# CONSENSUS RULES BASED ON DECISIVE FAMILIES

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Abstract. The notion of a decisive family of voters has played an important role in the analysis of various consensus functions defined on preference profiles. This role remains when the domain shifts to profiles of hierarchical classifications. The main result of this paper is a characterization of consensus rules defined on herarchies where the output clusters are determined by a decisive family of sets.

## 1. Introduction

In the classical theories of social choice and voting theory, the notions of decisiveness and decisive families play an important role. For example, game theoretic studies of voting as well as many proofs of Arrow's Theorem utilize these ideas. In this context, each voter contructs a preference binary relation (usually a weak order) on a given set of alternatives, by using some unspecified internal process. The situation involving classifications is somewhat different. Here a "voter" is often an algorithm that operates on information about the alternatives (usually involving similarity between alternatives) to produce a collection of subsets (the clusters) of the alternatives. These output set systems might be, for example, partitions (and hence have non-intersecting clusters), hierarchical classifications (with a tree-like structure), and weak hierarchies (that allow non-trivial overlapping). In each of these cases, consensus rules where the consensus output is determined by a family of decisive sets form important classes of consensus functions. (see [7], [4], [5] respectively) In this paper we focus on some work of Neumann [9] where he proved that a consensus rule $C$ on hierarchies is determined by a semidecisive family of sets if and only if $C$ satisfies an axiom of neutrality. Neumann's version of neutrality, which was renamed decisive neutrality in [2] in order to fit into a larger terminology

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scheme, is based on a standard view of a hierarchy as a set of clusters. Another view of a hierarchy is that of a ternary relation and thus consists of a set of triples, or triads. We will explore an analog to decisive neutrality from this point of view and give another characterization of consensus rules where the output clusters are determined by a decisive family of sets.

## 2. Terminology and notation

Let $S$ be a finite set with $n \geq 5$ elements. A **hierarchy** on $S$ is a collection $H$ of nonempty subsets of $S$ such that $S \in H$, $\{x\} \in H$ for all $x \in S$, and $A \cap B \in \{A, B, \emptyset\}$ for all $A, B \in H$. We will denote the set of all hierarchies on $S$ by $\mathcal{H}$ and call a set $X$ in a hierarchy $H$ for which $1 < |X| < n$ a **non-trivial cluster** of $H$. $H_\emptyset$ will denote the hierarchy with no non-trivial clusters. For any nontrivial subset $X$ of $S$ let $H_X = H_\emptyset \cup \{X\}$, so $H_X \in \mathcal{H}$ with $X$ as the only nontrivial cluster of $H_X$.

For each hierarchy $H$ there is an associated ternary relation $r_H$ on $S$ defined by $(a, b, c) \in r_H$ if and only if there exists $X \in H$, such that $a, b \in X$ and $c \notin X$ [1]. This relation is meant to capture the notion that $a$ and $b$ are more similar to each other than either element is to $c$, with respect to the hierarchy $H$. We will often write $ab \mid_H c$ instead of $(a, b, c) \in r_H$. The notation $abc \mid_H$ will be used if $\{(a, b, c), (c, a, b), (b, c, a)\} \cap r_H = \emptyset$. In general, the ordered triple $(a, b, c)$ is called a **triad**, or simply a **triple**.

The function that maps a hierarchy $H$ on $S$ to the ternary relation $r_H$ is injective [6]. In fact, a subset $X$ of $S$ belongs to $H$ if and only if $(a, b, c) \in r_H$ for all $a, b \in X$ and $c \notin X$. Thus by identifying $H$ with $r_H$, a hierarchy is precisely collection of triads.

A **consensus function** (on $\mathcal{H}$) is a map $C : \mathcal{H}^k \to \mathcal{H}$ where $k \geq 2$. Elements of $\mathcal{H}^k$, the $k$-fold Cartesian product, are called **profiles** and the conventional notation for profiles is $P = (H_1, ..., H_k), P' = (H'_1, ..., H'_k)$, and so on.

