Immobilizing hinged polygons

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Abstract: We study the problem of fixturing a chain of hinged objects in a given placement with frictionless point contacts. We define the notions of immobility and robust immobility, which are comparable to second and first order immobility for a single object [8, 7, 11, 12] robust immobility differs from immobility in that it additionally requires insensitivity to small perturbations of contacts. We show that $(p+2)$ frictionless point contacts can immobilize any chain of $p \neq 3$ polygons without parallel edges; it is unclear that five contacts can immobilize any three polygons in general, Any chain of p arbitrary polygons can be immobilized with at most $(p+3)$ contacts. We also show that $\lceil \frac{6}{5}(p+2) \rceil$ contacts suffice to robustly immobilize p polygons without parallel edges, and that $\lceil \frac{5}{4}(p+2) \rceil$ contacts can robustly immobilize p arbitrary polygons.

1 Introduction

Many manufacturing operations, such as machining and assembly, require the parts that are subjected to these operations to be fixtured, i.e., to be held in such a way that they can resist all external wrenches. Fixturing is a problem that is studied extensively, see e.g. [2, 3, 6, 15, 16, 17]. We consider the planar version of part fixturing (or immobilization), which appears e.g. in preventing all sliding motions of a part resting on a table. The concept of *form closure*, formulated by Reuleaux [9] in 1876, provides a sufficient condition for constraining, despite the application of possible external wrenches, all finite and infinitesimal motions of a rigid part by a set of contacts along its boundary. Any motion of a part in form closure has to violate the rigidity of the contacts. Markenscoff et al. [7] and Mishra et al. [8] independently showed that four frictionless point contacts are sufficient and often necessary to put any polygonal object in form closure. In fact, their result applies to almost any planar rigid part.

Czyzowicz et al. [4, 5] showed that three contacts can immobilize a polygon without parallel edges, and identified the conditions to be satisfied for the polygon to be immobilized with three contacts. It can be verified graphically if a given set of contacts satisfy the conditions. Rimon and Burdick [11, 12] also showed that three contacts

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can immobilize a rigid object, and this immobility is called *second-order immobility*. Second-order immobility analysis takes place in the configuration space of the part and regards the contacts as obstacles that limit the part's ability to move. A fundamental difference with half plane analysis by Reuleaux is that the second-order immobility analysis takes the curvature of possible motions into account, instead of only the directions. In their notion, first-order immobility is equivalent to form closure—the four contacts immobilize the object regardless of the curvature of the boundaries where the contacts touch. The inclusion of curvature effects is powerful enough to show that three frictionless contacts suffice to immobilize any polygonal part without parallel edges [10].

Most of the existing results on immobilization apply to rigid bodies, and hardly anything has been done for non-rigid objects such as assemblies or deformable shapes. As a first step in this direction, we study immobilization of an acyclic chain of objects connected to each other by hinges. This can be seen as a case study of immobilization of non-rigid objects. A hinge allow the two adjacent objects to rotate around it. We shall assume that the objects are polygonal, and the hinges are located at their vertices, but it seems that most of the result will carry over to more general objects. It is our aim to derive bounds on the number of contacts required to immobilize any chain of p hinged polygons in a priorly specified placement.

Our approach is graphical, but also bears some resemblance to second-order immobility analysis; we also analyze motions by identifying the areas where a point of the part can be placed locally with given point contacts on the boundary. We show that a chain of p polygons without parallel edges in a given placement can be immobilized by $(p+2)$ frictionless point contacts for all $p \neq 3$; in some cases, five contacts can can immobilize three polygons, but in general, it is unclear that five contacts can achieve it. We observe that the number of contacts required to immobilize a chain of p polygons equals the number of degrees of freedom of the chain. All the proofs are constructive in the sense that we give actual grasps with $(p+2)$ contacts for chains of p hinged polygons. Allowing for parallel edges leads to an increase in the number of contacts of one.

One observation about the first-order immobility is that any perturbation of any combination of the $(p+2)$ contacts maintains the immobility. This has motivated us to also investigate the number of point contacts required to obtain a more robust fixturing, which has the property—like form closure—that any contact can be perturbed slightly without destroying the immobility. We construct a robust immobility for a chain of p polygons with $\lceil \frac{6}{5}(p+2) \rceil$ contacts if the polygons have no parallel edges, and with $\lceil \frac{5}{4}(p+2) \rceil$ contacts if the polygons are allowed to have parallel edges. Informally speaking, we achieve robustness at the cost of one additional contact per five or four polygons.

The paper is organized as follows: We first introduce the concept of immobility and robust immobility in Section 2. In Section 3 and Section 4, we present how many fingers can immobilize or robustly immobilize a chain of p hinged polygons without or with parallel edges in a constructive way. Finally, we will summarize the results that we have and discuss further research topics in this direction in Section 6.

2 Immobility and robust immobility

Half-plane analysis was used by Reuleaux to check if an object is in form closure. Every infinitesimal motion in the plane can be seen as a rotation around a point in either counterclockwise or clockwise direction. When a contact is in the interior of a straight edge, the normal line divides centers of counterclockwise and clockwise rotations. The left side of the normal line has the centers of counterclockwise rotations and the right side has those of clockwise ones, when facing the interior of the object from the contact. (See Figure 1 (a).) In other words, the object can be rotated counterclockwise around a point on the left side of the normal line, and clockwise around a point on the right side.

When a contact is at a concave vertex, it induces two normals, because it is at both of the edges. The intersection region for the counterclockwise (clockwise) rotation induced by the two normals has the centers for the counterclockwise (clockwise) rotation, as in Figure 1 (b).

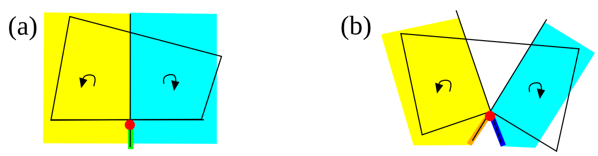

Figure 1: The half planes divide possible centers of rotations; (a) shows the situation when a contact is on an edge; and (b) shows when a contact is at a concave vertex.

