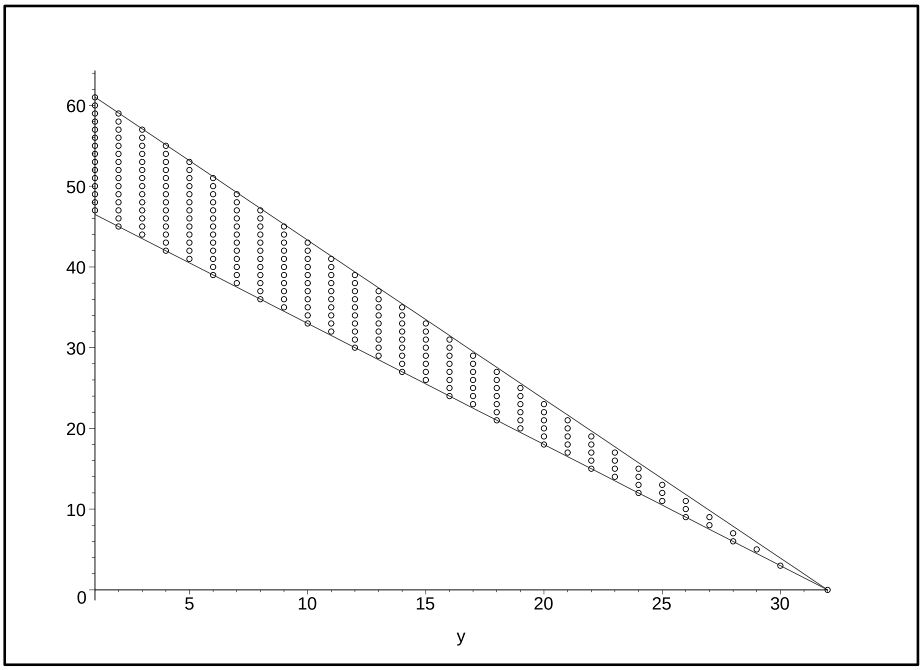
the polynomial parametric equations:

$$x = t^{32},\ y = t^{48} - t^{56} - t^{60} - t^{62} - t^{63}$$

With IPSOS we find that the points corresponding to non-zero monomials in the implicit equation (toric resultant) are delimited by the y-axis and the two lines:

$$x = (-(3/2)y + 48 \text{ and } \ x = -(63/32)y + 63.$$

Counting the points with integer coordinates inside (and on the sides) of the triangle, we see that there are 257 such points.



These points correspond exactly to the 257 non-zero monomials of the implicit equation (toric resultant).

References

1. C.L. Bajaj and S. Evans. Splines and geometric modeling. In J.E. Goodman and J. O'Rourke, editors, *Handbook of Discrete and Computational Geometry*, pages 833–850. CRC Press, Boca Raton, Florida, 1997.
2. L. Busé. Residual resultant over the projective plane and the implicitization problem. In *ISSAC (London ON, Canada)*, pages 48–55. ACM, 2001.
3. R.M. Corless, M.W. Giesbrecht, I.S. Kotsireas, and S.M. Watt. Numerical implicitization of parametric hypersurfaces with linear algebra. In *Artificial intelligence and symbolic computation (Madrid, 2000)*, pages 174–183. Springer, Berlin, 2001.

4. C. D'Andrea and I.Z. Emiris. Computing sparse projection operators. In *Symbolic computation: solving equations in algebra, geometry, and engineering (South Hadley, MA, 2000)*, pages 121–139. Amer. Math. Soc., Providence, RI, 2001.
5. T. Dokken. Approximate implicitization. In *Mathematical methods for curves and surfaces (Oslo 2000)*, Innov. Appl. Math., pages 81–102. Vanderbilt Univ. Press, Nashville, 2001.
6. Ioannis Z. Emiris and Ilias S. Kotsireas. Implicit polynomial support optimized for sparseness. In V. Kumar et al, editor, *Proceedings of ICCSA'2003*, LNCS 2669, pages 397–406, Montreal, Canada, 2003.
7. I.M. Gelfand, M.M. Kapranov, and A.V. Zelevinsky. *Discriminants, resultants, and multidimensional determinants.* Birkhäuser, Boston, 1994.
8. L. Gonzalez-Vega. Implicitization of parametric curves and surfaces by using multidimensional Newton formulae. *J. Symbolic Comput.*, 23(2-3):137–151, 1997. Parametric algebraic curves and applications (Albuquerque, NM, 1995).
9. Ilias S. Kotsireas and Edmond S.C. Lau. Implicitization of polynomial curves. In Z. Li and W. Sit, editors, *Proceedings of ASCM'2003, Beijing, P.R. China*, pages 217–226, 2003.
10. D. Manocha and J.F. Canny. Algorithm for implicitizing rational parametric surfaces. *Comput. Aided Geom. Design*, 9(1):25–50, 1992.
11. Thomas W. Sederberg, Takafumi Saito, Dongxu Qi, and Krzysztof S. Klimaszewski. Curve implicitization using moving lines. *Computer Aided Geometric Design*, 11(6):687–706, 1994.
12. B. Sturmfels. On the Newton polytope of the resultant. *J. Algebraic Combin.*, 3(2):207–236, 1994.
13. R. Zippel. *Effective Polynomial Computation.* Kluwer, Boston, 1993.