MAY'S THEOREM FOR TREES

F. R. MCMORRIS AND R. C. POWERS

1. Introduction

In 1952, Kenneth May gave an elegant characterization of simple majority decision based on a set with exactly two alternatives [9]. This work is a model of the classic voting situation where there is two candidates and the candidate with the most votes is declared the winner. May's theorem is a fundamental result in the area of social choice and it has inspired many extensions. See [2], [3], [4], [5], [8], and [10] for a sample of these results.

The goal of the current paper is to state and prove a version of May's theorem in the context of trees. In what follows, **tree** will mean a rooted tree with labelled leaves and unlabelled interior vertices, and no vertex except possibly the root can have degree 2. In the biological literature, such a tree T might represent the evolutionary history of the set S of species, with interior vertices of T representing ancestors of the species in S. Clearly the simplest nontrivial case is when $|S| = 3$. In this case, there are exactly 4 distinct trees with leaves labelled by the set S. It is within this context that we define a version of simple majority decision for trees and characterize it in terms of three conditions. There is a clear connection between our conditions and those given by May.

This paper is divided into four sections with this introduction being the first section. Section 2 is background material on May's work and includes the statement of May's Theorem. Section 3 contains the definition of majority decision for trees, and the main result of this paper is stated and proved in Section 4.

2. Background on May's Work

Let $S = \{x, y\}$ be a set with two alternatives. The binary relations $R_{-1} = \{(x,x),(y,y),(y,x)\}$, $R_0 = S \times S$, and $R_1 = \{(x,x),(y,y),(x,y)\}$ are the three weak orders on S. The relation R_{-1} represents the situation where y is strictly preferred to x, R_1 represents the situation where x is strictly preferred to y, and R_0 represents indifference between x and y.

Let $K = \{1, \ldots, k\}$ be a set with $k \geq 2$ individuals and let $\mathcal{W}(S)$ be the set $\{R_{-1}, R_0, R_1\}$. A function of the form

$$f : \mathcal{W}(S)^k \to \mathcal{W}(S)$$

is called a **group decision function** by May.

For any $p = (D_1, \ldots, D_k)$ in $\mathcal{W}(S)^k$ and for any $i \in \{-1, 0, 1\}$ let

$$N_p(i) = |\{D_j : D_j = R_i\}|.$$

That is, $N_p(i)$ is the number of times the relation R_i appears in the k-tuple p. It follows that $N_p(-1) + N_p(0) + N_p(1) = k$ and $N_p(i) \geq 0$ for each $i \in \{-1, 0, 1\}$.

The group decision function

$$M : \mathcal{W}(S)^k \to \mathcal{W}(S)$$

defined by

$$M(p) = \begin{cases} R_{-1} & \text{if } N_p(1) - N_p(-1) < 0 \\ R_1 & \text{if } N_p(1) - N_p(-1) > 0 \\ R_0 & \text{if } N_p(1) - N_p(-1) = 0 \end{cases}$$

for any k-tuple p is called, for obvious reasons, **simple majority decision**. The consensus weak order $M(p)$ has y strictly preferred to x if more individuals rank y strictly over x than x strictly over y. There is indifference between x and y if the number of individuals that strictly prefer y over x is the same as the number of individuals that strictly prefer x over y. Finally, $M(p)$ has x strictly preferred to y if the number of individuals that rank x striclty over y is more than the number of individuals that rank y strictly over x.

May simplified the notation used above as follows. The relation R_{-1} is identified with the number -1, the relation R_0 is identified with the number 0, and the relation R_1 is identified with 1. Using this identification we can think of a group decision function as a function with domain $\{-1, 0, 1\}^k$ and range $\{-1, 0, 1\}$.

Let $f : \{-1, 0, 1\}^k \to \{-1, 0, 1\}$ be a group decision function. Then reasonable properties that f may or may not satisfy are the following.

(A) For any k-tuple $p = (D_1, \ldots, D_k)$ and for any permutation α of K,

$$f(D_{\alpha(1)}, \ldots, D_{\alpha(k)}) = f(D_1, \ldots, D_k).$$

(N) For any k-tuple $p = (D_1, \ldots, D_k)$,

$$f(-D_1, \ldots, -D_k) = -f(D_1, \ldots, D_k).$$

(PR) For any k-tuples $p = (D_1, \ldots, D_k)$ and $p' = (D'_1, \ldots, D'_k)$,

$$\text{if } f(D_1, \ldots, D_k) \in \{0, 1\}, D'_i = D_i \text{ for all } i \neq i_0, \text{ and } D'_{i_0} > D_{i_0},$$

then

$$f(D'_1, \ldots, D'_k) = 1.$$

The conditions (A), (N), and (PR) correspond to conditions II, III, and IV given on pages 681 and 682 in [9]. Condition (A) states that f is a symmetric function of its arguments and thus individual voters are anonymous. Condition (N) is called **neutrality**. This axiom is motivated by the idea that the consensus outcome should not depend upon any labelling of the alternatives. Condition (PR) is called **positive responsiveness** since it reflects the notion that a group decision function should respond in a positive way to changes in individual preferences. If the consensus outcome $f(p)$ does not rank y strictly preferred to x and one individual i_0 changes their vote in a favorable way toward x, then the consensus outcome $f(p')$ should strictly prefer x to y.

