Characterizing Neutral Aggregation on Restricted Domains

Extended Abstract

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Abstract

Sen [6] proved that by confining voters to value restricted (acyclic) domains voting paradoxes (intransitive relations) can be avoided in aggregation by majority. We generalize this result to any neutral monotone aggregation. In addition, we show that acyclicity is neither necessary nor sufficient for transitivity for neutral non-monotone aggregation: we construct a cyclic transitive domain and introduce *strong acyclicity* as a sufficient condition for transitivity. We also show that strong acyclicity is necessary if repeated transitivity is sought. Finally, we present a cyclic domain repeatedly transitively aggregatable by a non neutral function.

1 Introduction

The concept of restricted preference domains was first introduced by Black [2]. In the paper from 1948 he showed that 'single peaked' domains are transitive for majority namely by restricting the voters to these domains aggregation by majority will always produce a transitive binary relation. The importance of this concept became more eminent two years later with the publication of Arrow's seminal work [1]. Arrow specified the basic requirements from a social welfare function (SWF): the *Pareto* condition, *independence of irrelevant alternatives* (IIA) and unrestricted domain. He showed that an aggregation satisfying these requirements generates an intransitive relation for at least one profile of voter preferences. Later work by Sen and Pattanaik [6], [7] showed that for any uneven number of voters a necessary and sufficient condition for majority to produce a transitive relation is a condition on the domain they called 'value restriction' or 'acyclicity'.[1]

How important is domain acyclicity for SWFs other than majority? Maskin [4] proved that domain acyclicity is a necessary but insufficient condition for transitivity under a neutral and symmetric SWF that is not majority. Maskin

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[1] For an extensive discussion see Sen's book [5]; for an historical perspective see Gaertner [3].

conjectured that acyclicity is necessary but insufficient for transitivity under a neutral SWF without dummies i.e. the symmetry requirement can be substituted with a weaker requirement that every voter has some influence. In this paper we show that acyclicity is necessary and sufficient for transitivity under any neutral monotone SWF and that acyclicity is preserved in the image and therefore the domain is transitive for repeated aggregation. We shall give an explicit construction of a non monotone neutral SWF called Anti Dictator and a cyclic domain on which it is transitive, thus refuting the conjecture. But we show that Anti Dictator is the only SWF transitive on an acyclic domain and that this domain is not transitive for repeated aggregation. For any other non monotone SWF we show that transitivity requires a condition called *strong acyclicity* and that this condition suffices for repeated aggregation for all neutral SWFs.

2 Preliminaries

We begin by briefly describing the model we will be using. A *voting game* G is a tuple $([n], \mathcal{W})$ where $[n] = [1, \ldots, n]$ is a set of voters and $\mathcal{W}$ is a set of coalitions (subsets of $[n]$) such that $\emptyset \notin \mathcal{W}$, $[n] \in \mathcal{W}$. The set $\mathcal{W}$ designates the winning coalitions. G is a *simple voting game* if either $S \in \mathcal{W}$ or $[n] - S \in \mathcal{W}$ for every coalition $S \subset [n]$. A game is *monotone* if $S \in \mathcal{W}$ and $S \subset T$ imply $T \in \mathcal{W}$. A simple monotone voting game is equivalent to a strong simple game as defined in [8]. Note that we do not require monotonicity in the definition of voting game, indeed non monotonicity is essential for the construction of the example refuting Maskin's conjecture.

A voter is *influential* or *effective* if his or her vote may have some impact on the outcome. In a voting game G this would mean that the voter is a *pivot* for at least one coalition, namely $S \notin \mathcal{W}$ and $S \cup \{i\} \in \mathcal{W}$ for some coalition $S \subset [n] - \{i\}$.

Let $G = ([n], \mathcal{W})$ and $G' = ([n'], \mathcal{W}')$ be voting games. We say that G *embeds* G' if there exists a surjective function $\varphi : [n] \to [n']$ such that $S \in \mathcal{W}'$ iff $\varphi^{-1}(S) \in \mathcal{W}$ for every $S \subset [n']$. We denote this by $G' = G \circ \varphi^{-1}$. Let Maj_3, $Prty_3$ and $AntiD_3$ denote Majority, Parity and Anti Dictator on three voters (see table 1).