If $H \in \mathcal{H}$ and $X$ is a proper subset of $S$, then $H|_X$ denotes the hierarchy whose nontrivial clusters are the nonempty distinct elements of $\{A \cap X : A$ is a nontrivial cluster of $H$ and $1 < |A \cap X| < n\}$. In addition, $H|_X - X$ is the hierarchy $H|_X$

without the cluster $X$. Note that

$$ab \mid_H c \text{ if and only if } H|_{\{a,b,c\}} - \{a,b,c\} = H_{\{a,b\}}.$$

This notion of restriction extends to profiles in a natural way. Specifically, for any profile $P = (H_1, \ldots, H_k)$ and subset $X$ of $S$,

$$P|_X = (H_1|_X, \ldots, H_k|_X)$$

and

$$P|_X - X = (H_1|_X - X, ..., H_k|_X - X).$$

Let $K = \{1, \ldots, k\}$. For any consensus function $C$ on $\mathcal{H}$, profile $P$, cluster $X$, and triple $(a,b,c)$ let

$$K_X(P) = \{i \in K : X \in H_i\},$$

$$K_{(a,b,c)}(P) = \{i \in K : ab \mid_{H_i} c\},$$

and let

$$U_C = \{I : I = K_{(a,b,c)}(P) \text{ and } ab \mid_{C(P)} c \text{ for some profile } P \text{ and triple } (a,b,c)\}.$$

So $K_X(P)$ and $K_{(a,b,c)}(P)$ identify the hierarchies in the input that contain the cluster $X$ and the triad $(a,b,c)$, respectively. The set $U_C$ contains all possible subsets of $K_{(a,b,c)}(P)$ where the triad $(a,b,c)$ belongs to the consensus output $C(P)$.

## 3. Consensus based on semidecisive and decisive families

**Definition 1.** *A nonempty subset $\Sigma$ of $2^K$ is called a* **semidecisive family** *on $K$ if, for all $I, J \in \Sigma$, $I \cap J \neq \emptyset$.*

*A nonempty subset $\Sigma$ of $2^K$ is called a* **decisive family** *on $K$ if it is semidecisive and, for all $I \in \Sigma$, $I \subseteq J \subseteq K$ implies $J \in \Sigma$.*

**Example 1.** *(i) For a fixed $j \in K$, then $\Sigma = \{I \subseteq K : j \in I\}$ is a decisive family on $K$, and is called a dictatorial family on $K$.*

*(ii) Let $\ell$ be an integer such that $\ell > \frac{k}{2}$. Then $\Sigma_\ell = \{I \subseteq K : \ell \leq |I|\}$ is a decisive family on $K$, and is called a quota family on $K$.*

*(iii) Let $\ell_1$ and $\ell_2$ be integers such that $\frac{k}{2} < \ell_1 \leq \ell_2 < k$. Then $\Sigma = \{I \subseteq K : \ell_1 \leq |I| \leq \ell_2\}$ is a semidecisive family on $K$ which is not decisive.*

We now give a precise definition of what is means for a consensus function on $\mathcal{H}$ to be determined by either a semidecisive or decisive family of sets.

**Definition 2.** *If $\Sigma$ is a semidecisive family on $K$, then define $M_\Sigma : \mathcal{H}^k \to \mathcal{H}$ as follows: for a profile $P = (H_1, \ldots, H_k)$, a nontrivial cluster $X$ belongs to $M_\Sigma(P)$ if and only if $\{i \in K : X \in H_i\} \in \Sigma$.*

That $M_\Sigma(P)$ is a well-defined hierarchy follows immediately from the definition of semidecisive family and the fact that it is understood that $S \in M_\Sigma(P)$ and $\{x\} \in M_\Sigma(P)$ for all $x \in S$. Note that when $\Sigma$ is decisive, the set of nontrivial clusters of $M_\Sigma(P)$ can be expressed as

$$M_\Sigma(P) = \bigcup_{I \in \Sigma} \left[ \bigcap_{i \in I} H_i \right].$$

**Example 2.** *Let $l = \lfloor \frac{k}{2} \rfloor + 1$. Then $M_{\Sigma_l} = Maj$, the majority-rule consensus function on hierarchies* [3]*. That is, $X \in M_{\Sigma_l}(P)$ if and only if $X$ belongs to a strict majority of the hierarchies in the profile $P$.*

Neumann [9] and McMorris & Neumann [4] characterized $M_\Sigma : \mathcal{H}^k \to \mathcal{H}$ where $\Sigma$ is semidecisive and decisive respectively. The characterizing properties relied on the cluster as the basic unit of a hierarchy. However, viewing a hierarchy as a set of triples does not allow the properties to translate directly. For example, the cluster based axiom called **decisive neutrality** (**DN**) (see [2, pp. 55-56] and [8]) states that for any profiles $P$ and $P'$ and for any clusters $X$ and $Y$, $K_X(P) = K_Y(P)$ implies that $X \in C(P)$ if and only if $Y \in C(P')$. A direct translation of this axiom, where clusters are replaced by triads, doesn't work very well. For example, for $Maj : \mathcal{H}(S)^3 \to \mathcal{H}(S)$, the majority rule consensus function, if $P = (H_{\{a,b\}}, H_{\{a,b\}}, H_\emptyset)$ and $P' = (H_{\{a,x,y\}}, H_{\{x,y\}}, H_\emptyset)$, then $K_{(a,b,c)}(P) = K_{(x,y,z)}(P')$, $ab|_{Maj(P)}c$ holds and $xy|_{Maj(P')}z$ fails. This example leads us to the following analog of decisive neutrality.

