Control of Coupled Slow and Fast Dynamics

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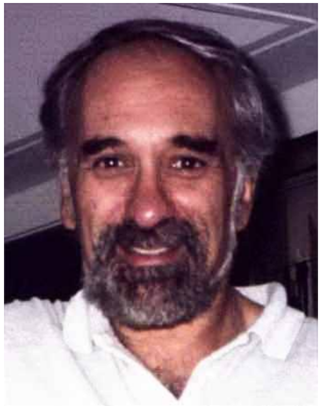
Presentations in:

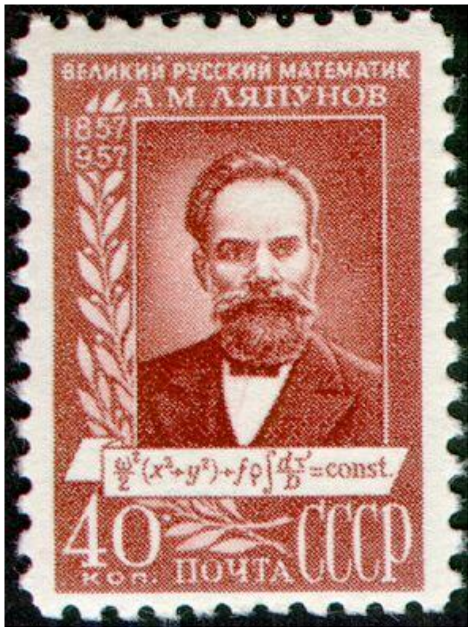
DIMACS Workshop on Perspectives and Future Directions in Systems and Control Theory = **Sontagfest**

May 23, Piscataway

Happy Birthday Eduardo

Many happy returns!




ВЕЛИКИЙ РУССКИЙ МАТЕМАТИК
А.М. ЛЯПУНОВ
1857
1957
40
КОП.
ПОЧТА
СССР


750
CENTO ANNI DI RADIO
J.C.MAXWELL
SAN MARINO

Where do coupled slow-fast systems occur?
Everywhere!

Natural phenomena and engineering design:

Hydropower Production, Nuclear Reactions, Aircraft Design, Flight Control, Optical Communiction …

Issues include:

Regulation, Feedback Design, Stabilization, Optimal Control, …

A model: Singularly perturbed control systems:

$$\text{minimize} \quad \int_a^b c(x, y, u)\,dt$$

$$\text{subject to} \quad \frac{dx}{dt} = f(x, y, u)$$

$$\epsilon \frac{dy}{dt} = g(x, y, u)$$

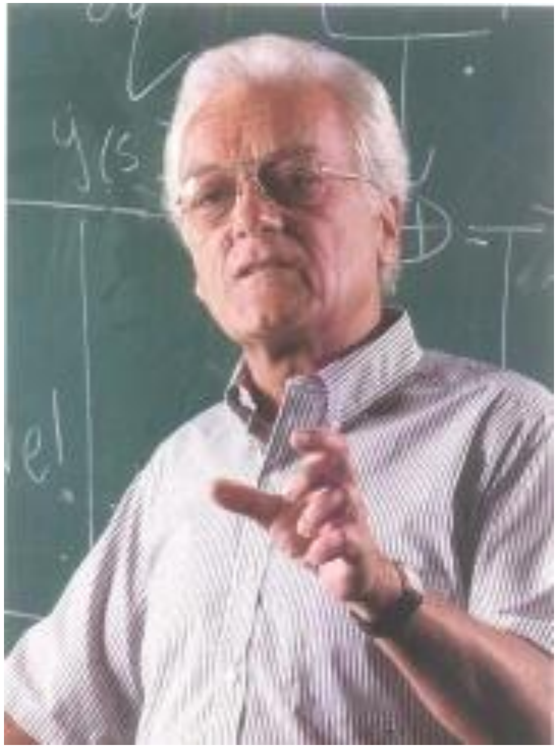
$$x(a) = x_0$$

$$y(a) = y_0$$

Where: x in R^n the slow and y in R^m the fast, variables

Of interest: The behavior of the system as $\epsilon \to 0$

Petar Kokotovic



Andrei Nikolayevich Tikhonov

1906 -1993

The order reduction method (Petar Kokotovic et al.)

