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Condorcet domains and distributive lattices

Extended abstract

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The problem to get a collective preference from various voter's preferences on n alternatives (candidates, issues, decisions, outcomes…) is an old problem, since it appeared as soon as multicandidates elections had occurred, and it seems have been discussed at least since Ramon Lull (in *Blanquerna* 1283).. In 1785, Condorcet published his *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix* where he proposed to use the majority rule on the pairs of alternatives: alternative y is preferred by the majority to alternative x -denoted by $xR_{MAJ}y$- if the number of voters preferring y to x is greater than the number of voters preferring x to y. This book contains the first examples of what has come to be called the "cyclical majorities" (Dodgson 1876) or the "Condorcet effect" (Guilbaud 1952) or the "Paradox of Voting": when the voters express their preferences by means of linear orders on the set of alternatives, the majority relation of these orders can contain cycles. The simplest example is obtained with 3 alternatives x, y, z and 3 voters of which the set of preferences is a 3-*cyclic set* like xyz, yzx and zxy[1]. The majority relation of these three preferences is the 3-cycle $xR_{MAJ}yR_{MAJ}zR_{MAJ}x$.

The attempts made by Condorcet himself, Lhuillier, Daunou, Morales, Nanson or Dodgson (better known as Lewis Caroll) in the eighteenth or nineteenth centuries to overcome this problem consisted to modify majority rule or to adopt another aggregation rules. In 1948, Black initiated another way to escape the Condorcet effect. He proved that this effect cannot occur if the preferences of the voters are restricted to a subset of all possible linear orders, namely the set of the so-called *single-peaked* linear orders. After Arrow's 1951 famous book containing his impossibility theorem (which, in fact, leads to a dual and radical way to escape Condorcet effect, see Monjardet 1977), works in social choice theory have begun to develop. In particular, one have found other

[1] We denote a linear order by a permutation, where xyz means $x<y<z$, and we say that the least preferred alternative x has the first rank, the middle element y the second rank and the best preferred element z the third rank.

Condorcet domains, i.e. sets of linear orders where the Condorcet effect cannot occur[2]. In fact, the simplest and more general way to prevent Condorcet effect is to forbid 3-cyclic sets in the domain D of linear orders allowed for preferences'voters : for every 3-set of alternatives, the restrictions of the linear orders of D to this set don't contain a 3-cyclic set. This condition has been given by Ward (1965) under the name of *Latin-Square-Lessness* and is equivalent (in the case of linear orders) to Sen's *Value Restricted-Preferences* condition (1966). This last condition says that, for every 3-set of alternatives, there exists an alternative which is either never ranked first or never ranked second or never ranked third in the restrictions of the linear orders of D to these alternatives. It have been shown (at least by Kreweras as soon as 1962) that the number of single-peaked linear orders on a set of n alternatives is 2^{n-1}, so a little number compared to the n! possible linear orders. And many other Condorcet domains found in the sixties and seventies have no more elements (see Romero 1978, Raynaud 1981 or Arrow and Raynaud 1986). Let us denote by f(n) the maximum cardinality of a Condorcet domain (on a set of n alternatives). It is not clear when has been raised for the first time the natural question "how large can be Condorcet domains ?", i.e. the problem of determining f(n). However, one finds as soon as 1980 in Kim and Roush's book a result disproving Craven's conjecture $f(n) = 2^{n-1}$ (1992!): $f(n) \geq 2^{n-1} + 2^{n-3}-1$ ($> 2^{n-1}$ for $n \geq 4$).
In fact, the problem of determining f(n) has shown daunting. The first serious attempts to determine it or rather to find good least or upper bounds were made by Abello. He (with Johnson in his first paper) proved that $f(n) \geq 3(2^{n-2})-4$ ($> 2^{n-1} + 2^{n-3}-1$, for $n \geq 5$) by constructing Condorcet domains with this size. Such constructions follow from two key observations. Let us denote by L_n the set of all linear orders on n alternatives endowed with the partial order which makes it the *permutoèdre lattice* (Guilbaud and Rosenstiehl, 1963). Firstly, the linear orders in a maximal chain of L_n contain 4n(n-1)(n-2)/3 ordered triples xyz. Secondly, the linear orders in a Condorcet domain of L_n contain at most 4n(n-1)(n-2)/3 ordered triples xyz. Now, Abello's algorithm to construct Condorcet domains of size $3(2^{n-2})-4$ starts from some maximal chains of L_n and adds to such a chain all the linear orders which don't increase the set of ordered triples present in this chain. Abello shows also that these Condorcet domains are upper semi-modular lattices of the permutoèdre lattice.

[2] Many of these works bear on the case where individual preferences are weak orders, but here, we consider only the case where they are linear orders. Condorcet domains have been also called *transitive simple majority domains* or *consistent sets* (Abello and Jonhson,1984), *majority-consistent set*s (Craven, 1996) or *acyclic sets* (Fishburn,1997)

Come back to Black's single-peaked linear orders. In his 1952 pape*r Theories of the general interest and the logical problem of aggregation* Guilbaud observes that this set of linear orders has a distributive lattice structure ; in fact, this set is a *covering sublattice* of the permutoèdre lattice L_n, what means that the covering relation in this sublattice is the same as the covering relation in L_n. Other covering distributive sublattices of the permutoèdre lattice were given in Frey (1971) and Frey and Barbut (1971) and shown to be Condorcet domains. This led to Chameni-Nembua's result (1989) answering a question that I asked him for his thesis: any covering distributive sublattice of the permutoèdre lattice is a Condorcet domain. This result led to find for n = 6 such a sublattice of size 45 (so, surpassing the best Abello and Johnson's lower bound known at this date, namely 44). This last example was sent to Fishburn who was working on Condorcet domains for some time and found quickly a construction, generalizing it (see his 1997 paper). He called this construction the *alternating scheme* since it is based on alternating value restrictions: an element middle in an ordered triple is either never ranked first or never ranked third. The size of a Condorcet domain satisfying the alternating scheme is greater than $3(2^{n-2})-4$ for $n \geq 5$ (for instance, for n = 10, it has size 1069 against 764).
Are there relations between Abello's upper semimodular lattices, Chameni-Nembua's distributive lattices and Fishburn's alternating scheme ? The answer is brought by a recent Galambos and Reiner's work, where they rediscover (quite independently) and generalize another Guilbaud's observation in his 1952 paper. Guilbaud have showed that the distributive lattice structure of the single-peaked orders comes from that they correspond to the filters of a partial order defined on the set of all ordered pairs of alternatives. Galambos and Reiner show that Abello's upper semimodular lattices are in fact Chameni-Nembua's distributive lattices and that they are obtained as lattices of ideals of a partial order defined on the set of all ordered pairs of alternatives. Condorcet domains obtained by Fishburn's alternating scheme are a special case.
In our talk, after recalling the above history of Condorcet domains, we will specify some links between maximal chains and covering distributive sublattices of L_n.

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