A geometric approach to judgement aggregation

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1 Introduction

The problem of judgement aggregation consists in aggregating individual judgments on an agenda of logically interconnected propositions into a collective set of judgments on these propositions. This relatively new literature (see List and Puppe (2007) for a survey) is centred on problems like the discursive dilemma which are structurally similar to paradoxes and problems in social choice theory like the Condorcet paradox and Arrow's general possibility theorem. Saari (1995) has successfully introduced a geometric approach to the analysis of such paradoxes the extension of which to judgment aggregation seems promising.

A major difference of judgement aggregation to social choice theory lies in the representation of the information involved. While binary relations over a set of alternatives are a natural representation of preferences, judgments are typically represented by sets of propositions or by vectors of their valuations, where the logical interconnections between these propositions determine the set of feasible valuations. E.g. the agenda $(p, q, p \wedge q)$ is associated the set of feasible valuations $\{(0,0,0),(1,0,0),(0,1,0),(1,1,1)\}$.

In this paper we want to use Saari's tools to analyse results in judgement aggregation. In particular we will first use Saari's representation cubes to provide a geometric presentation of profiles and majority rule outcomes. Applying Saari's idea of a profile decomposition we will show what can go wrong in certain domains of judgment aggregation and how problems can be avoided with the help of domain restrictions. Moreover, we will show that usual qualified majorities can not resolve such paradoxical situations.

2 Formal Framework

Let J be the set of propositions on which judgements have to be made. Most problems in the literature on judgment aggregation can be formulated with the help of vectors of binary valuations $x = (x^1, x^2, ..., x^{|J|}) \in X \subseteq \{0,1\}^{|J|}$, where $x^j = 1$ means that proposition j is believed and X denotes the set of all feasible (typically: logically consistent) valuations.

A profile of individual judgments is represented by a vector $\mathbf{p} = (p_1, p_2, ..., p_{|X|})$ which associates with every binary valuation $x_k \in X$ the fraction p_k of individuals with this valuation. This is an anonymous representation of voters' preferences as only the distribution but not the actual number of preferences is required.

A judgment aggregation rule is a mapping f that associates with every profile $\mathbf{p} = (p_1, p_2, ..., p_{|X|})$ a valuation $f(\mathbf{p}) \in \{0,1\}^{|J|}$. If $f(\mathbf{p}) \in X$ for all $\mathbf{p}$ we will call f consistent.

Saari (1995) analysed preference aggregation using a geometric approach. For the simplest setting consider three alternatives a, b, c. This gives rise to three pairwise comparisons, namely between a and b, b and c and c and a. A "1" for the first issue (i.e. the comparison between a and b) means that a is preferred to b, written $a \succ b$. On the other hand, a "0" indicates the opposite preference, i.e. $b \succ a$. By using $x_j = \frac{\sum_{i \in N} x_j^i}{n}$ a preference profile maps into a point $x \in [0,1]^m$ in the hypercube with dimension 3 (the number of pairwise comparisons). See figure 1.

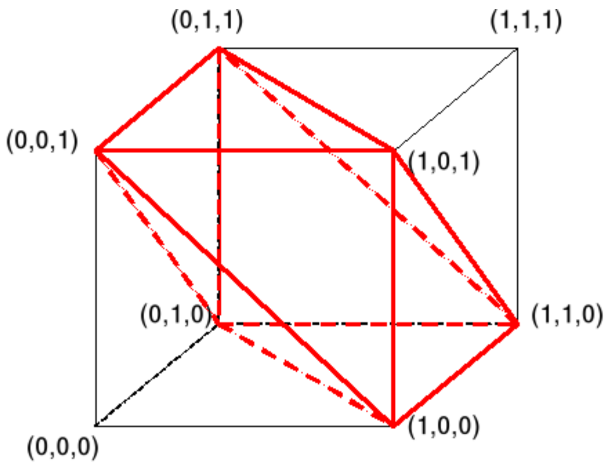

Figure 1: Saari's representation cube

In figure 1, the vertex $(0,0,1)$ thus represents the preference where $b \succ a$, $c \succ b$ and $c \succ a$ or - for simplicity - the ranking cba. As there are eight vertices but only six rational rankings of the three alternatives, there are two vertices representing irrational voters with cyclic preferences, namely $(0,0,0)$ and $(1,1,1)$. If we exclude those vertices, we see that the convex hull of the remaining six vertices is the representation polytope, i.e. every preference profile maps into a point in this polytope.