**Definition 3.** *Let $C$ be a consensus function on $\mathcal{H}$. Then $C$ satisfies* **triad neutrality** *(***TN***) if the following three conditions hold.*

*1. For all profiles $P, P'$ and all triples $(a, b, c), (x, y, z)$,*

$$K_{(a,b,c)}(P) = K_{(x,y,z)}(P') \text{ and } ab|_{C(P)}c$$

*imply that there exists a profile $P''$ such that*

$$P''|_{\{x,y,z\}} - \{x,y,z\} = P'|_{\{x,y,z\}} - \{x,y,z\} \text{ and } xy|_{C(P'')}z.$$

2. *For any profiles $P, P'$, nontrivial cluster $X$ and any triple $(x,y,z)$,*

$$K_X(P) = K_{(x,y,z)}(P') \text{ and } X \in C(P)$$

*imply that there exists a profile $P''$ such that*

$$P''|_{\{x,y,z\}} - \{x,y,z\} = P'|_{\{x,y,z\}} - \{x,y,z\} \text{ and } xy|_{C(P'')}z.$$

3. *For any profiles $P, P'$, nontrivial cluster $X$ and any triple $(x,y,z)$,*

$$K_X(P) = K_{(x,y,z)}(P') \text{ and } xy|_{C(P')}z$$

*imply that $X \in C(P)$.*

Our other key property is the following.

**Definition 4.** *$C$ satisfies* **weak independence** *(***WI***) if for any profiles $P, P'$ and all triples $(a,b,c)$,*

$$P|_{\{a,b,c\}} - \{a,b,c\} = P'|_{\{a,b,c\}} - \{a,b,c\} \text{ and } ab \;|_{C(P)}\; c$$

*imply*

$$ab \;|_{C(P')}\; c \text{ or } abc \;|_{C(P')}.$$

The idea behind (WI) is that profile agreement need not imply that the outputs agree but, at least, the output hierachies should be compatible. (Two hierarchies are **compatible** if their union is a hierarchy.) This axiom was introduced in [10].

## 4. The Main Result

A consensus function $C$ on $\mathcal{H}$ is said to be **nontrivial** if there exists a profile $P$ such that $C(P) \neq H_\emptyset$. In this case, $U_C \neq \emptyset$.

As mentioned above, consensus functions based on decisive families were characterized by McMorris and Neumann [4] (see also Theorem 4.9 in [2]) using cluster based axioms. Theorem 1, our main result, provides a different type of characterization with an emphasis on triads.

**Theorem 1.** *If $\Sigma$ is a decisive family on $K$, then $M_\Sigma$ is a nontrivial consensus rule satisfying (***WI***) and (***TN***). Conversely, if a nontrivial consensus rule $C$ satisfies (***WI***) and (***TN***), then $C = M_\Sigma$ for some decisive family $\Sigma$ on $K$.*

Part of the proof of the main result depends on showing that $U_C$ is a semidecisive family and that $C = M_{U_C}$. The following theorem shows that there is a close connection between the axiom (**WI**) and the condition that $U_C$ is a semidecisive family.

**Theorem 2.** *If $U_C$ is a semidecisive family, then $C$ satisfies (***WI***). Conversely, if a nontrivial rule $C$ satisfies item 1 in (***TN***) and (***WI***), then $U_C$ is a semidecisive family.*

**Proof of Theorem 2.** If $C$ does not satisfy (**WI**), then there exist profiles $P, P'$ and a triple $(a, b, c)$ such that

$$P|_{\{a,b,c\}} - \{a, b, c\} = P'|_{\{a,b,c\}} - \{a, b, c\} \text{ and } ab \ |_{C(P)} \ c$$

and

$$ac \ |_{C(P')} \ b \text{ or } bc \ |_{C(P')} \ a.$$

Observe that $K_{(a,b,c)}(P)$ and either $K_{(a,c,b)}(P')$ or $K_{(b,c,a)}(P')$ produce two disjoint sets in $U_C$ contrary to the fact that $U_C$ is a semidecisive family.