We now can state May's result.

Theorem 1. *A group decision function is the method of simple majority decision if and only if it satisfies (A), (N), and (PR).*

3. Trees with 3 Leaves

As we have noted, May studied majority decision for two alternatives, which is the simplest non-trivial case for weak orders. Since our goal is to prove a version of May's result for trees, we too restrict our attention to the simplest non-trivial case for trees; namely when $|S| = 3$. For $S = \{x, y, z\}$, and $\{u, v\} \subset S$, let $T_{\{u,v\}}$ denote the tree with one non-root vertex of degree three adjacent to the root, u, and v. Let $T_\emptyset$ be the tree whose only internal vertex is the root.

Let $\mathcal{T}(S)$ be the set $\{T_{\{x,y\}}, T_{\{x,z\}}, T_{\{y,z\}}, T_\emptyset\}$ of all trees with the leaves labelled by the elements of S. We will call a function of the form

$$C : \mathcal{T}(S)^k \to \mathcal{T}(S)$$

a **consensus function** to conform with current useage [6] . An element $P = (T_1, \ldots, T_k)$ in $\mathcal{T}(S)^k$ is called a **profile** and the output $C(P)$ is called a **consensus tree**. For any profile $P = (T_1, \ldots, T_k)$ and for any two element subset $\{u, v\}$ of S, let

$$N_P(uv) = |\{T_i : T_i = T_{\{u,v\}}\}|.$$

Also, let

$$N_P(\emptyset) = |\{T_i : T_i = T_\emptyset\}|.$$

So $N_P(xy) + N_P(xz) + N_P(yz) + N_P(\emptyset) = k$. The consensus function

$$Maj : \mathcal{T}(S)^k \to \mathcal{T}(S)$$

defined by

$$Maj(P) = \begin{cases} T_{\{u,v\}} & \text{if } N_p(uv) > \frac{k}{2} \\ T_\emptyset & \text{otherwise} \end{cases}$$

is called **majority rule** [7]. This consensus function is well known but it is not the best analog of simple majority decision *sensu* May. We feel that a better candidate is the consensus function

$$M : \mathcal{T}(S)^k \to \mathcal{T}(S)$$

defined by

$$M(P) = \begin{cases} T_{\{u,v\}} & \text{if } N_p(uv) > \max\{N_P(uw), N_P(vw)\} \\ T_\emptyset & \text{otherwise} \end{cases}$$

where $\{u, v, w\} = \{x, y, z\}$. It is easy to see that if $Maj(P) = T_{\{u,v\}}$ for some two element subset $\{u, v\}$ of S, then $M(P) = Maj(P)$. The converse is not true. For example, if $P = (T_1, \ldots, T_k)$ such that $T_1 = T_{\{x,y\}}$ and $T_i = T_\emptyset$ for all $i \neq 1$ in K, then $M(P) = T_{\{x,y\}}$ and $Maj(P) = T_\emptyset$. For the remainder of this paper the function M will be called **majority decision**.

4. Main Result Summary

We introduce translations of the conditions (A), (N), and (PR) [(A)$^+$, (N)$^+$, (PR)$^+$] to the context of trees and prove

Theorem 2. *The consensus function $C : \mathcal{T}(S)^k \to \mathcal{T}(S)$ is simple majority decision if and only if C satisfies (A)$^+$, (N)$^+$, and (PR)$^+$.*

References

[1] K. J. Arrow, *Social Choice and Individual Values*, Wiley, New York 2nd ed. (1963).

[2] G. Asan and M. R. Sanver, Another characterization of the majority rule, *Economic Letters* **75** 409-413 (2002).

[3] D. E. Campbell, A characterization of simple majority rule for restricted domains, *Economic Letters* **28** 307-310 (1988).

[4] D. E. Campbell and J. S. Kelly, A simple characterization of majority rule, *Economic Theory* **15** 689-700 (2000).

[5] E. Cantillon and A. Rangel, A graphical analysis of some basic results in social choice, *Social Choice and Welfare* **19** 587-611 (2002).

[6] W. H. E. Day and F. R. McMorris, *Axiomatic Consensus Theory in Group Choice and Biomathematics*, SIAM Frontiers of Applied Mathematics, vol. 29, Philadelphia, PA, (2003).

[7] T. Margush and F. R. McMorris, Consensus n-trees, *Bulletin of Mathematical Biology* **42** No. 2 239-244 (1981).

[8] E. Maskin, Majority rule, social welfare functions, and game forms, in: K. Basu, P.K. Pattanaik and K. Suzumura, eds., *Choice, Welfare and Development*, Festschrift for Amartya Sen, Clarendon Press, Oxford (1995).

[9] K. O. May, A set of independent necessary and sufficient conditions for simple majority decision, *Econometrica* **20** 680-684 (1952).

[10] G. Woeginger, A new characterization of the majority rule, *Economic Letters* **81** 89-94 (2003).

Department of Applied Mathematics, Illinois Institute of Technology, Chicago, IL 60616, USA

Department of Mathematics, University of Louisville, Louisville, KY 40292, USA