One difference of form closure and second-order immobility is in how the normal lines are treated. Strictly speaking, any rotation is possible around the points on the half line below and including the contact, while no rotation is possible around the points on the rest of the line. Even then, half plane analysis cannot distinguish a subtle case. (For more details, refer Czyzowicz et al. [5].) Form-closure analysis, contrary to second-order immobility analysis, does not take advantage of this observation; it conservatively assumes that any rotation is possible about any point on the line.

When the regions induced by the contacts holding an object have an empty intersection, the object is said to be immobilized. Four contacts are necessary and sufficient to achieve first-order immobility or form closure, while three are often enough for second-order immobility.

All the existing notions and analyses apply to rigid objects. It is particularly difficult to generalize Reuleaux's form closure analysis to explain the immobility of a chain of hinged polygons. Thus, we propose an intuitive analysis of immobilization in the two-dimensional space of the part itself, by considering motions of specific points of the objects.

We will identify the free areas where the hinged vertices can move locally with point contacts on the object-boundaries. When the two free areas of the hinged vertex of adjacent parts touch each other, and when it cannot move to another position without breaking the rigidity of the body or the contacts, we say that the parts are immobilized.

Now we would like to address one intuitive and essential difference between first-order immobility (form closure) and second-order immobility from a practical viewpoint: slight perturbations of frictionless contacts along the edges can maintain immobility, which is unlikely for the second-order immobility. This motivates us to define robust immobility.

Definition 2.1 *We say that $\mathcal{B}$ is* **robustly immobilized** *when $\mathcal{B}$ is immobilized and there exists a real number $\epsilon > 0$ for each contact in the interior of an edge, such that any perturbation of the contact in the ϵ-interval in both directions along the edge still keeps $\mathcal{B}$ immobilized.*

Lemma 2.1 *A two dimensional polygon $\mathcal{B}$ in form closure is robustly immobilized.*

Proof: All objects can be put in form closure with at most four contacts [7, 8]. When we use half-plane analysis, no three normals induced by the contacts meet at one point. This means that the intersection points of each pair of normals have non-zero distances.

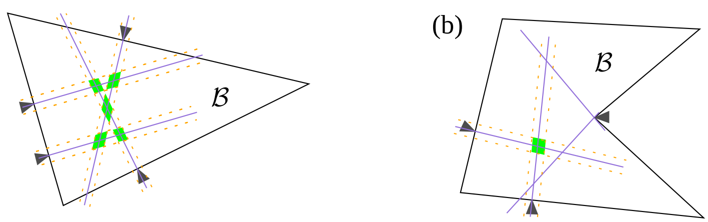

Figure 2: Two dimensional objects in form closure with (a) four and (b) three contacts for an $\epsilon > 0$.

When two normals are perturbed along the edges, the intersection points are in a quadrilateral region defined by the perturbed line boundaries. (See the shaded quadrilateral regions in Figure 2.) Note that we do not perturb a contact at a concave vertex. Since the intersection points have non-zero distances from one another, we can always find an interval for each contact along which the contact can be shifted while keeping the polygon immobilized. ◻

Lemma 2.1 and the result of Markenscoff et al. [7], Mishra et al. [8], Rimon and Burdick [10] and van der Stappen et al. [14] produce the following two lemmas.

Lemma 2.2 *Any polygonal part can be robustly immobilized with four frictionless point contacts.*

Lemma 2.3 *Any polygonal part without parallel edges can be immobilized with three frictionless point contacts.*

Now we introduce some notions and notations used in this paper. Let $(\mathcal{B}_1, \mathcal{B}_2, \cdots, \mathcal{B}_p)$ denote an acyclic chain of p hinged polygons from the left. We assume that the two edges of a single polygon incident to a hinge are not collinear. Let v_i denote the hinge connecting $\mathcal{B}_i$ and $\mathcal{B}_{i+1}$. An arrangement of the contacts holding p hinged polygons is represented as $(n_1, n_2, \cdots, n_p)$, where n_i denotes the number of the contacts holding $\mathcal{B}_i$ ($1 \leqslant i \leqslant p$). An immobility or robust immobility with $(n_1, n_2, \cdots, n_p)$ contact arrangement is called $(n_1, n_2, \cdots, n_p)$ finger configuration. We also assume that the two edges of different polygons incident to a hinge do not coincide nor overlap.

When a maximal inscribed circle of a polygon touches the polygon, and the supporting lines of the touching edges contains the circle in a bounded (usually triangular) region, which is always the case for polygons without parallel edges, placing contacts at the touching points immobilizes the polygon. Placing two contacts on either sides of one of the touching points gives a robust immobility. A maximal inscribed circle can be computed in $O(n)$-time, where n is the number of the vertices.

When a maximal inscribed circle of a polygon does not have this property stated above, the polygon has two parallel edges. In this case, placing contacts at each of the four edges incident to a pair of the furthest vertices of the polygon immobilizes it robustly. This finger configuration can again be computed in $O(n)$-time.

3 Immobility

A polygon without parallel edges can be immobilized with three point contacts, while a polygon with parallel edges may need one more contact to be immobilized. Likewise, it turns out that a chain of p polygons with parallel edges in general needs more contacts. First, we will start with immobility of hinged polygons without parallel edges, and then hinged arbitrary polygons.

3.1 Polygons without parallel edges

We will subsequently discuss the immobilization of chains of two, three and four polygons without parallel edges. The immobility of a single polygon, and of chains of two and four polygons serve as building blocks for the immobility of longer chains.

3.1.1 Two polygons without parallel edges

There are two ways of immobilizing two polygons depending on the nature (convex or concave) of the hinged vertices. Let α be the angle at a vertex v of a polygon. First we look at the behavior of one polygon when two point contacts are placed along two adjacent edges. It is a generalization of a result in [1].