The limit as $\epsilon \to 0$ is depicted by $\epsilon = 0$ namely, by:

$$\text{minimize} \quad \int_a^b c(x,y,u)dt$$

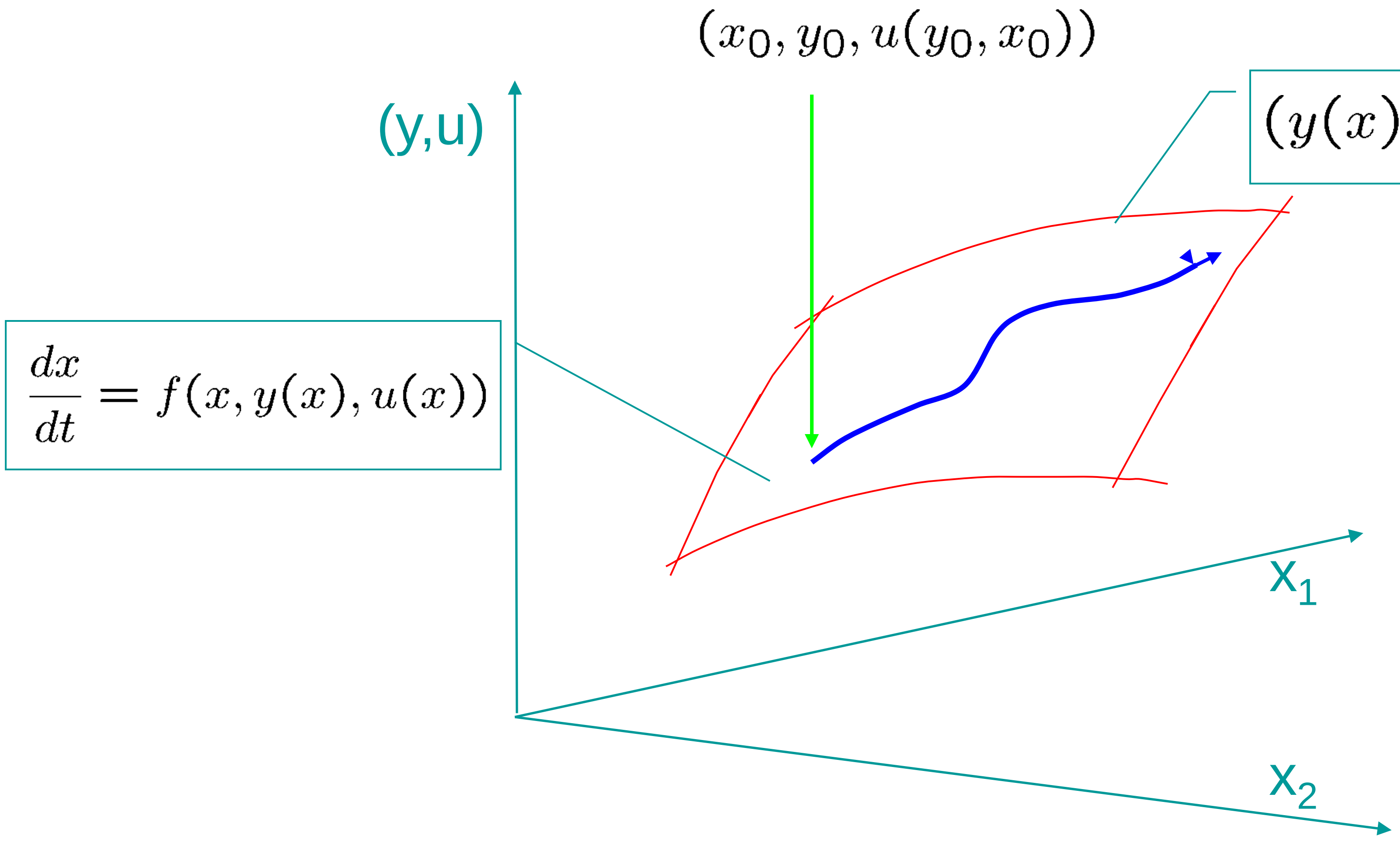
$$\text{subject to} \quad \frac{dx}{dt} = f(x,y,u)$$