Lemma 1. *Let G be a simple voting game on $n > 3$ effective voters.*

1. *If G is monotone then it embeds Maj_3.*
2. *If G is non monotone and $G \neq AntiD_n$ then it embeds $Prty_3$.*

Let $[m]$ be a set of $m > 2$ alternatives. Designate the set of all complete antisymmetric binary relations on $[m]$ by Δ and the set of all linear orders $\Omega \subset \Delta$. In our model a preference is a linear order (we disregard indifference) and a *domain* is a subset of Ω. An n voter *social welfare function* (SWF) on domain $\mathfrak{C} \subset \Omega$ is a function $f : \mathfrak{C}^n \to \Delta$ such that any $P = f(P_1, \ldots, P_n)$ satisfies *independence of irrelevant alternatives* (IIA) : the preference of P on

v_1	v_2	v_3	Maj_3	$Prty_3$	D_3^1	D_3^2	D_3^3	$AntiD_3^1$	$AntiD_3^2$	$AntiD_3^3$
0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	1	0	0	1	1	1	0
0	1	0	0	1	0	1	0	1	0	1
0	1	1	1	0	0	1	1	1	0	0
1	0	0	0	1	1	0	0	0	1	1
1	0	1	1	0	1	0	1	0	1	0
1	1	0	1	0	1	1	0	0	0	1
1	1	1	1	1	1	1	1	1	1	1

Table 1: The three place voting games. 1 for v_1, v_2, v_3 means the variable is in a coalition; 1 in the other columns means the coalition is a winning one

alternatives $a, b \in [m]$ depends only on the individual preferences of each voter between these two alternatives, and the *Pareto* condition: if all voters prefer a to b then so does P. It is implied by these conditions that a function f is a SWF iff there exists a collection of voting games $\{G_{ab}\}_{a,b\in[m]}$ such that aPb iff $\{j \in [n] : aP_jb\}$ is a winning coalition in G_{ab}. Notice that such a collection must satisfy $\mathcal{W}_{ba} = \{[n] - S : S \in \mathcal{W}_{ab} - \{[n]\}\} \cup \{[n]\}$. A SWF is *neutral* if $G_{ab} = G$ for all $a, b \in [m]$ and G is a simple voting game, in this case we shall occasionally identify f with G. A voter $k \in [n]$ is influential in f if it is influential in G_{ab} for at least one pair of alternatives.

Let $P \in \Omega$ be a linear order on m alternatives, we denote by $P(a_1, \ldots, a_k)$ the order induced by P on alternatives $a_1, \ldots, a_k \in [m]$, thus $P(a, b) = [ab]$ if aPb and $P(a, b, c) = [abc]$ if aPb, bPc and aPc. Let $\mathfrak{C}(a_1, \ldots, a_k)$ denote the domain of orders on $\{a_1, \ldots, a_k\}$ induced by $\mathfrak{C}$. A domain is called *cyclic* if there exist three alternatives $a, b, c \in [m]$ such that $\mathfrak{C}(a, b, c)$ contains a *cycle* namely a set of the form $\{[abc], [cab], [bca]\}$ (the complement cycle $\{[acb], [bac], [cba]\}$ is called the *anti cycle*).

Let $\mathfrak{C} \subset \Omega$ be a preference domain with a SWF f. The *image* $Im(f)$ is the set of all binary relations generated from preferences in the domain. The Pareto principle implies $\mathfrak{C} \subset Im(f)$. We say that $\mathfrak{C}$ is *transitive* for f if the image is a domain of transitive relations, namely $Im(f) \subset \Omega$.

Let f and f' be neutral SWF defined by G and G' respectively such that there is an embedding $G' = G \circ \varphi^{-1}$. For any $P = f'(P_1, \ldots, P_{n'})$ then by definition $f(P_{\varphi(1)}, \ldots, P_{\varphi(n)}) = f'(P_1, \ldots, P_{n'}) = P$ and consequently $Im(f') \subset Im(f)$. This shows transitivity for an n-place neutral monotone non dictatorial aggregation implies transitivity for Maj_3 and transitivity for a neutral non monotone aggregation implies transitivity for $Prty_3$ or $AntiD_3$.

3 Neutral Monotone Aggregation

Sen [6] showed that acyclic domains are transitive for majority; we generalize this result to any neutral monotone SWF.

Theorem 1. *Let f be a neutral monotone non dictatorial SWF, and let $\mathfrak{C}$ be a domain of linear orders.*

1. *$\mathfrak{C}$ is transitive iff it is acyclic.*

2. *$\mathfrak{C}$ acyclic implies $Im(f)$ is acyclic.*

This shows that acyclicity not only guaranties transitivity for majority but for any monotone neutral SWF and shows that the image remains acyclic. Thus in a complex multi-tier voting process we know that as long as the voters in the lowest level are restricted to an acyclic preference domain and on each tier the local committees vote via a neutral monotone SWF the *repeated* vote will be transitive.

4 A Cyclic Transitive Domain

We refute Maskin's conjecture by constructing a neutral SWF without dummies that is transitive on a cyclic domain.

Theorem 2. *There exists a cyclic domain $\mathfrak{C}$ and a neutral SWF f with no dummy voters such that $Im(f) \subset \Omega$.*

proof: For the domain we take $\mathfrak{C} = \{[\pi^j(1) \dots \pi^j(m)] : j = 0, \dots m-1\}$ where $\pi^j(i) = i + j \ mod \ m$ called the *unicyclic* domain. The *anti dictator game* on n voters $AntiD_n$ is defined by $\mathcal{W} = 2^{[n]-\{1\}} \cup \{[n]\}$ the proper subsets of $[n]$ that do not include voter 1 (the 'anti dictator') and $[n]$. Every voter apart form 1 is a pivot for the coalition $[n]$ and 1 is a pivot for any other non empty coalition therefore no voter is a dummy.