Lemma 3.1 *Let v be a vertex of a polygon, and let two point contacts $\mathcal{A}_1$ and $\mathcal{A}_2$ be placed on the edges e_1 and $e2$ incident to v respectively. Let $\mathcal{C}$ be the unique circle through $\mathcal{A}_1$, $\mathcal{A}_2$ and v. The area where v can locally move around under the constraint of $\mathcal{A}_1$ and $\mathcal{A}_2$ is the interior and the boundary of $\mathcal{C}$ when v is convex, and the exterior and the boundary of $\mathcal{C}$ when v is concave.*

Proof: First we look at the case when v is a convex vertex. Assume that v can reach outside of $\mathcal{C}$ under the restriction of $\mathcal{A}_1$ and $\mathcal{A}_2$ by translation and rotation from the current configuration q. Let z be the point outside of $\mathcal{C}$ as in Figure 3 (a). Let v' be the intersection point of the line $\overline{\mathcal{A}_1 z}$ and $\mathcal{C}$. It is a well known geometrical fact that the angle $\angle \mathcal{A}_1 v' \mathcal{A}_2 = \angle \mathcal{A}_1 v \mathcal{A}_2 = \alpha$. A simple trigonometric calculation shows that $\angle \mathcal{A}_1 z \mathcal{A}_2 < \angle \mathcal{A}_1 v \mathcal{A}_2$, thus $\angle \mathcal{A}_1 z \mathcal{A}_2 < \alpha$, which is a contradiction. Therefore, the vertex v of $\mathcal{B}$ can only move locally inside or along $\mathcal{C}$.
We now look at the case when v is a concave vertex. Assume that v can be placed inside of $\mathcal{C}$ under the restriction of $\mathcal{A}_1$ and $\mathcal{A}_2$ by translation and rotation from the current configuration q. Let z be the point inside of $\mathcal{C}$ as in figure 3 (b). Let v' be the intersection point of the boundary of $\mathcal{C}$ and the supporting line of $\overline{\mathcal{A}_1 z}$. Likewise, we can see that $\angle \mathcal{A}_1 v' \mathcal{A}_2 = \angle \mathcal{A}_1 v \mathcal{A}_2 = \alpha$. A simple trigonometric calculation shows that $\angle \mathcal{A}_1 z \mathcal{A}_2 > \angle \mathcal{A}_1 v \mathcal{A}_2$, thus $\angle \mathcal{A}_1 z \mathcal{A}_2 > \alpha$, which is a contradiction. Therefore, the vertex v of $\mathcal{B}$ can only move locally outside or along $\mathcal{C}$. □

Now we show how to immobilize two hinged polygons with four contacts. Let e_1 and e_1' be the two edges of $\mathcal{B}_1$ that are incident to the hinged vertex v_1, and e_2 and e_2'

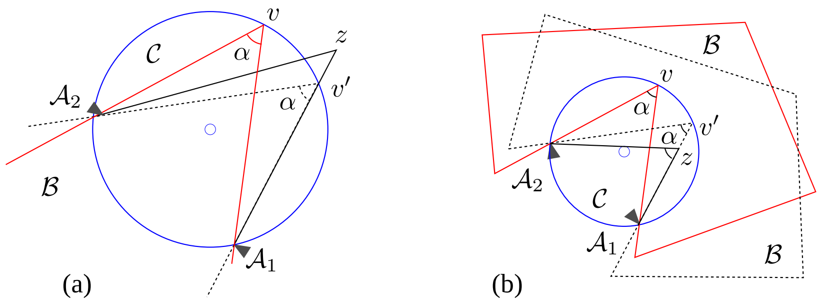

Figure 3: The free area where v can locally move around under the restriction of the two contacts $\mathcal{A}_1$ and $\mathcal{A}_2$ is: (a) the interior and the boundary of $\mathcal{C}$ when v is convex, and (b) the exterior and the boundary of $\mathcal{C}$ when v is concave.

be such edges of $\mathcal{B}_2$. We first focus on the case when v_1 is a convex vertex of both $\mathcal{B}_1$ and $\mathcal{B}_2$. (See Figure 4 (a).) Choose a line l containing v_1 such that e_1 and e'_1 are strictly on one side of l, and that e_2 and e'_2 are strictly on the other side. Let l' be the perpendicular line of l at v_1. Take a circle $\mathcal{C}_1$ for $\mathcal{B}_1$ that satisfies the following two conditions:

1. The circle $\mathcal{C}_i$ is centered at a point on l' such that $\mathcal{C}_i$ touches l at the hinged vertex v_1, and

2. $\mathcal{C}_i$ intersects e_i and e'_i in their interiors.

Take a circle $\mathcal{C}_2$ for $\mathcal{B}_2$ that satisfies the above two conditions. Place four contacts at the intersection points of the circles and the polygons.

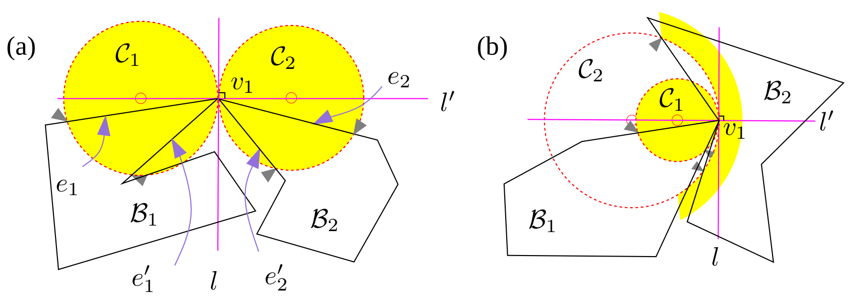

Figure 4: Two hinged polygons with four contacts.

When one hinged vertex is concave, the construction is the same as in the previous case except for a few details. Without loss of generality, assume that the hinged vertex of $\mathcal{B}_2$ is concave. First, the line l is chosen such that all the adjacent edges are strictly on one side of l—see Figure 4 (b). Second, take a circle $\mathcal{C}_2$ for $\mathcal{B}_2$ that satisfies the two conditions given above. Finally take a circle $\mathcal{C}_1$ such that $\mathcal{C}_1$ is smaller than $\mathcal{C}_2$.

Lemma 3.2 *Four contacts can immobilize two hinged polygons.*

Proof: We call the area where the hinged vertex can move freely *free area*. Lemma 3.1 says that the free area is inside or outside of the circle, and the circular arc is the boundary. In any case, the two free areas $\mathcal{C}_1$ and $\mathcal{C}_2$ touch each other at v_1. Note that, when v_1 is at a point on the boundary, $\mathcal{B}_1$ has a unique configuration, that is, it cannot rotate nor translate maintaining the current position of v_1 without breaking the rigidity of the contacts. And the same is true for $\mathcal{B}_2$. This implies that $\mathcal{B}_1$ and $\mathcal{B}_2$ can be only at a certain configuration to remain connected by a hinge in the plane of the polygons. Therefore, the two hinged polygons are immobilized with four contacts. $\square$

In general, two hinged polygons can not be immobilized with less than four contacts; when one of the polygons has only one contact, rotation of this polygon around the hinge away from the contact is possible in most cases.