$$0 = g(x,y,u)$$

$$x(a) = x_0$$

$$y(a) = y_0$$

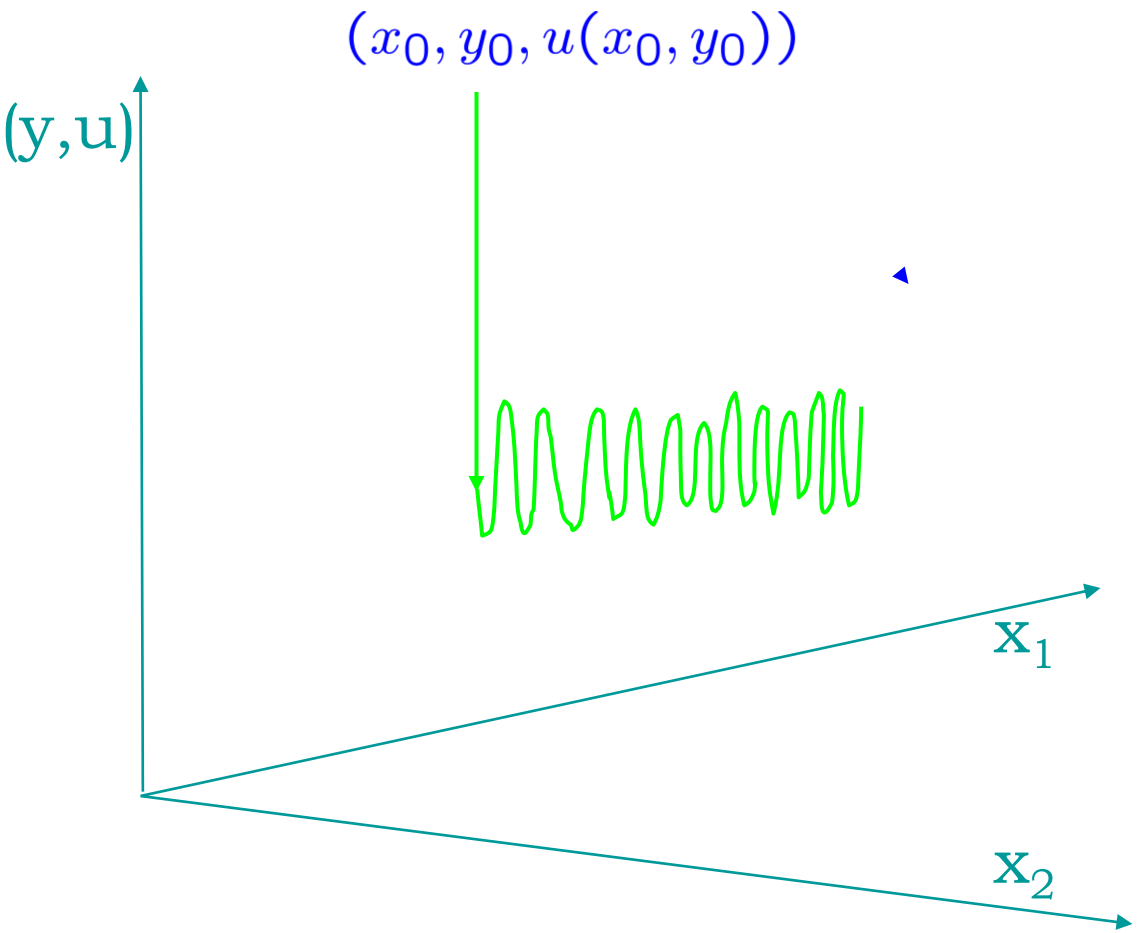
The solution method:



BUT

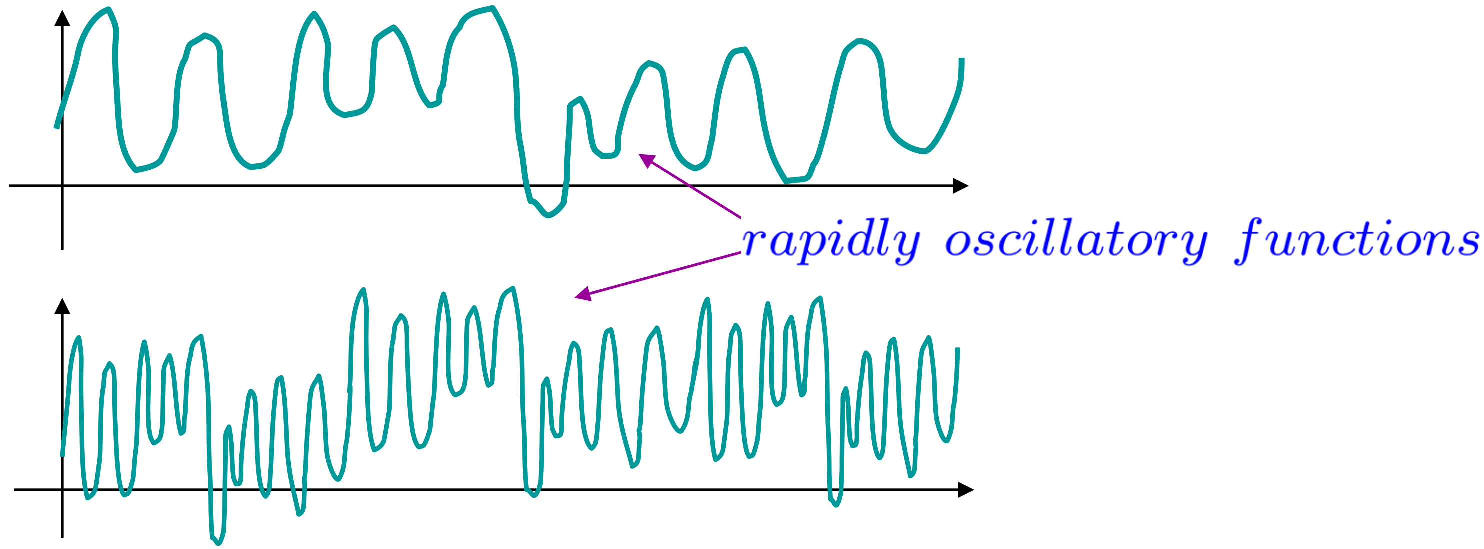
The general situation:

There is no reason why, in general, the optimal fast solution will converge and not, say, oscillate!

