Universal Rigidity of Bipartite Graphs

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The molecule problem: existence

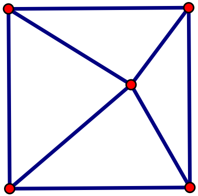
Basic Molecule Problem: Given a graph G and edge lengths $\{e_{ij} = e_{ji}\}$ for G, is there a configuration $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_n)$ in $\mathbb{R}^d$ such that $|\mathbf{p}_i - \mathbf{p}_j| = e_{ij}$ for all edges ij of G?

Easier Molecule Problem: Same as above, but allow the realization $\mathbf{p}$ to be in any higher dimensional Euclidean space $\mathbb{R}^D \supset \mathbb{R}^d$.

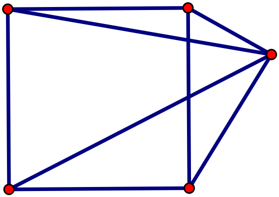
Universal rigidity: uniqueness

Given a graph G and a corresponding configuration $\mathbf{p}$ in $\mathbb{R}^d$, we say that $(G, \mathbf{p})$ is *universally rigid* if for any other configuration $\mathbf{q}$ in any $\mathbb{R}^D \supset \mathbb{R}^d$, with corresponding edge lengths in G the same, then $\mathbf{q}$ is congruent to $\mathbf{p}$. That is, ALL edge lengths of $\mathbf{q}$ are the same as the corresponding edge lengths of $\mathbf{p}$.

Planar examples



Not universally rigid



Universally rigid

Back to the molecule problem

A direct consequence of the definition of universal rigidity is:

Theorem

If $(G, \mathbf{p})$ is universally rigid, then the molecule problem, given its edge lengths, is approximately solvable by semi-definite programming (SDP).

The determination of universal rigidity is a much more tractable problem on its own.

Theorem (Connelly-Gortler 2014)

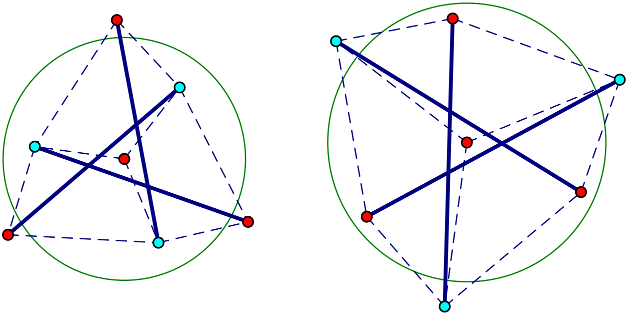
Given universally rigid $(G, \mathbf{p})$, in general, it is possible to find a certificate that guarantees that it is universally rigid.

The certificate above is a sequence of positive semi-definite matrices, whose ranks sum to $n - d - 1$ and another calculation that rules out affine motions.

An interesting example: Complete bipartite graphs

Theorem (Connelly, Gortler 2015)

If $(K(n,m),(\mathbf{p},\mathbf{q}))$ *is a complete bipartite framework in* $\mathbb{R}^d$*, with* $m+n \geq d+2$*, such that the partition vertices* $(\mathbf{p},\mathbf{q})$ *are strictly separated by a quadric, then it is not universally rigid.*



Universally Rigid

Not Universally Rigid

Conic/quadric separation

Our quadric is of the form $\{\mathbf{x} \in \mathbb{R}^d \mid \hat{\mathbf{x}}^t A \hat{\mathbf{x}} = 0\}$, where A is a $(d+1)$-by-$(d+1)$ symmetric matrix, $\hat{\mathbf{x}}$ is the vector $\mathbf{x}$ with a 1 added as an extra coordinate, and $\hat{\mathbf{x}}^t$ is its transpose.

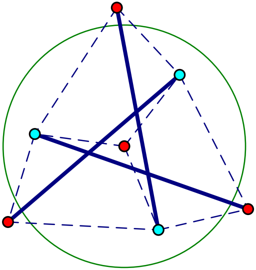
Definition

If $\mathbf{p} = (\mathbf{p_1}, \ldots, \mathbf{p_n})$ and $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_m)$ are two configurations of points in $\mathbb{R}^d$, we say that they are *strictly separated by a quadric*, given by a matrix A, if for each $i = 1, \ldots, n$ and $j = 1, \ldots, m$,

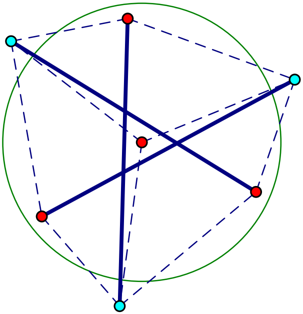
$$\hat{\mathbf{q}}_j^t A \hat{\mathbf{q}}_j < 0 < \hat{\mathbf{p}}_i^t A \hat{\mathbf{p}}_i.$$

Symmetry Helps

If the configuration is symmetric about a point with a sufficiently large symmetry group the conic (or quadric in 3D) must also be symmetric by averaging, and this simplifies the calculation of universal rigidity considerably.