3.1.2 Four polygons without parallel edges

Four polygons without parallel edges can be immobilized with six contacts by immobilizing the first and the last polygons. The contact arrangement is $(3, 0, 0, 3)$. The following Lemma shows why they are immobilized.

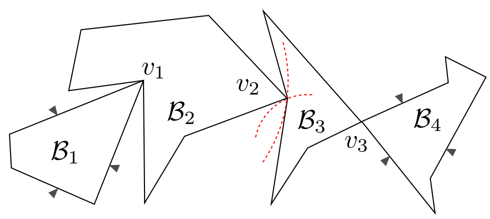

Figure 5: Four hinged objects without parallel edges can be immobilized with six contacts.

Lemma 3.3 *Two adjacent polygons $\mathcal{B}_2$ and $\mathcal{B}_3$ are immobilized if $\mathcal{B}_1$ and $\mathcal{B}_4$ are immobilized.*

Proof: Since the hinges v_1 and v_3 is fixed, $\mathcal{B}_2$ and $\mathcal{B}_3$ can only rotate around v_1 and v_3 respectively. Vertex v_2 of $\mathcal{B}_2$ and $\mathcal{B}_3$ can move along the circular arcs in dotted lines respectively as in Figure 5. Only at the intersection point, v_2 can lie such that the distance $|\overline{v_1 v_2}|$ and $|\overline{v_2 v_3}|$ are preserved at the same time. $\square$

3.1.3 Immobilizing $p \geqslant 5$ polygons without parallel edges

To immobilize five and six hinged polygons, the contacts can be arranged starting from the immobilization of four polygons: $(3, 0, 0, 3)$. Replace the polygon held with three contacts with two polygons held with four contacts. Thus the contact arrangements for five and six polygons are $(3, 0, 0, 2, 2)$ and $(2, 2, 0, 0, 2, 2)$ respectively. Seven polygons still can be immobilized with nine contacts using a contact arrangement of $(3, 0, 0, 3, 0, 0, 3)$.

Now we will immobilize a chain of $p \geqslant 8$ hinged polygons using the contact arrangements for two and four hinged polygons. From the right end of the chain, cut off a trailing multiple of four polygons until four, five, six or seven polygons are left. Immobilize these left polygons as described above, and immobilize the trailing quadruples using the arrangement $(0, 0, 2, 2)$ repeatedly.

Theorem 3.1 *A chain of p hinged polygons without parallel edges can be immobilized with $(p+2)$ contacts, when $p \neq 3$, which is a tight bound. Six contacts can immobilize three polygons without parallel edges.*

3.2 Immobility of polygons with parallel edges

With parallel edges in the polygons, three contacts cannot immobilize a polygon any more, but one polygon with parallel edges can be immobilized with four contacts by Lemma 2.2. On the other hand, four contacts can still immobilize two arbitrary polygons (Lemma 3.2), because the proof of Lemma 3.2 does not use the condition that the polygons have no parallel edges.

Three arbitrary polygons can be immobilized with six $(= (p+3))$ contacts as follows. Immobilize the first two polygons $\mathcal{B}_1$ and $\mathcal{B}_2$ with $(2,2)$ contact arrangement. The last polygon $\mathcal{B}_3$ will rotate around the hinge. Immobilize $\mathcal{B}_3$ with two contacts, by placing one on each of the incident edges to the hinged vertex. The contact arrangement is $(2,2,2)$. (See Figure 6.)

Lemma 3.4 *Six contacts can immobilize three polygons with parallel edges.*

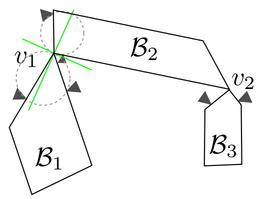

Figure 6: A contact arrangement that immobilizes three arbitrary polygons.

Four polygons with parallel edges can be immobilized with seven $(= (p+3))$ contacts, using $(2,2,0,3)$ contact arrangement. Immobilize the first two polygons $\mathcal{B}_1$ and $\mathcal{B}_2$ with four contacts. Take a maximal inscribed circle of the last polygon $\mathcal{B}_4$. If the touching points of the circle and $\mathcal{B}_4$ gives an immobility contact arrangement, we are done. Otherwise, two of the touching edges are parallel edges. Translate the circle along these two parallel edges so that it touches $\mathcal{B}_4$ at two points. Place a contact $\mathcal{A}_1$ at one touching point, and $\mathcal{A}_2$ and $\mathcal{A}_3$ on both sides of the other touching point like $\mathcal{B}_4$ in Figure 7.

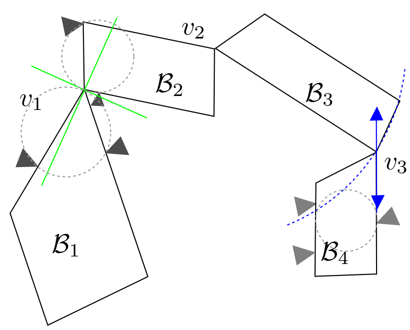

Figure 7: Four arbitrary polygons can be immobilized with seven contacts.

Lemma 3.5 *Seven contacts can immobilize four polygons with parallel edges.*

Proof: The vertex v_3 of $\mathcal{B}_3$ can rotate around v_2, while v_3 of $\mathcal{B}_4$ can move along a line parallel to the parallel edges. Remember that the hinge of $\mathcal{B}_3$ and $\mathcal{B}_4$ follows a circular arc and a line segment respectively. The intersection of the curve and the line segment is unique and at this configuration, $\mathcal{B}_3$ and $\mathcal{B}_4$ cannot change their configurations without breaking the rigidity of the polygons or the contacts. Therefore, the three polygons are immobilized. ◻

Now we will look at how a most $(p+3)$ contacts can immobilize p polygons with parallel edges. From the right end of the chain, cut off a trailing multiple of four polygons until at most four polygons are left. Immobilize these left polygons as described in Lemma 3.2, 3.4 and 3.5. Immobilize the trailing quadruples using the arrangement of $(0, 0, 2, 2)$ repeatedly. Note that the contact arrangement of $(0, 0, 2, 2)$ can still be used for each quadruple on the right, because even with parallel edges, the contact arrangement of $(2, 2)$ still works for two arbitrary polygons (Lemma 3.2). Hence when one polygon or three, four polygons are remained on the left after cutting off multiples of quadruple, $(p+3)$ contacts can immobilize the p polygons; when two polygons are remained on the left, $(p+2)$ contacts can immobilize the p polygons.