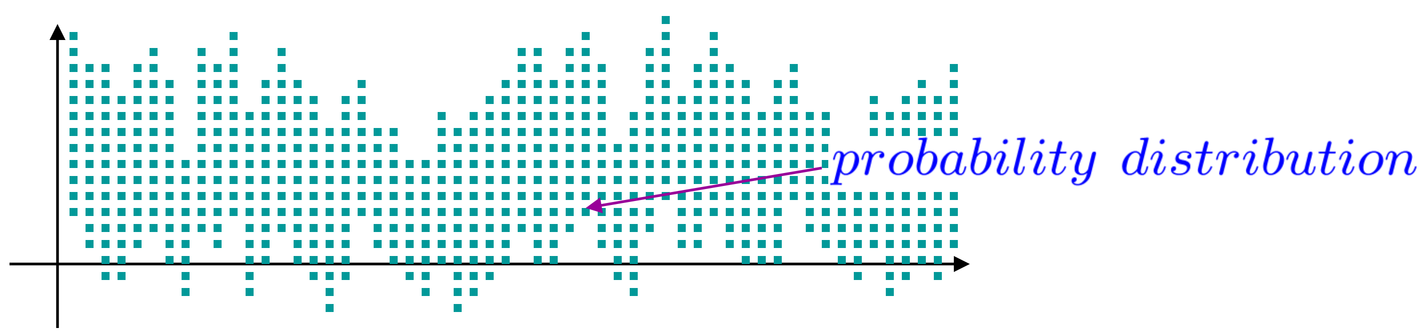


The remedy: Young measures

The limit of a sequence of highly oscillatory functions

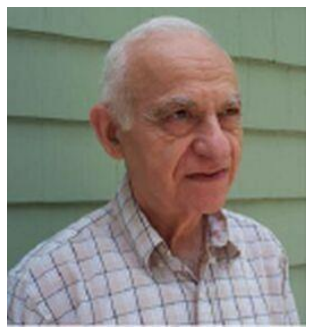


is the probability-valued map, the Young measure



Prior uses of Young Measures in differential equations and control:

L.C. Young: Generalized Curves in the Calculus of Variations

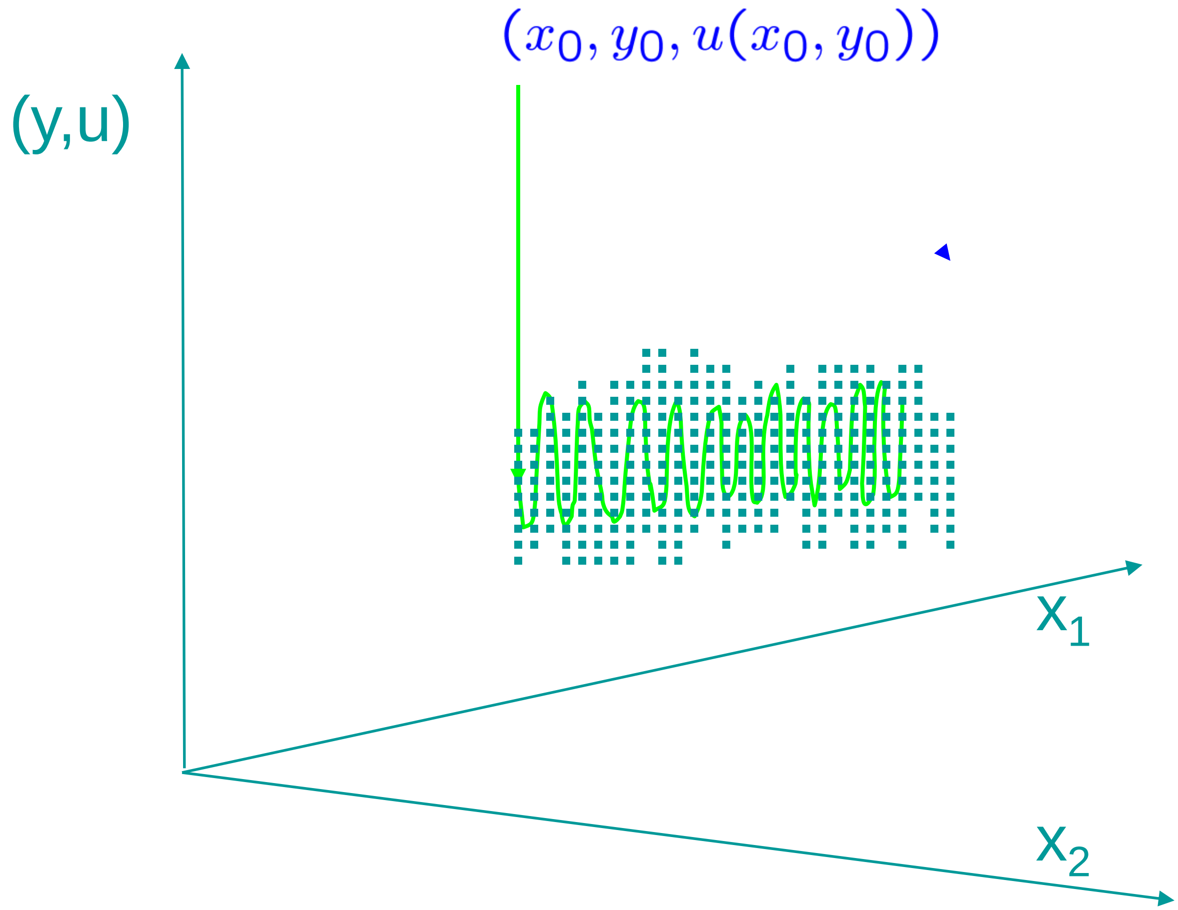


Jack Warga: Relaxed Controls

Luc Tartar: PDEs Compensated Compactness

John Ball Material Science

The situation in the singularly perturbed case:



The limit solution: $(x(t), \mu(x(t)))$

A very useful property:

An optimal solution always exists!

(under a boundedness condition)

There is a structure:

The values of the Young measure are:

invariant measures

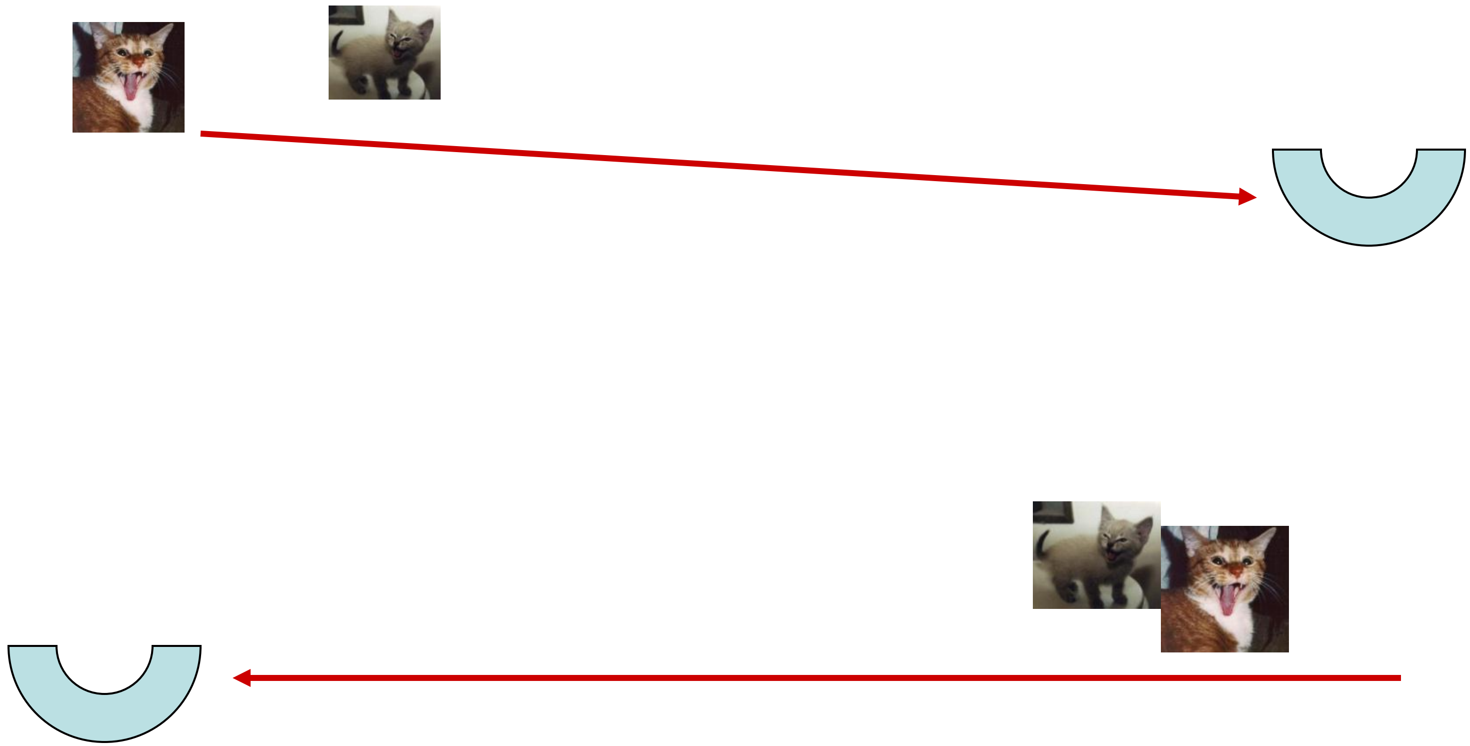
of the (fast state, control) dynamics !

A study of these invariant measures:

As invariant measures of multi-valued maps
J.P. Aubin, H. Frankowaska, A. Lasota.
Z. Artstein.