3 Majority (In)consistency and Domain Restrictions

The same 3-dimensional hypercube can be used for a simple judgement aggregation problem with 3 propositions (issues), i.e. $|J| = 3$. For simple majority voting on the issues, every profile of individual judgements on J is mapped into a point in the hypercube. Its Euclidean distance to the respective vertices determines the majority outcome. This means that the hypercube can be partitioned into 8 equally sized subcubes each determining the majority outcome for profiles mapped into those subcubes. E.g. in figure 2 the shaded subcube, determined by the diagonal $\left[(\frac{1}{2},\frac{1}{2},\frac{1}{2}),(1,0,1)\right]$ consists of all points $x(\mathbf{p}$ that are of closest Euclidean distance to the vertex $(1,0,1)$ and hence lead to a majority outcome of $(1,0,1)$. For $d_E(x,y)$ denoting the Euclidean distance between $x,y \in \{0,1\}^{|J|}$, we can also think of the majority valuation x^M as the

$$argmin_{x\in\{0,1\}^{|J|}} \sum_{k=1}^{|X|} p_k d_E(x^k,x)$$

.

Consider the following set of propositions $\mathcal{P} = \{p, q, p \wedge q\}$ with associated domain of feasible individual valuations $X = \{(0,0,0),(1,0,0),(0,1,0),(1,1,1)\}$. The four feasible vertices in the hypercube determine the representation polytope as seen in figure 3.

Given X, consider the profile $\mathbf{p} = (0,\frac{1}{3},\frac{1}{3},\frac{1}{3})$, i.e. no voter has valuation $(0,0,0)$, one third of the voters has valuation $(1,0,0)$, and so on. As this maps into the point $x = (\frac{2}{3},\frac{2}{3},\frac{1}{3})$ - a point whose closest vertex is $(1,1,0)$ - the representation polytope obviously passes through one subcube representing a majority outcome not in the domain. That such an incsonsistency can occur in general can also be seen from the following lemma:

Lemma 1 *Given any vertex $x \in \{0,1\}^{|J|}$, there exist 3 vertices a,b,c such that for some linear combinations of those vertices there is a point in the x-subcube.*

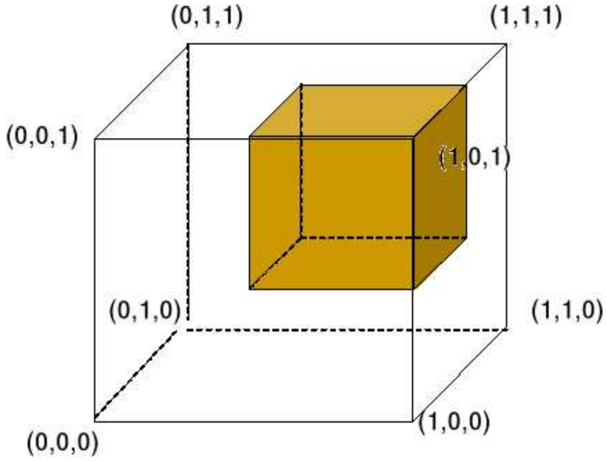

Figure 2: Majority subcube

For $|J| = 3$, these 3 vertices necessarily need to be the neigbors of that vertex, i.e. they are only allowed to differ from it in one issue. Given that, we can now provide a simple result for the occurrence of majority consistency, i.e. what X needs to look like to guarantee that the majority outcomes are themselves in X.

Proposition 1 *For $|J| = 3$, the set of valuations X is majority consistent iff for any triple of vertices in the domain with a common neighbor, this common neighbor is also contained in the domain.*

To analyse those paradoxical outcomes and suggest restrictions to overcome those, we will use a tool developed by Saari (1995). Consider two indviduals with the respective valations $(1, 0, 0)$ and $(0, 1, 1)$. They are exact opposites, so from a majority rule point of view those two valuations cancel out. Hence this implies that for any two opposite feasible valuations in X, we can cancel the valuation held by the smaller number of individuals. This leads to a reduced profile, the majority outcome of which is identical to the majority outcome of the original profile.