Theorem 3.2 *A chain of p hinged arbitrary polygons can be immobilized with at most $(p+3)$ contacts.*

The number of contacts $(p+3)$ is tight in the sense that there exist p polygons that cannot be immobilized with less than $(p+3)$ contacts. For example, when there is one polygon left after cutting off multiples of quadruples, $(p+3)$ contacts are necessary to immobilize them by Lemma 2.2.

4 Robust immobility

Like in the case of immobility, we have different results for robust immobility of chains of polygons with and without parallel edges.

4.1 Robust immobility of polygons without parallel edges

Some building blocks will be also used here to achieve robust immobility, which are the contact arrangements for one polygon, two, three, and four hinged polygons. Lemma 2.3 says that any polygon can be robustly immobilized with four contacts. We proceed to show how to robustly immobilize two, three and four polygons.

4.1.1 Two polygons without parallel edges

A contact arrangement for the robust immobility of two polygons with five contacts can be constructed from that for two polygons with four contacts. Note that both hinged vertices cannot be concave at the same time. Without loss of generality, let $\mathcal{B}_2$ have a convex hinged vertex. Let e_2 and e_2' be the edges of $\mathcal{B}_2$ incident to the hinge v, and e_1 and e_1' be those of $\mathcal{B}_1$ incident to v as in Figure 8. Line l is chosen in the same way as in the immobilization of two polygons in Section 3. Take a circle that touches l_1 at v and that intersects e_1 and e_1' in their interiors. Place two contacts at these intersections. Rotate line l around v clockwise and counter-clockwise so that the two perturbed lines l_1 and l_2 do not overlap $\mathcal{B}_2$ locally. Take a circle that touches l_1 at v and that intersects e_2 and e_2' in their interiors. Call the intersection points $\mathcal{A}_1$ and $\mathcal{A}_2$. Take another circle that satisfies the following three conditions: (i) it touches l_2 at v, (ii) it intersects e_2 and e_2' in their interiors, and (iii) it passes through one of $\mathcal{A}_1$ and

$\mathcal{A}_2$ (it is $\mathcal{A}_1$ in Figure 8). Place the contacts at the intersection points of the edges and the two circles.

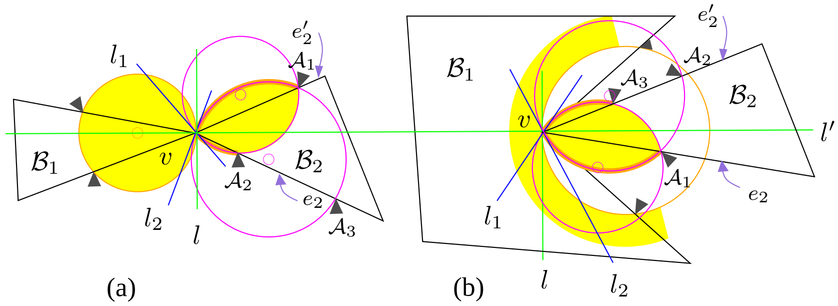

Figure 8: Two hinged polygons are robustly immobilized with five contacts, (a) when both of the hinged vertices are convex, and (b) when one hinged vertex is concave.

Lemma 4.1 *Five contacts can robustly immobilize two hinged polygons.*

Proof: The two contacts on $\mathcal{B}_1$ makes the boundary of the free area where v of $\mathcal{B}_1$ can move around a circular arc. The three contacts on $\mathcal{B}_2$ define two circles with v: one passing through $\mathcal{A}_1$, $\mathcal{A}_2$ and v, and another through $\mathcal{A}_1$, $\mathcal{A}_3$ and v. (See Figure 8 (a) and (b).) The thick arcs from the two circles make the partial boundary of the free area where v of $\mathcal{B}_2$ can move around. The two free areas touch each other at a single point on their boundaries, and it is easy to show that the two polygons are immobilized using the arguments explained in Lemma 3.2.

There exists a set of perturbations of all contacts on the two polygons that define different circles such that the two areas still touch each other at one point on their boundaries. Thus, the two polygons are robustly immobilized. ◻

4.1.2 Three polygons without parallel edges

Now, we show how to robustly immobilize three polygons. Compute maximal inscribed circles $\mathcal{C}_1$ and $\mathcal{C}_3$ of $\mathcal{B}_1$ and $\mathcal{B}_3$ respectively; let c_1 and c_3 be their centers. Let l_1 and l_2 be the lines at the hinges v_1 and v_2 that are perpendicular to the line $\overline{v_1 v_2}$ as in Figure 9 (a). We assume two things for a while for simplicity. First, the touching points of the circles and the polygons are in the interior of the edges like $\mathcal{B}_1$ in Figure 9 (a). Second, the line segments $\overline{c_1 v_1}$ and $\overline{v_2 c_3}$ are not collinear with $\overline{v_1 v_2}$.

When $\mathcal{B}_1$ is immobilized with three contacts, the three normals induced by the contacts meet at one point. Remember that the polygons do not have parallel edges. If we perturb one contact, the normals form a triangular region, consisting of centers of possible rotations only in one direction: clockwise or counterclockwise. Infinitesimal rotations of v_1 of $\mathcal{B}_1$ around a point p in this triangular region move v_1 in a direction along a half-line emanating from v_1 orthogonal to the line $\overline{p v_1}$. All these half-lines lie in a wedge-like region—the shaded region on the left side of l_1 in Figure 9 (a). It is important that we can always choose the direction of rotations, so that v_1 move strictly towards the left or the right side of l_1. For $\mathcal{B}_1$ in Figure 9 (a), v_1 can be only on the left side of l_1. The two boundary lines of the shaded wedge region are perpendicular to the lines through v_1 and tangent to the triangle. Notice that the wedge region does not include any segment of l_1 except v_1; otherwise, $\mathcal{B}_1$ can rotate around v_2. We construct the same contact arrangement for $\mathcal{B}_3$ so that the wedge region for v_2 lies strictly on the right side of l_2.