Characterization via dual variables
A. Leizarowirz, V. Gaitsgory

An illustration



The questions: when should the switch be made? How should this be carried out when the speed is very fast?

An example – after V. Veliov 1996

$$\text{maximize} \quad \int_0^1 |y_1(t) - 2y_2(t)|dt$$

$$\text{subject to} \quad \epsilon \frac{dy_1}{dt} = -y_1 + u$$

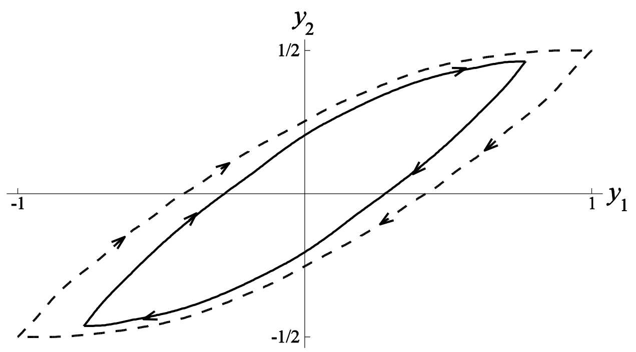
$$\epsilon \frac{dy_2}{dt} = -2y_2 + u$$

$$u \in [-1, 1]$$

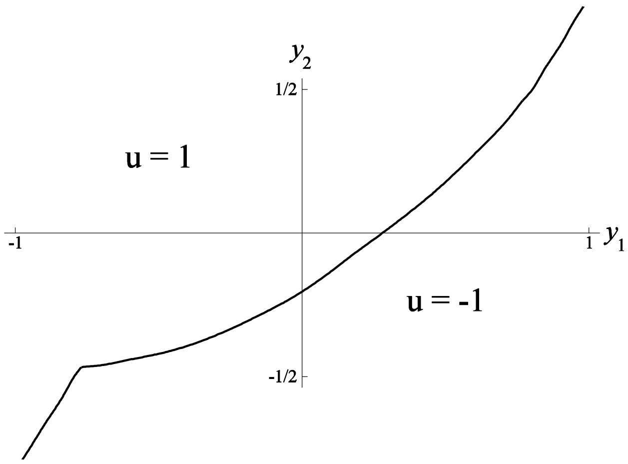
Applying an order reduction (i.e. plugging $\epsilon = 0$) yields zero value. Clearly one can do better!

The solution:

The limit strategy as $\epsilon \to 0$ can be expressed as a bang-bang feedback $u(y_1, y_2)$ resulting in:



Limit occupational measure



The bang-bang feedback

The general limit solution is of the form:

$$(x(t), \mu(x(t)))$$

Where: $x(t)$ solves the averaging equation

$$\frac{dx}{dt} = \int_{Y \times U} f(x,y,u)\mu(x)(dy \times du);$$

$\mu(x)(dy \times du)$ is an invariant measure (when x is fixed) of

$$\frac{dy}{ds} = g(x,y,u)$$

And the limit cost is

$$\int_a^b \int_{Y \times U} c(x(t),y,u)\mu(x(t)(dy \times du)\ dt$$

Notice, the limit distributions are the **control variables**, replacing the equilibrium points in the classical case

A special case:

$$(x(t), \mu(x(t)))$$

The state variable $x(t)$ is one dimensional

Then the Kokotovic approach applies !!

(Joint work with Arie Leizarowitz, 2002)

Another special case:

$$(x(t), \mu(x(t)))$$

The state variable $x(t)$ is two-dimensional

Then it is enough to consider invariant measures on a periodic trajectory (a sort of Poicare-Bendixson result) !!

(Joint work with Ido Bright, 2010)

Some Propaganda:

The method has been applied by Z.A. and collaborators to a variety of applications, including:

- Stability and Stabilization
- Relaxed Controls
- Elimination of randomization
- Game theoretic considerations
- Quantitative analysis for singular perturbations
- Applications to averaging
- Invariant measures for set-valued maps
- Tracking systems
- Linear-quadratic problems
- Infinite horizon
- Time-varying systems (including fast time-varying)
- Optimization via Lagrange multipliers
- Value function via Hamilton-Jacobi equations
- Linear systems, bang-bang

For a discussion of some of these issue please check papers listed in my web page.