Universally Rigid



Not Universally Rigid

Stress Matrices

Any set of scalars $\omega_{ij} = \omega_{ji}$ associated to all the pairs of points of a configuration is called a *stress*, and in the context of an associated graph G, we assume that $\omega_{ij} = 0$ for non-edges i, j of G. The matrix of the quadratic form

$$\sum_{i<j} \omega_{ij}(x_i - x_j)^2$$

is called the *stress matrix* Ω, where the row and column sums are 0, and the i, j entry is $-\omega_{ij}$ for $i \neq j$. We say Ω is an (equilibrium) stress for a configuration $\mathbf{p}$ if for each vertex i

$$\sum_{j} \omega_{ij}(\mathbf{p}_j - \mathbf{p}_i) = 0.$$

Basic Results

The following fundamental theorem is a basic tool and the first step used to establish universal rigidity.

Theorem

Let $(G, \mathbf{p})$ be a framework whose affine span of $\mathbf{p}$ is all of $\mathbb{R}^d$, with an equilibrium stress ω and stress matrix Ω. Suppose further

(i) *Ω is positive semi-definite (PSD).*

(ii) *The rank of Ω is $N - d - 1$.*

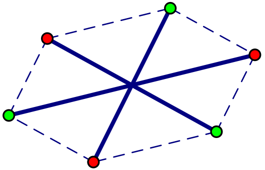
(iii) *The edge directions of $(G, \mathbf{p})$ do not lie on a conic at infinity.*

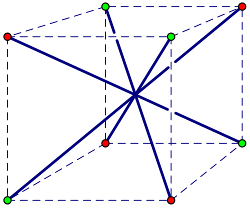
Then $(G, \mathbf{p})$ is universally rigid.

A framework satisfying these properties are called *super stable* (and is clearly universally rigid).

A Small Bipartite Example

When the number of vertices and edges of the graph G is small, one very simple example is $K(d+1, d+1)$ is $\mathbb{R}^d$, where one partition $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_{d+1})$ is just chosen to be affine independent with the origin in the centroid of its convex hull, while $\mathbf{q} = -\mathbf{p} = (-\mathbf{p}_1, \ldots, -\mathbf{p}_{d+1})$.





These are all super stable frameworks.

Rigidity Definitions

Definition

We say a framework $(G, \mathbf{p})$ is *globally rigid* in $\mathbb{R}^d$ if any other framework $(G, \mathbf{q})$ equivalent to $(G, \mathbf{p})$ in $\mathbb{R}^d$ is such that $\mathbf{q}$ is congruent to $\mathbf{p}$.

Definition

We say a framework $(G, \mathbf{p})$ is *locally rigid* in $\mathbb{R}^d$ if every other framework $(G, \mathbf{q})$, equivalent to $(G, \mathbf{p})$ in an open neighborhood $U_{\mathbf{p}}$ of $\mathbf{p}$ in configuration space, is congruent to $(G, \mathbf{p})$.

Definition

We say a framework $(G, \mathbf{p})$ is *infinitesimally rigid* spanning $\mathbb{R}^d$ if there is an open neighborhood $U_{\mathbf{p}}$ of $\mathbf{p}$ in configuration space such that for all configurations $\mathbf{q}, \mathbf{q}' \in U_{\mathbf{p}}$, $(G, \mathbf{q})$ is equivalent to $(G, \mathbf{q}')$ if and only if $\mathbf{q}$ is congruent to $\mathbf{q}'$.

Connecting Theorems

Definition

We say a configuration $\mathbf{p}$ is *generic* in $\mathbb{R}^d$ if the coefficients of $\mathbf{p}$ satisfy no non-zero polynomial with integer coefficients.