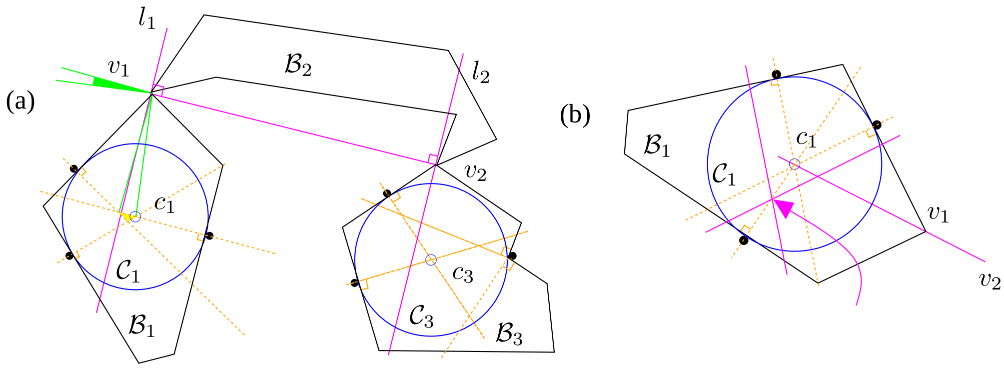

Figure 9: (a) A robust immobility of three hinged polygons with six contacts when c_1, v_1 and v_2 are not collinear. (b) When c_1, v_1 and v_2 are collinear, a new position of c_1 can be computed.

Now let us look at the case when one of the contacts touches a vertex. Without loss of generality, let $\mathcal{B}_3$ have such a contact arrangement (see Figure 9 (a)). In this case, the vertex must be concave, and the contact arrangement for $\mathcal{B}_3$ achieves form closure [13]. The other polygon $\mathcal{B}_1$ can be in the same situation; otherwise, it can be held with three contacts such that the hinge v_1 can move away from $\mathcal{B}_3$, thus towards the left side of l_1, as in Figure 9 (a).

Now we look at the case when one of c_1 or c_3 is collinear with $\overline{v_1 v_2}$. Suppose that c_1 is collinear with $\overline{v_1 v_2}$, and that $\mathcal{C}_1$ touches $\mathcal{B}_1$ at some edges in the interior only. The meeting point of the three normals can be perturbed by moving two normals together along the third one as in Figure 9 (b). After perturbing the center c_1, we can use the same method described before to robustly immobilize the three polygons.

Lemma 4.2 *Six contacts can robustly immobilize three hinged polygons without parallel edges.*

Proof: To maintain the distance between v_1 and v_2 of $\mathcal{B}_2$, v_1 and v_2 should stay at the apexes of the wedges. Since $\mathcal{B}_1$ and $\mathcal{B}_3$ cannot rotate around v_2 or v_1, they are immobilized. There exists a set of perturbation intervals of the contacts which keeps the three polygons immobilized, therefore, they are robustly immobilized. ◻

4.1.3 Four polygons without parallel edges

Four hinged polygons can be robustly immobilized with eight contacts by robustly immobilizing the first and the last polygons with four contacts respectively; the contact arrangement is $(4, 0, 0, 4)$.

Lemma 4.3 *Two adjacent polygons $\mathcal{B}_2$ and $\mathcal{B}_3$ are robustly immobilized if $\mathcal{B}_1$ and $\mathcal{B}_4$ are robustly immobilized.*

Proof: The two polygons in the middle are immobilized by Lemma 3.3. Since the neighbors are robustly immobilized, the whole chain is robustly immobilized. ◻

4.1.4 Robust immobility of $p \geqslant 5$ polygons without parallel edges

Five hinged polygons can be robustly immobilized with nine contacts by robustly immobilizing the first polygon with four contacts, and the last two polygons with five contacts. The contact arrangement is $(4, 0, 0, 3, 2)$. Since the first and the last polygons are in robust immobility, the whole chain of polygons is robustly immobilized according to Lemma 4.3.

Robustly immobilizing $p \geqslant 5$ polygons can be executed as follows. From the right end of the chain, cut off a trailing multiple of five polygons, until at most five polygons are left. These left polygons can be robustly immobilized as describe in Lemma 4.1, 4.2 and 4.3. Each group of five polygons can be immobilized with the contact arrangement of $(0, 0, 3, 0, 3)$.

Theorem 4.1 *A chain of p hinged polygons without parallel edges can be robustly immobilized with $\lceil \frac{6}{5}(p+2) \rceil$ contacts.*

4.2 Robust immobility of polygons with parallel edges

We will use the Lemmas 2.2, 4.1, 4.2 and 4.3 as building blocks. Except the contact arrangement of $(3, 0, 3)$ for three polygons (Lemma 4.2), all of these can be used for polygons with parallel edges We proceed to show how to robustly immobilize three polygons.

Take maximal inscribed circles for $\mathcal{B}_1$ and $\mathcal{B}_3$. If one of them allows a contact arrangement for immobility by placing contacts at the touching points, we can use $(4, 0, 3)$ or $(3, 0, 4)$ contact arrangement. The construction for the three contacts is the same as in Lemma 4.2. Assume the contrary. Two of the touching edges of $\mathcal{B}_3$ and its maximal inscribed circle are parallel edges. Translate the circle along these two parallel edges so that it touches $\mathcal{B}_4$ at two points. Place a contact $\mathcal{A}_1$ at one touching point, and $\mathcal{A}_2$ and $\mathcal{A}_3$ on both sides of the other touching point like $\mathcal{B}_3$ in Figure 10. Now robustly immobilize $\mathcal{B}_1$ with four fingers.

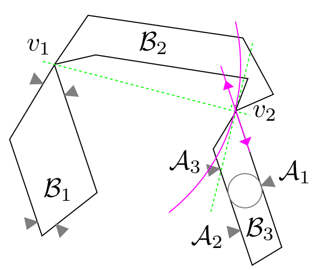

Figure 10: Seven fingers can immobilize three arbitrary polygons.