E.g. given $X = \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 1)\}\}$, let (p_1, p_2, p_3, p_4) be the shares of individuals holding each of the respective valuations where $\sum_i p_i = 1$. As $(0, 0, 0)$ and $(1, 1, 1)$ are exact opposites, the reduced profile will have a share of 0 for the valuation held by the smaller number of individuals. In the case of $p_1 > p_4$ such a reduced profile will be $\mathbf{p}' = (\frac{p_1 - p_4}{p_1 + p_2 + p_3 - p_4}, \frac{p_2}{p_1 + p_2 + p_3 - p_4}, \frac{p_3}{p_1 + p_2 + p_3 - p_4}, 0)$. Hence the reduced profile maps into one of the following two planes represented in figure 4.

Now we want to determine whether there are consistency conditions, i.e. what sort of profiles do guarantee majority consistency in the sense of a

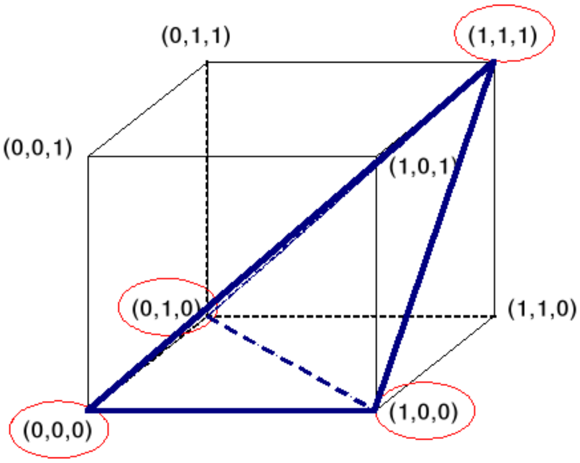

Figure 3: representation polytope

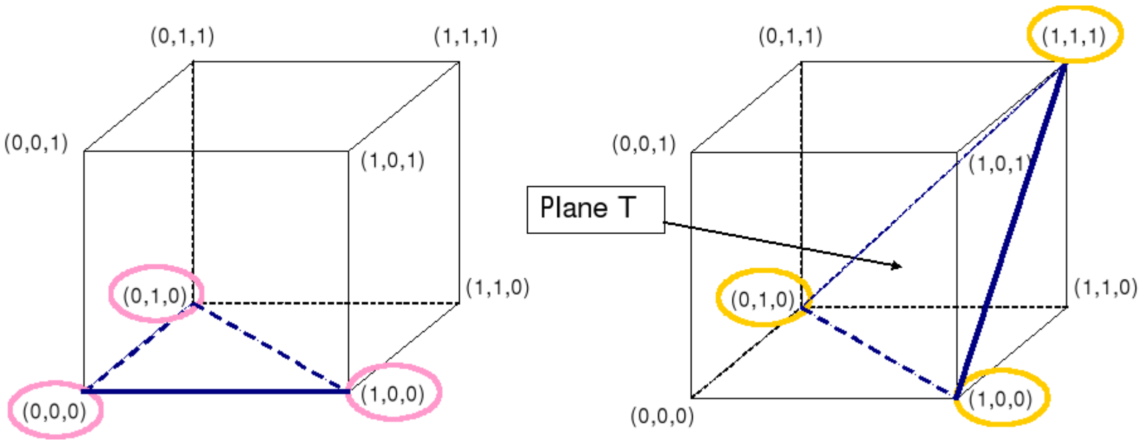

Figure 4: planes

majority outcome in X. Let $a_i = \frac{p_i}{\sum_{i=2}^{4} p_i}$, for $i = 2, 3, 4$. Then $\alpha = (a_2 + a_4, a_3 + a_4, a_4) \in T$. By definition, $\mathbf{p} = (1 - p_1)(0, a_2, a_3, a_4) + p_1(1, 0, 0, 0)$. By linearity, $x(\mathbf{p}) = (1 - p_1)\alpha + p_1(0, 0, 0) = (1 - p_1)\alpha$, where $x(\mathbf{p}) \in [0, 1]^{|J|}$ is the vector of average values over all individual valuations on each issue.

So, geometrically any profile can be plotted via a point in T, its connection to the $(0, 0, 0)$ vertex and a weight p_1. The following figure 5 shows plane T, the shaded area of which represents the troublesome points, i.e. the cut with the $(1, 1, 0)$-subcube and hence those points where a profile leads to an inconsistent majority outcome.