Theorem (Gortler-Thurston)

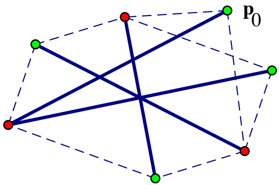
If a given framework $(G, \mathbf{p})$ in $\mathbb{R}^d$ is infinitesimally rigid and globally rigid, then $(G, \mathbf{q})$ in $\mathbb{R}^d$ is globally rigid at any generic configuration $\mathbf{q}$.

Theorem (Connelly-Gortler-Theran 2016)

If a given framework $(G, \mathbf{p})$ in $\mathbb{R}^d$ is globally rigid, then there is a generic configuration $\mathbf{q}$ in $\mathbb{R}^d$ such that $(G, \mathbf{q})$ is super stable and infinitesimally rigid.

Trilateration

For any given framework $(G, \mathbf{p})$ in $\mathbb{R}^d$ a trilaterization is an attachment of another vertex, say $\mathbf{p}_0$ to $d+1$ other vertices of $(G, \mathbf{p})$ such that $\mathbf{p}_0$ and the other attaching vertices are in *general position* (no $d+1$ of the $d+2$ vertices lie in a hyperplane).



This is an example of super stable $K(3,3)$ trilaterated from $\mathbf{p}_0$.

Proposition (easy)

Trilaterization preserves universal, global, and infinitesimal rigidity.

Basic Results

The following is the starting point for showing non-universal rigidity.

Theorem (Alfakih 2011)

If $(G, \mathbf{p})$ is a universally rigid framework with N vertices whose affine span is d dimensional, $d \leq N - 2$, then $(G, \mathbf{p})$ has a non-zero PSD equilibrium stress matrix Ω.

Note that the rank of the stress matrix Ω implied above can be quite low, even one-dimensional. This is also the starting point for the one-dimensional case of Jordan and Nguyen.

Proof of the bipartite theorem

Let A be the $(d+1)$-by-$(d+1)$ symmetric matrix for the separating quadric for our bipartite theorem, and let ω be an equilibrium stress for $(K(n,m),(\mathbf{p},\mathbf{q}))$ with stress matrix Ω. For any vertex $\mathbf{q}_j$ in one partition, the equilibrium condition can be written, for each $j = 1,\ldots,m$ as

$$\sum_{i=1}^{n} \omega_{ij}(\hat{\mathbf{p}}_i - \hat{\mathbf{q}}_j) = 0,$$

or equivalently

$$\sum_{i=1}^{n} \omega_{ij}\hat{\mathbf{p}}_i = \left(\sum_{i=1}^{n} \omega_{ij}\right)\hat{\mathbf{q}}_j = \mu_j\hat{\mathbf{q}}_j.$$

Then taking the transpose of this equation, and multiplying on the right by $A\mathbf{q}_j$, we get

$$\sum_{i=1}^{n} \omega_{ij}\hat{\mathbf{p}}_i^t A\hat{\mathbf{q}}_j = \mu_j\hat{\mathbf{q}}_j^t A\hat{\mathbf{q}}_j.$$

Similarly for the $\mathbf{p}_i$. ($\hat{\mathbf{x}}$ is the vector $\mathbf{x}$ with a 1 added as an extra coordinate.)

Proof of the bipartite theorem

$$\sum_{j=1}^{m} \mu_j \hat{\mathbf{q}}_j^t A \hat{\mathbf{q}}_j = \sum_{ij} \omega_{ij} \hat{\mathbf{p}}_i^t A \hat{\mathbf{q}}_j = \sum_{ij} \omega_{ij} \hat{\mathbf{q}}_i^t A \hat{\mathbf{p}}_j = \sum_{i=1}^{n} \lambda_j \hat{\mathbf{p}}_i^t A \hat{\mathbf{p}}_i.$$

By Alfakih's Theorem, if $(K(n,m),(\mathbf{p},\mathbf{q}))$ were universally rigid, then there would be an equilibrium stress with a stress matrix Ω that would be PSD and non-zero. Then $\mu_j \geq 0$ for all $j = 1,\ldots,m$, $\lambda_i \geq 0$ for all $i = 1,\ldots,n$, and we would have at least one positive diagonal term. But then the equation above would contradict the quadric separation condition in the definition of quadric/conic separation. Thus $(K(n,m),(\mathbf{p},\mathbf{q}))$ is not universally rigid.

