Can Polynomiography be useful in Computational Geometry?

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Having studied the polynomial root-finding problem for well over a decade, I have arrived at "polynomiography," defined to be the art and science of visualization in approximation of zeros of complex polynomials, via fractal and non-fractal images created using the convergence properties of iteration functions. As polynomials are the most fundamental class of functions in virtually every branch of science and mathematics, I anticipate that polynomiography will find a wide range of scientific applications. It is also a fantastic tool from the educational, artistic, and commercial point of view.

Here I wish to propose the potential applicability of polynomiography in the field of computational geometry. To this end I will first describe a fundamental family of iteration function, called the the *Basic Family*. Given a complex polynomial $p(z)$, set $D_0(z) \equiv 1$, and for each natural number $m \geq 1$, define the Toeplitz determinant

$$D_m(z) = \det \begin{pmatrix} p'(z) & \frac{p''(z)}{2!} & \dots & \frac{p^{(m-1)}(z)}{(m-1)!} & \frac{p^{(m)}(z)}{(m)!} \\ p(z) & p'(z) & \ddots & \ddots & \frac{p^{(m-1)}(z)}{(m-1)!} \\ 0 & p(z) & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \frac{p''(z)}{2!} \\ 0 & 0 & \dots & p(z) & p'(z) \end{pmatrix}.$$

It can be shown that $D_m(z)$ is also computable recursively:

$$D_m(z) = \sum_{i=1}^{n} (-1)^{i-1} \frac{p^{i-1}(z) p^{(i)}(z)}{i!} D_{m-i}(z), \quad D_j = 0, \quad j < 0.$$

Now for each $m \geq 2$, define the iteration function

$$B_m(z) \equiv z - p(z) \ \ \frac{D_{m-2}(z)}{D_{m-1}(z)}.$$

The first member, $B_2(z)$, is Newton's iteration function $z - p(z)/p'(z)$. For many results and deep properties of this family, including multipoint variations, and its connections to a determinantal generalization of Taylor's theorem see [1]-[3]. For its artistic and educational potentials see [4]. One of the interesting properties of the Basic Family is the following connection to Voronoi regions: the basins of attractions of the roots of a complex polynomial, as computed with respect to $B_m(z)$, $m = 2, 3, \dots$, converge to the Voronoi regions of the roots of $p(z)$. Not only this property results in the creation of beautiful images, only by inputing a few numbers into our software, but it suggests that in order to compute or approximate the Voronoi regions of a given set of points one may convert the problem into a polynomial root-finding one.

Figures 1 and 2 present several fractal images that confirm the theoretical convergence results: as m increases, the basins of attractions to the roots, as computed with respect to the iteration function $B_m(z)$, converge to the Voronoi regions of the roots. Thus the regions with chaotic behavior shrink to the boundaries of the Voronoi regions.

Figure 1 considers a polynomial with a random set of roots, depicted as dots. The figure shows the evolution of the basins of attraction of the roots to the Voronoi regions as m assumes the values $2, 4, 10$, and 50. Figure 2 shows the basins of attractions for the polynomial $p(z) = z^4 - 1$, corresponding to different values of m. In both figures in the case of $m = 2$, which corresponds to Newton's method, the basins of attractions are chaotic. However, these regions rapidly improve by increasing m.

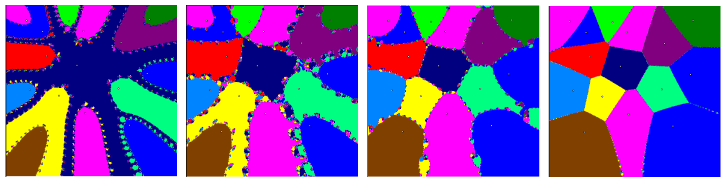

Figure 1: Evolution of basins of attraction to Voronoi regions via $B_m(z)$: random points, $m = 2, 4, 10, 50$ (left to right).

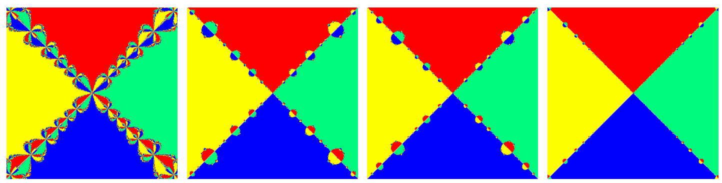

Figure 2: Evolution of basins of attraction to Voronoi regions via $B_m(z)$: $p(z) = z^4 - 1$, $m = 2, 3, 4, 50$ (left to right).

The connection between Voronoi regions and polynomial root-finding can perhaps lead to some applications in computational geometry, visually, practically, or pedagogically. As an example consider the problem of computing the Voronoi regions of a set of points given explicitly, and the roots of a polynomial given implicitly via its coefficients. One way to do this problem is to first approximate the roots and then compute the Voronoi regions of the combined points. A second approach that may have its own advantages is to convert the explicit set of points into a polynomial, form the product of the two polynomials, and subsequently make use of the Basic Family. The latter approach may also result in the saving of space in representing the set of points. As an example $z^n - 1$ is a very compact description of n points, the n roots of unity. The use of iteration function also allows a layering of the points within each Voronoi region.

References

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