Lemma 4.4 *Seven point contacts can robustly immobilize three arbitrary polygons.*

Proof: This chain is immobilized with the same argument as in Lemma 3.5. The first polygon is already in robust immobility, and the three fingers can be perturbed along the parallel edges of $\mathcal{B}_3$ without changing the line along which v_2 moves. Therefore, the seven finger arrangement immobilize the three polygons robustly. ◽

Now we show how to robustly immobilize $p \geqslant 5$ arbitrary polygons. From the right end of the chain, cut off a trailing multiple of four polygons until at most four polygons are left. Each quadruple can be robustly immobilized by the contact arrangement of $(0, 0, 3, 2)$. The remaining one, two, three or four polygons can be robustly immobilized as described above with four, five, seven and eight fingers respectively.

Theorem 4.2 *A chain of p arbitrary polygons can be robustly immobilized with $\lceil \frac{5}{4}(p+2) \rceil$ contacts.*

5 Immobilizing other types of hinged polygons

In some cases, hinged polygons may not form an open chain. It can be in a single cycle, a tree, a general graph, or an open chain with one polygon at the end attached to a wall. Here we will consider the cases when the hinged polygons form a cycle and when an end of the chain is attached to a wall. The previous results can easily be used for these cases.

First let's look at the case of a cycle. A cycle needs at least two polygons as in Figure 11. Two polygons forming a cycle can be considered as one polygon, hence three and four point contacts can immobilize or robustly immobilize them as in Figure 11 (a) and (b).

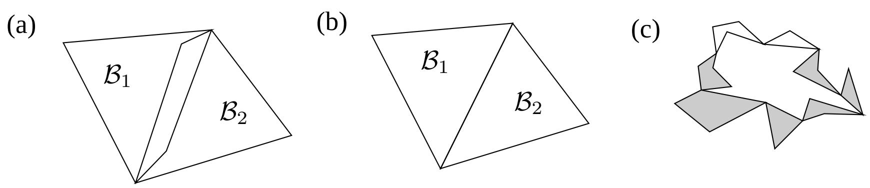

Figure 11: (a) Two polygons forming a cycle. (b) Two polygons forming a cycle: this can be considered to be one polygon. (c) A cycle with more than two polygons.

When $p \geqslant 3$ polygons form a cycle, this can be divided as two groups: any two adjacent polygons, and the rest. (See Figure 11 (c).) The rest polygons can be seen as a free chain of $(p-2)$ hinged polygons—the gray polygons in Figure 11 (c). Immobilizing or robustly immobilizing the chain of gray polygons immobilizes the whole cycle. This leads us to the next theorem.

Theorem 5.1 *A cycle of p hinged polygons without parallel edges can be immobilized with p point contacts, when $p \geqslant 3, p \neq 5$; two and five polygons can be immobilized with three and six point contacts respectively. A cycle of p hinged polygons without parallel edges can be robustly immobilized with $\lceil \frac{6}{5}p \rceil$ point contacts; two and five polygons can be immobilized with four and six contacts repectively.*

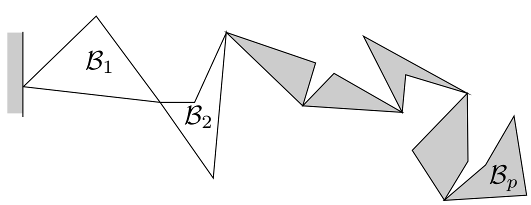

Figure 12: A chain of p polyogns one end of which is attached to a wall.

Figure 12 shows the case when one end of a chain is attached to a wall. Let $\mathcal{B}_1, \mathcal{B}_2, \cdots, \mathcal{B}_p$ be the polygons from the one attached to the wall. This can be (robustly) immobilized in a similar way. Skip the two polygons $\mathcal{B}_1$ and $\mathcal{B}_2$, and (robustly) immobilize the rest. The number of the point contacts needed for immobility is $p-2+2=p$ when $p \geqslant 3$ and $p \neq 5$; six contacts are needed for five polygons. Likewise, the number for robust immobility for $p \geqslant 3$ polygons is $\lceil \frac{6}{5}(p-2+2) \rceil = \lceil \frac{6}{5}p \rceil$.

When one polygon is attached to a wall, one contact can immobilize it, if a polygon has an edge whose internal edge normal goes through the vertex attached to the wall. The reason is as follows. The polygon $\mathcal{B}_1$ can only rotate around the vertex v attached to the wall. The rotation of $\mathcal{B}_1$ relative to the point contact $\mathcal{A}$ can be seen as rotation of $\mathcal{A}$ around v. Place $\mathcal{A}$ at the edge whose normal goes through v as in Figure 13 (a). Because the edge is tangent to the circle around v, the rotation of $\mathcal{A}$ violates the rigidity of the polygon and the contact.

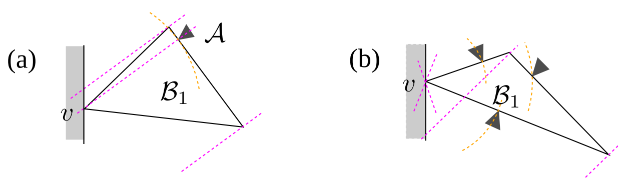

Figure 13: (a) This polygon can be immobilized with one point contact. (b) This polygon cannot be immobilized with one point contact.

Unfortunately, two contacts are needed to immobilize a polygon attached to a wall in general. The polygon in Figure 13 (b) does not have any such edge. This leads us to the next lemma.

Lemma 5.1 *Two point contacts can immobilize and robustly immobilize one polygon attached to a wall.*

Let v_1 be the hinge between $\mathcal{B}_1$ and $\mathcal{B}_2$. For two contacts to immobilize two polygons attached to a wall, the last polygon $\mathcal{B}_2$ must satisfy the following condition: (i) if $\mathcal{B}_2$ has two edges whose normals meet at the line $\overline{v_1 v_2}$ and (ii) if the half-planes induced by these edges have counterclockwise rotational centers above $\overline{v_1 v_2}$, and clockwise rotational centers below $\overline{v_1 v_2}$, two point contacts can immobilize the two polygons. (See Figure 14 (a).)