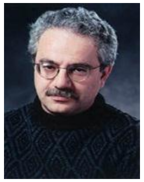
Collaborators on various applications of Young measures:

- Alexander Vigodner*, now in New York, NY
- Vladimir Gaitsgory, Adelaida, Australia
- Marshall Slemrod, Madison, WI
- Cristian Popa*, now in NY (Deutsche Bank)
- Michael Grinfeld, Glasgow, Scotland
- Arie Leizarowitz, Haifa, Israel
- Yannis Kevrekidis, Princeton, NJ
- Edriss Titi, Rehovot, Israel
- Jasmine Linshiz*, Rehovot, Israel
- C. William Gear, Princeton NJ
- Ido Bright*, Rehovot, Israel

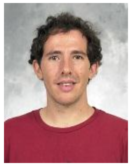





1953-2010







Work in progress:

Singular perturbations of control systems without split to slow and fast coordinates:

The perturbed system:

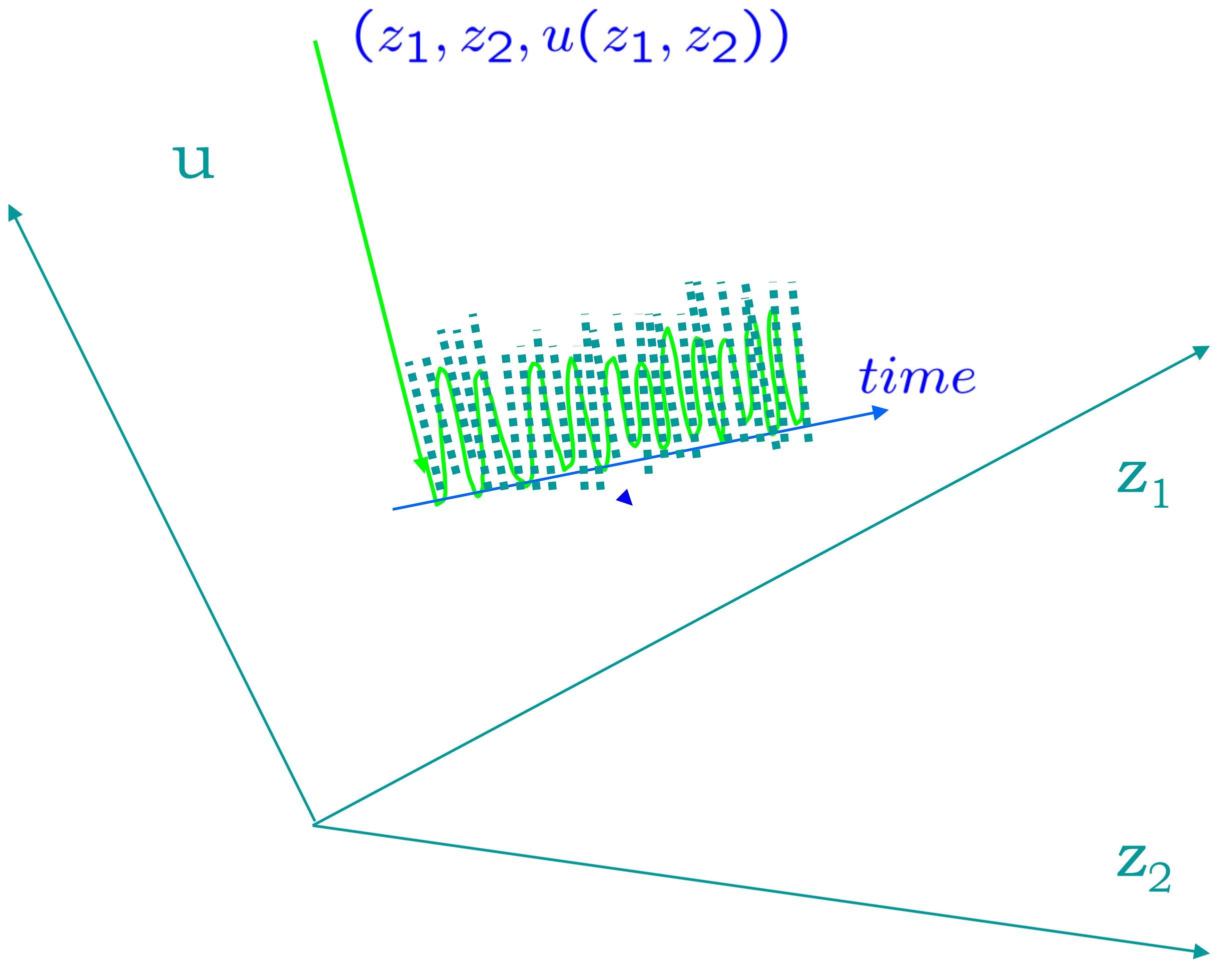
$$\frac{dz}{dt} = G(z,u) + \frac{1}{\epsilon}F(z,u)$$

Compare with:

$$\frac{dx}{dt} = f(x,y,u)$$

$$\epsilon\frac{dy}{dt} = g(x,y,u)$$

The general situation without slow-fast split:



Identifying slow and fast contributions :

Fast equation:

$$\frac{dz}{dt} = \frac{1}{\epsilon} F(z,u)$$

The perturbed system:

$$\frac{dz}{dt} = G(z,u) + \frac{1}{\epsilon} F(z,u)$$

The limit solution:

As $\epsilon \to 0$ the limit (in the sense of Young measures) of the solution of the perturbed system:

$$\frac{dz}{dt} = G(z,u) + \frac{1}{\epsilon}F(z,u)$$

is an invariant measure of the fast equation:

$$\frac{dz}{dt} = \frac{1}{\epsilon}F(z,u)$$

drifted by the the slow component.