Now it is easy to see what form of (domain) restrictions can avoid such inconsistent majority outcomes. Either $p_1 \leq p_4$, because then reduced pro-

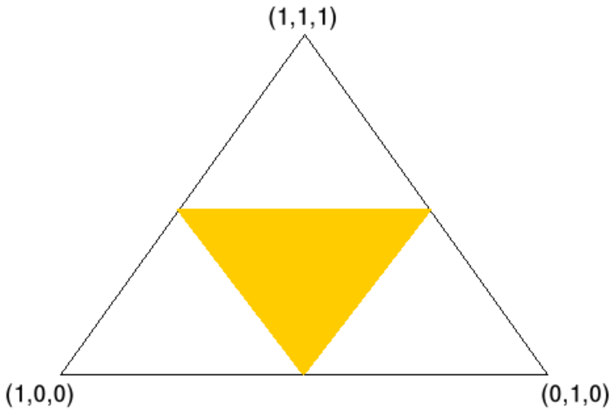

Figure 5: plane T

files are mapped into the plane indicated on the left of figure 4. Otherwise, if $p_4 > p_1$, we need to find all profiles that map into points outside the shaded area. More precisely those restrictions are the following:

- $\frac{p_2}{\sum_{i=2}^{4} p_i} > \frac{1}{2}$
- $\frac{p_3}{\sum_{i=2}^{4} p_i} > \frac{1}{2}$
- $\frac{p_4}{\sum_{i=2}^{4} p_i} > \frac{1}{2}$

One interesting feature of those restrictions is that they are based on the space of profiles which is more general than restrictions on the space of valuations which is usually used in classical domain restrictions.

Moreover, this framework also opens the analysis of various paradoxical situations as stated in the following proposition:

Proposition 2 *There exist profiles such that there is almost unanimous agreement on one issue and still an inconsistent majority outcome is obtained.*

Intuitively this can be seen by looking at figure 5 and considering points right at the corner of the shaded triangle. Those represent profiles that lead to an incosistent majority outcome but still provide almost unanimous agreement on one proposition.

4 Codomain Restrictions and Distance-Based Aggregation

Besides restrictions on the space of profiles, there is an alternative way to guarantee collective rationality, namely via restricting the set of outcomes to collectively rational outcomes. One possibility to do this is by using distance-based aggregation rules. In analogy to a well-known procedure in social choice theory (Kemeny [1], Pigozzi ([4]) introduced such an approach to judgment aggregation. In principle a distance-based aggregation rule determines the social valuation as the valuation that minimizes the sum of distances to the individual valuations. Formally, given the profile of individual valuations $(x^1, x^2, ..., x^n)$, the social valuation is the consistent and complete valuation x that minimizes the sum of distances to the individual valuations, i.e.

$$f(\mathbf{p}) = argmin_{x \in X} \sum_{i=1}^{n} d(x, x^i)$$

.

The most commonly used distance function is the Hamming distance, which counts the number of issues on which two valuations disagree, i.e. for $x = (1, 0, 0)$ and $x' = (1, 1, 1)$, $d(x, x') = 2$.

Now given our geometric approach, there is a simple geometric explanation of this distance-based aggregation rule. As could be seen in 5, all problematic profiles lead to a point in the shaded triangle. However, one option is to divide the triangle into three sub-triangles as in 6.