For the converse, observe that $U_C \neq \emptyset$ since $C$ is nontrivial. Assume that there exist sets $I$ and $J$ belonging to $U_C$ such that $I \cap J = \emptyset$. By the definition of $U_C$, there exist profiles $P$ and $P'$ and elements $a, b, c, x, y, z \in S$ (may not all be distinct) such that $ab \ |_{C(P)} \ c$, $xy \ |_{C(P')} \ z$, $I = \{i \in K : ab \ |_{H_i} \ c\}$, and $J = \{i \in K : xy \ |_{H'_i} \ z\}$. Construct a profile $Q$ such that $I = \{i \in K : ab \ |_{Q_i} \ c\}$, $J = \{i \in K : ac \ |_{Q'_i} \ b\}$, and $Q_i = H_\emptyset$ for all $i \notin I \cup J$. Since $I = \{i \in K : ab \ |_{H_i} \ c\}$ and $ab \ |_{C(P)} \ c$ it follows from item 1 in (**TN**) that there exists a profile $Q'$ such that

$$Q'|_{\{a,b,c\}} - \{a, b, c\} = Q|_{\{a,b,c\}} - \{a, b, c\} \text{ and } ab|_{C(Q')}c.$$

Similarly, since $J = \{i \in K : xy \ |_{H_i} \ z\}$ and $xy \ |_{C(P')} \ z$ it follows from item 1 in (**TN**) that there exists a profile $Q''$ such that

$$Q''|_{\{a,b,c\}} - \{a, b, c\} = Q|_{\{a,b,c\}} - \{a, b, c\} \text{ and } ac|_{C(Q'')}b.$$

This leads to

$$Q''|_{\{a,b,c\}} - \{a,b,c\} = Q'|_{\{a,b,c\}} - \{a,b,c\}$$

such that $ab|_{C(Q')}c$ and $ac|_{C(Q'')}b$ contrary to (**WI**). □

**Proof of Theorem 1.** Let $\Sigma$ be a semidecisive family on $K$. By definition, a semidecisive family $\Sigma$ is nonempty and so $M_\Sigma$ is nontrivial. Our first goal is to that $M_\Sigma$ satisfies (**WI**). Let $I \in \Sigma$ and define a profile $P$ as follows: $H_i = H_{\{a,b\}}$ for all $i \in I$ and $H_i = H_\emptyset$ otherwise. Observe that $K_{\{a,b\}}(P) = I$ and so $\{a,b\} \in M_\Sigma(P)$. Therefore, for any $c \in S \setminus \{a,b\}$, $ab \mid_{C(P)} c$ and so $I = K_{\{a,b\}}(P) = K_{(a,b,c)}(P) \in U_{M_\Sigma}$. Thus, $\Sigma \subseteq U_{M_\Sigma}$. Since $\Sigma$ is a semidecisive family it follows that $U_{M_\Sigma}$ is a semidecisive family. Therefore, by Theorem 2, $M_\Sigma$ satisfies (**WI**).

To prove that $M_\Sigma$ satisfies item 1 in (**TN**), assume $K_{(a,b,c)}(P) = K_{(x,y,z)}(P')$ and that $ab|_{C(P)}c$. Then there exists $X \in M_\Sigma(P)$ such that $a, b \in X$, $c \notin X$, and $K_X(P) \in \Sigma$. Observe that $K_X(P) \subseteq K_{(a,b,c)}(P)$. Define $P''$ as follows: $H''_i = H_{\{x,y,w\}}$ for all $i \in K_{(a,b,c)}(P) \setminus K_X(P)$ where $w \in S \setminus \{x,y,z\}$ and $H''_i = H'_i|_{\{x,y,z\}} - \{x,y,z\}$ otherwise. Notice that $K_{\{x,y\}}(P'') = K_X(P) \in \Sigma$ and so $\{x,y\} \in M_\Sigma(P)$. Thus $xy \mid_{C(P'')} z$. Finally , observe that $P''|_{\{x,y,z\}} - \{x,y,z\} = P'|_{\{x,y,z\}} - \{x,y,z\}$.