The trajectory of invariant measures:

The drift (change in time) of the measures $\mu_0(t)$ is determined by generalized moments, or observables, preferably first integrals of the fast equation:

$$v = v(t)$$

The dynamics of the observables satisfies:

$$\frac{dv}{dt} = \int_{R^n} (\nabla v)(z) \cdot G(z,u)\mu_0(t)(dz)$$

The novelty: The observables are not part of the state space.

An example (without a control)*:

With W. Gear, I. Kevrekidis, E. Titi, M. Slemrod

$$\frac{dU_k}{dt} + \frac{1}{2h\epsilon} U_k (U_{k+1} - U_{k-1}) = \frac{1}{h^2}(U_{k+1} - 2U_k + U_{k-1})$$

$$k = 1, 2, \ldots, 2n$$

With periodic boundary conditions. This is the Lax-Goodman discretization of the KdV-Burgers

$$u_t + u(u_x + \frac{h^2}{6} u_{xxx}) = \epsilon u_{xx}$$

$$0 \leq x \leq 2\pi$$

with periodic boundary conditions

* from a paper to appear in SIAM J. Numerical Analysis

The first integrals of the fast equation:

$$\frac{dU_k}{dt} + \frac{1}{2h\epsilon} U_k (U_{k+1} - U_{k-1}) = 0$$

$$k = 1, 2, \ldots, 2n$$

are the traces of the so called Lax pairs – these are computable even polynomials

Computing the dynamics of these time-varying polynomial enables the construction of the drift of the invariant measures

Computational results for an invariant measure*:

$$\frac{dU_k}{dt} + \frac{1}{2h\epsilon} U_k(U_{k+1} - U_{k-1}) = 0$$

$$k = 1, 2, \ldots, 6$$

for the limit as $\epsilon \to 0$

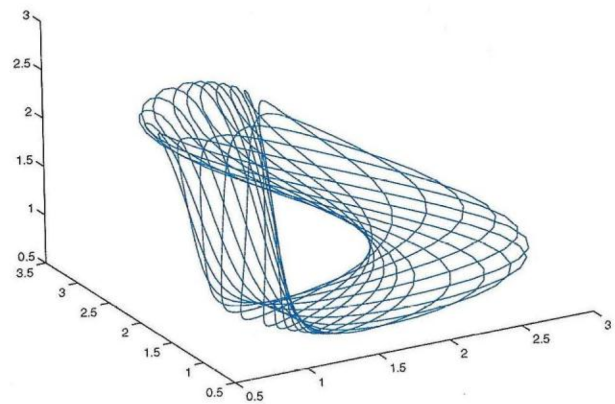


FIGURE 1. Torus for the case $N = 6$ of system (3.1). Initial values were [1 1 1 3 2 1].

* from the paper to appear in SIAM J. Numerical Analysis

Computational results for the drifted measure*:

$$\frac{dU_k}{dt} + \frac{1}{2h\epsilon} U_k (U_{k+1} - U_{k-1}) = \frac{1}{h^2}(U_{k+1} - 2U_k + U_{k-1})$$

$$k = 1, 2, \ldots, 6$$

for the limit as $\epsilon \to 0$

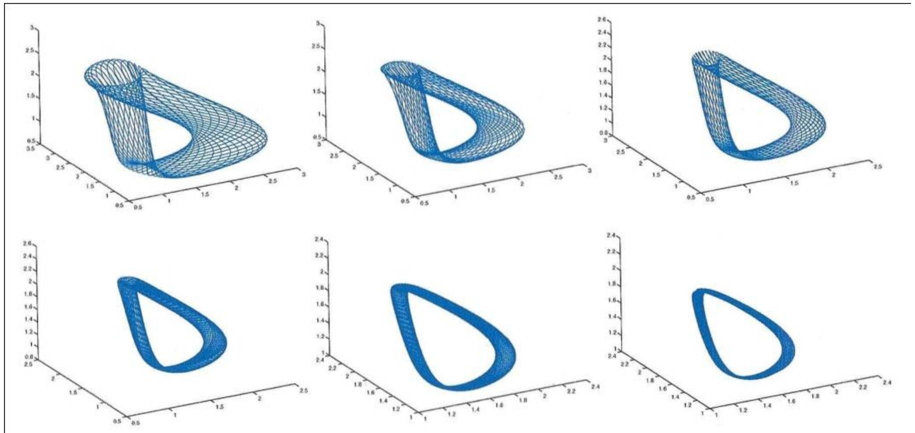


FIGURE 7. Evolution of Tori of system (6.1) for initial condition [1 1 1 1 4 1].

* from the paper to appear in SIAM J. Numerical Analysis

The End

Thanks for the attention

All the best, Eduardo !