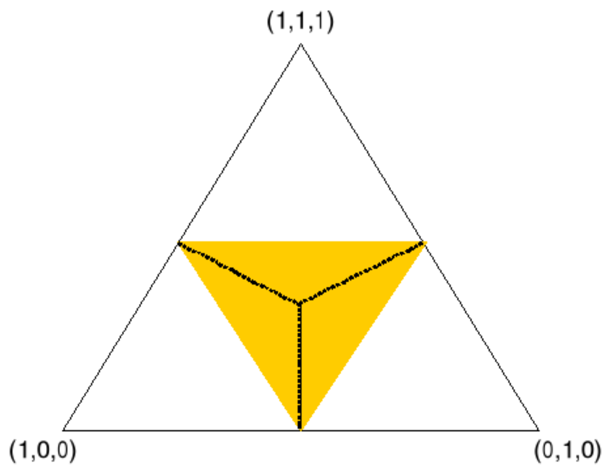

Figure 6: Distances

The point in the middle is exactly the barycenter point of the triangle $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Using these additional lines, we now have divided the triangle into three areas, points in which are characterized by being of smallest Euclidean distance to the vertex of the proper triangle w.r.t. the points within the shaded triangle. So points in the south-western part of the shaded triangle will be closest to the $(1, 0, 0)$ vertex. This, however, is identical to saying that for any point in the shaded triangle, switch the majority valuation on the issue which is closest to the 50-50 threshold (see Merlin and Saari [3]).

Example 1 *Let* $X = \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 1)\}$ *and* $\mathbf{p} = (0.1, 0.35, 0.3, 0.25)$. *This leads to* $x(\mathbf{p}) = (0.6, 0.55, 0.25)$ *and hence an inconsistent majority outcome* $(1, 1, 0)$. *Looking at picture 5 we see that* $\alpha \in T$ *lies in the south-western shaded triangle. Thus, according to our distance-based aggregation rule, the outcome will be the consistent valuation* $(1, 0, 0$ *as* α *is closest to the* $(1, 0, 0)$ *vertex. However, this can also be seen as swithching the valuation on the issue which is closest to the 50-50 threshold, which - in* $x(\mathbf{p})$ *- is obviously issue 2.*

5 Likelihood of Inconsistency

After having studied various possibilites to avoid inconsistent majority outcomes, we now want to use the geometric framework to anaylize the frequency of occurence of such inconsistencies. Given that only 4 vertices are feasible individual valuations, any majority outcome in the representation cube is determined by a unique profile. Consider again the situation $X = \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 1)\}\}$. Then for any vector of shares of individual valuations $\mathbf{p} = (p_1, p_2, p_3, p_4)$ we get the following: $x_1 = p_2 + p_4$, $x_2 = p_3 + p_4$, $x_3 = p_4$, $1 = p_1 + p_2 + p_3 + p_4$. As those are 4 equations with 4 unknowns there exists a unique solution. The volume of certain subspaces now indiciates the likelyhood of occurrence of certain outcomes. Consider first the volume of the representation cube, V: $V = \frac{1}{2} \cdot 1 \cdot \frac{1}{3} = \frac{1}{6}$. On the other hand, points leading to inconsistent majority outcomes are located in the tetraeder determined by the points $\left[(\frac{1}{2}, \frac{1}{2}, 0), (1, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 1, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\right]$. The volume of this tetraeder is $\frac{1}{24}$ (see figure 7).

So its volume relative to the volume of the whole representation polytope is $\frac{1}{4}$ and hence we can say that - under the assumption of equal probabilities of individuals holding each feasible valuation - the probabily of an outcome being inconsistent is 25 percent.

Of course, different domains allow for different probabilities. E.g. consider $X = \{(0, 0, 1), (1, 0, 0), (0, 1, 0), (1, 1, 1)\}\}$. Then $x_1 = p_2 + p_4$, $x_2 = p_3 + p_4$,

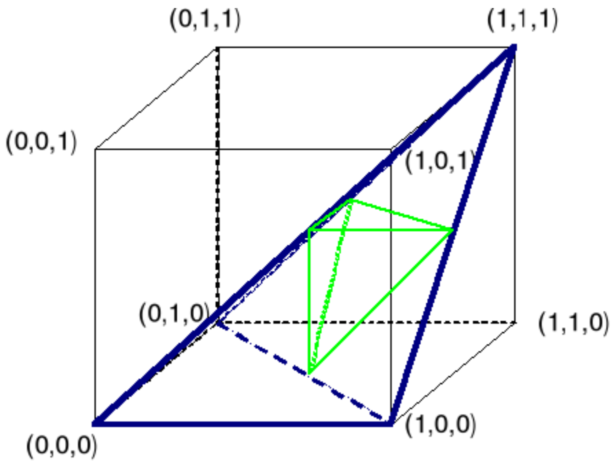

Figure 7: Distances

$x_3 = p_1 + p_4$ and $p_1 + p_2 + p_3 + p_4 = 1$. Again we get a unique solution. Making the same calculations as before, we get a probability of inconsistent outcomes of 50 percent.

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