To prove that $M_\Sigma$ satisfies item 2 in (**TN**), assume $K_X(P) = K_{(x,y,z)}(P')$ and $X \in M_\Sigma(P)$. Then $K_X(P) \in \Sigma$. Define $P''$ by $P'' = P'|_{\{x,y,z\}} - \{x,y,z\}$ and observe that $P''|_{\{x,y,z\}} - \{x,y,z\} = P'|_{\{x,y,z\}} - \{x,y,z\}$. Moreover, $K_{\{x,y\}}(P'') = K_{(x,y,z)}(P') = K_X(P) \in \Sigma$ and so $\{x,y\} \in M_\Sigma(P'')$. Thus, $xy \mid_{M_\Sigma(P'')} z$.

Up to this point it should be noted that all we needed was that $\Sigma$ is a semidecisve family. In the next part of the proof we will need to know that $\Sigma$ is actually a decisive family.

To prove that $M_\Sigma$ satisfies item 3 in (**TN**), assume $K_X(P) = K_{(x,y,z)}(P')$ and $xy \mid_{M_\Sigma(P')} z$. Then there exists $Y \in M_\Sigma(P')$ such that $x, y \in Y$, $z \notin Y$, and $K_Y(P') \in \Sigma$. Observe that $K_Y(P') \subseteq K_{(x,y,z)}(P') = K_X(P)$. Since $K_Y(P') \in \Sigma$ and $\Sigma$ is a decisive family it follows that $K_X(P) \in \Sigma$ and so $X \in M_\Sigma(P)$.

For the converse assume that $C$ is a nontrivial consensus rule satisfying (**WI**) and (**TN**). By Theorem 2, we know that $U_C$ is a semidecisive family. Our goal is to show that $U_C$ is a decisive family and that $C = M_{U_C}$. Suppose $X \in C(P)$ for some profile $P$ and nontrivial cluster $X$. Let $(x,y,z)$ be any triple. Choose a profile $P'$ such that $K_X(P) = K_{(x,y,z)}(P')$. Since $X \in C(P)$ it follows from item 2 in (**TN**) that there exists a profile $P''$ such that $P''|_{\{x,y,z\}} - \{x,y,z\} = P'|_{\{x,y,z\}} - \{x,y,z\}$ and $xy|_{C(P'')}z$. Notice that $K_X(P) = K_{(x,y,z)}(P'') = K_{(x,y,z)}(P')$. Also, notice that $K_{(x,y,z)}(P'') \in U_C$. It follows that $X \in M_{U_C}(P)$ and so $C(P) \subseteq M_{U_C}(P)$.

Now let $Y \in M_{U_C}(P)$ and note that $K_Y(P) \in U_C$. So there exists a profile $P'$ and a triple $(x, yz)$ such that $K_Y(P) = K_{(x,y,z)}(P')$ and $xy|_{C(P')}z$. It follows from item 3 in (**TN**) that $Y \in C(P)$. At this stage, we know that $C(P) = M_{U_C}(P)$ for any profile $P$.

The last step is to show that $U_C$ is a decisive family. Let $I \in U_C$ and suppose $I \subseteq J \subseteq K$. Define a profile $P$ as follows: $H_i = H_{\{x,y\}}$ for all $i \in I$; $H_i = H_{\{x,y,w\}}$ for all $i \in J \setminus I$; $H_i = H_\emptyset$ otherwise. Since $K_{\{x,y\}}(P) = I \in U_C$ and $C(P) = M_{U_C}(P)$ it follows that $\{x,y\} \in C(P)$. Therefore, $xy \,|_{C(P)}\, z$ for any $z \in S \setminus \{x,y,w\}$. Since $K_{(x,y,z)}(P) = J$ it follows that $J \in U_C$ and we're done. □

It turns out that not all the conditions of Theorem 1 are independent. In fact, it can be shown that if $C$ is a nontrivial consensus rule satisfying items 2 and 3 in (**TN**), then $C$ satisfies item 1 in (**TN**) and (**WI**). On the other hand, it is not possible to drop either item 2 or item 3 in (**TN**) and still prove that $C = M_{U_C}$.

We conclude with an example showing why item 3 in (**TN**) is needed for the main result.

**Example 3.** *Define $C : \mathcal{H}(S)^3 \to \mathcal{H}(S)$ as follows: $C(P) = H_X$ if $P = (H_X, H_X, H_X)$ and $|X| = n-1$; $C(P) = H_\emptyset$ otherwise. So $C$ outputs a nontrivial hierarchy only at $n$ profiles and $U_C = \{K\}$. It can be verified that $C$ satisfies items 1 and 2 in (*__TN__*). Since $U_C = \{K\}$ is a semidecisive family and $C$ satisfies item 1 in (*__TN__*) it follows from Theorem 2 that $C$ satisfies (*__WI__*).*

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