The Verenose Map

- Let $\mathcal{M}_d$ be the $(d+1)(d+2)/2$ dimensional space of $(d+1)$-by-$(d+1)$ symmetric matrices, which we call the *matrix space*.
- Define the map $\mathcal{V} : \mathbb{R}^d \to \mathcal{M}_d$ by $\mathcal{V}(\mathbf{v}) = \hat{\mathbf{v}}\hat{\mathbf{v}}^t$, which is a $(d+1)$-by-$(d+1)$ symmetric matrix, with the lower right-hand coordinate 1, where $\hat{\mathbf{v}}$ is the vector with an extra coordinate of 1 added at the bottom.
- $\mathcal{V}(\mathbb{R}^d)$ is a d-dimensional algebraic set embedded in a $(d+1)(d+2)/2 - 1$ dimensional linear subspace. The function $\mathcal{V}$ is called the *Veronese map*.

The Veronese Map

The effect of $\mathcal{V}$ is to transfer quadratic conditions in $\mathbb{R}^d$ to linear conditions in a $(d+1)(d+2)/2-1$ dimensional linear subspace of $\mathcal{M}_d$.

Proposition

In $\mathbb{R}^d$ the vertices of the configurations $\mathbf{p}$ and $\mathbf{q}$ can be strictly separated by a quadric , if and only if the matrix configurations $\mathcal{V}(\mathbf{p})$ and $\mathcal{V}(\mathbf{q})$ can be strictly separated by a hyperplane in $\mathcal{M}_d$.

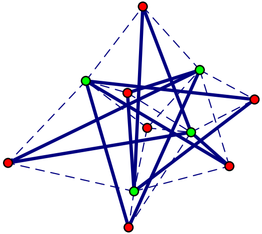
So in the plane the vertices of the partitions of a bipartite graph $K(4,3)$ can be strictly separated by a conic if and only if their Veronese images can be linearly separated by a hyperplane in a 5-dimensional linear space.

Non-separability Predicts Universal Rigidity

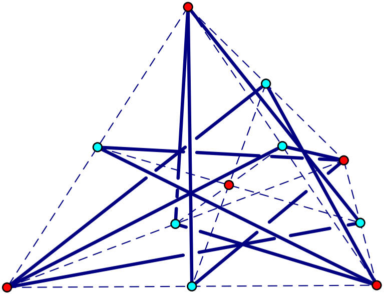
Theorem (Connelly, Gortler 2015)

If the convex hull of $\mathcal{V}(\mathbf{p})$ *and* $\mathcal{V}(\mathbf{q})$ *intersect in the relative interior of each set, then* $(K(m,n),(\mathbf{p},\mathbf{q}))$ *is universally rigid.*

Some examples in $\mathbb{R}^3$:



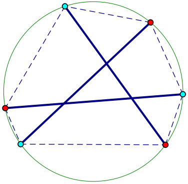
K(7,4)



K(6,5)

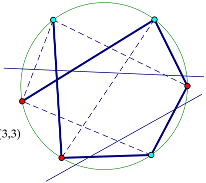
Lower Dimensional Spans

When the configuration lies on a conic (or quadric) the separation criteria determine the universal rigidity.

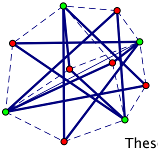


Universally Rigid

K(3,3)

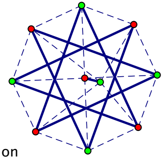


Not Universally Rigid

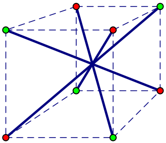


K(6,4)

These lie on one quadric.



K(5,5)



K(4,4)

This lies on the intersection of 3 quadrics.

Back to the Molecule Problem

- The generic case: This is where there is no algebraic relation among the coordinates of the configuration. Here the conic/quadric separation criterion determines whether its configuration can be determined with SDP.
- The generic realization of $K(5,5)$ in $\mathbb{R}^3$ has another realization with the same edge lengths. So SDP fails to give a 3 dimensional realization, or if it does, it may not be the "right" one. But when its vertices lie on a quadric and cannot be separated with a quadric, it can be universally rigid.
- Instead of general position or generic position, it is useful to assume that the Veronese realization is in general position. So SDP success is determined by the separability.

Acknowledgements

A paper "On universally rigid frameworks on the line" by Tibor Jordán and Viet-Hang Nguyen was a big impetus for our results here. They essentially gave the separation criterion for the case when the bipartite graph is in the line.

An old paper "When is a bipartite graph a rigid framework" by Ethan Bolker and Ben Roth was very insightful and laid the groundwork for this paper.