Let $\mathcal{C}$ be the circle around v that v_1 follows, and let l be the half-plane whose line is the tangent line of $\mathcal{C}$ at v_1, and the half-plane does not contain $\mathcal{C}$. If $\mathcal{B}_2$ does not satisfy the conditions (i) and (ii) described above, the immediate rotation of $\mathcal{B}_2$ around some point in the wedges will not confine the immediate motion of v_1 in the half-plane l. Note that the condition (ii) is not necessary, but the condition (i) must be satisfied.

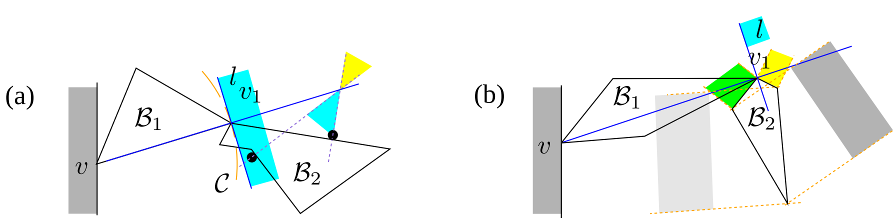

Figure 14: (a) These two polygons can be immobilized with two contacts. (b) These two polygons cannot be immobilized with two contacts.

Unfortunately, $\mathcal{B}_2$ may not have such an edge pair that satisfy the condition (i), like the polygon $\mathcal{B}_2$ in Figure 14 (b). Thus we need three point contacts to immobilize them. To robustly immobilize them, surprisingly we also need three point contacts; use a similar method to that for three polygons in Section 4. This leads us to the next lemma.

Lemma 5.2 *Three point contacts can immobilize two polygons attached to a wall, and four can robustly immobilize them.*

The next theorem summarizes the results so far.

Theorem 5.2 *When a chain of $p \geqslant 3, p \neq 5$ hinged polygons without parallel edges has an end attached to a wall, it can be immobilized with p point contacts; one, two and five polygons can be immobilized with two, three and six $(= p + 1)$ point contacts respectively. Moreover, it can be robustly immobilized with $\lceil \frac{6}{5}p \rceil$ point contacts.*

6 Discussion

We have shown that $(p + 2)$ contacts can immobilize a chain of $p\,(\neq 3)$ polygons without parallel edges, and that at most $(p + 3)$ contacts can immobilize p arbitrary polygons. We also showed that $\lceil \frac{6}{5}(p+2) \rceil$ contacts can robustly immobilize p polygons when the polygons have no parallel edges and that $\lceil \frac{5}{4}(p + 2) \rceil$ contacts can robustly immobilize p general polygons. Immobilizing three polygons without parallel edges with less than six contacts remains open.

We believe that $(p + 2)$ contacts are necessary to immobilize the hinged polygons, because the chain has $(p + 2)$ degrees of freedom. Two dimensional and three dimensional objects can be immobilized robustly with four and seven point contacts respectively, which is the degrees of freedom plus one. Therefore, we think that $(p+3)$ fingers might be able to immobilize the chain.

Throughout the paper, we have assumed that the placement in which the chain has to be immobilized is given. The number of contacts required for immobilization is expected to be smaller when the placement can be chosen freely. In the future, we intend to study whether or not this is indeed the case. We also plan to work on immobilizing other types of hinges and more general structures of connected polygons other than chains.

References

[1] R-P. Berretty. *Geometric design of part feeders*. PhD thesis, Institute of Information and Computing Sciences, Utrecht University, 2000.

[2] R. Brost and K. Goldberg. A complete algorithm for designing planar fixtures using modular components. In *IEEE Transactions on Robotics and Automation*, volume 12, pages 31–46, 1996.

[3] P. D. Q. Campbell. Basic fixture design, 1994.

[4] J. Czyzowicz, I. Stojmenovic, and J. Urrutia. Immobilizing a polytope. In *Algorithms and Data structures – Proc. 2nd Workshop*, volume 519 of *Lecture Notes in Computer Science*, pages 214–227, 1991.

[5] J. Czyzowicz, I. Stojmenovic, and J. Urrutia. Immobilizing a shape. *International Journal of Computational Geometry and Applications*, 9(2):181–206, 1999.

[6] E. Hoffman. Jig and fixture design, 1991.

[7] X. Markenscoff, L. Ni, and C. H. Papadimitriou. The geometry of grasping. *International Journal of Robotics Research*, 9(1):61–74, 1990.

[8] B. Mishra, J. T. Schwartz, and M. Sharir. On the existence and synthesis of multifinger positive grips. *Algorithmica*, 2:541–558, 1987.

[9] F. Reuleaux. *The Kinematics of Machinery*. Macmilly and Company, 1876. Republished by Dover in 1963.

[10] E. Rimon and J. W. Burdick. New bounds on the number of frictionless fingers required to immobilize planar objects. *J. of Robotic Systems*, 12(6):433–451, 1995.

[11] E. Rimon and J. W. Burdick. Mobility of bodies in contact—part I: A second-order mobility index for multiple-finger graps. *IEEE Transactions on Robotics and Automation*, 14:696–708, 1998.

[12] E. Rimon and J. W. Burdick. Mobility of bodies in contact—part II: How forces are generated by curvature effects. *IEEE Transactions on Robotics and Automation*, 14:709–717, 1998.

[13] A.F. van der Stappen. On the existence of form-closure configurations on a grid. In *IEEE/RSJ International Conference On Intelligent Robots and Systems (IROS)*, pages 1237–1242, 2000.

[14] A.F. van der Stappen, C. Wentink, and M.H. Overmars. Computing immobilizing grasps of polygonal parts. *International Journal of Robotics Research*, 19(5):467–479, 2000.

[15] A. S. Wallack and J. Canny. Planning for modular and hybrid fixtures. *Algorithmica*, 19(1–2):40–60, 1997.

[16] C. Wentink, A.F. van der Stappen, and M.H. Overmars. Algorithms for fixture design. In J-P. Laumond and M.H. Overmars, editors, *Algorithms for Robotic Motion and Manipulation*, pages 321–346. A.K. Peters, 1997.

[17] J. V. Woodworth. Grinding and lapping – tools, processes and fixtures, 1907.