Let f be the neutral SWF defined by $AntiD_n$. Transitivity is a relation on triples hence it suffices to show that $\mathfrak{C}(a,b,c) = \{[abc],[cab],[bca]\}$ is transitive under f for any $a,b,c \in [m]$. Let $P_1, \dots, P_n$ be a profile and $P = f(P_1, \dots, P_n)$, it follows from the definition that if P_1 agrees with P_j on $a, b \in [m]$ for all $j > 1$ i.e. $\{j : aP_jb\} = [n]$ then $aPb \equiv aP_1b$ otherwise since, $\{j : aP_jb\} \subsetneqq [n]$ is a winning coalition iff 1 is not in the coalition, $bPa \equiv aP_1b$. We may assume w.l.g that $P_1(a,b,c) = [abc]$. If $\{P_1(a,b,c), \dots, P_n(a,b,c)\} = \{[abc]\}$ then the profile agrees on the pairs of $\{a,b,c\}$ hence $P(a,b,c) = P_1(a,b,c) = [abc]$. If $\{P_1(a,b,c), \dots, P_n(a,b,c)\} = \{[abc],[cab]\}$ then the profile agrees with P_1 on $[ab]$ and disagrees on $[ac]$ and $[cb]$. From the definition of $AntiD_3$ it follows that $cPaPb$ hence $P(a,b,c) = [cab]$. In the same manner it follows that $\{P_1(a,b,c), \dots, P_n(a,b,c)\} = \{[abc],[bca]\}$ implies $P(a,b,c) = [bca]$. In the case $\{P_1(a,b,c), \dots, P_n(a,b,c)\} = \mathfrak{C}(a,b,c)$ the profile disagrees with P_1 on all the pairs in $\{a,b,c\}$ hence $P(a,b,c)$ is an inversion of $P_1(a,b,c)$ namely $P(a,b,c) = [cba]$. It follows that P is transitive for any triple $a,b,c \in [m]$ thus P is transitive $\square$

We observe in the latter case that $P = [cba] \notin \mathfrak{C}(a,b,c)$ which implies $P \notin \mathfrak{C}$. Moreover let $Q = f(P, Q_2, \dots, Q_n)$ where $\{Q_2(a,b,c), \dots, Q_n(a,b,c)\} =$

$\{[cab], [bca]\}$, P and $Q_2, \ldots, Q_n$ agree on $[ca]$ and disagree on $[ba]$ and $[cb]$ therefore $aQbQcQa$ namely Q is intransitive. Consequently $Im(AntiD_3)$ is intransitive for $AntiD_3$. This shows that if voters in a committee are restricted to the unicyclic domain then a voting process defined by $AntiD_n$ will not produce any paradoxes. However, if this process is repeated on a multi-tiered voting system the same restriction is insufficient since paradoxes may appear in second tier committees. Notice that the intransitivity of Q above also implies that an acyclic domain $\mathfrak{C}$ such that $\{[cab], [bca], [cba]\} \subset \mathfrak{C}(a, b, c)$ is intransitive for $AntiD_n$.

5 Neutral Non Monotone Aggregation

In the previous section we saw that the image of a transitive domain may be intransitive. We introduce a condition that strengthens the acyclicity requirement and ensures repeated transitivity.

Definition 1. *A domain is called* ***strongly acyclic*** *if* $[abc], [cab] \in \mathfrak{C}(a, b, c)$ *implies* $[acb], [bca] \notin \mathfrak{C}(a, b, c)$.

Obviously strong acyclicity implies acyclicity, thus monotone SWF are repeatedly transitive for such domains. It can be shown that strong acyclicity is preserved in the image.

Theorem 3. *Let f be a neutral non monotone SWF and let $\mathfrak{C}$ be a domain of linear orders.*

1. *If $f \neq AntiD_n$ then:*
   - (a) *$\mathfrak{C}$ is transitive iff it is strongly acyclic.*
   - (b) *$\mathfrak{C}$ strongly acyclic implies $Im(f)$ strongly acyclic.*
2. *If $f = AntiD_n$ then:*
   - (a) *$\mathfrak{C}$ is transitive if it is mixed unicyclic and strongly acyclic.*
   - (b) *$Im(f)$ is transitive only if $\mathfrak{C}$ is strongly acyclic.*

The theorem shows that strongly acyclic domains are repeatedly transitive for any neutral SWF.

To summarize: dictatorial SWFs are transitive on any domain, non dictatorial monotone SWFs are transitive only on acyclic domains. Apart from $AntiD_n$ non monotone SWFs are transitive only on strongly acyclic domains. $AntiD_n$ is transitive on mixed unicyclic/acyclic domains but repeatedly transitive only on strongly acyclic domains.

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